EFFECTIVE DIVISORS ON $\overline{\mathcal{M}}_g$ AND A COUNTEREXAMPLE TO THE SLOPE CONJECTURE

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1. Introduction

The purpose of this note is to prove two statements on the slopes of effective divisors on the moduli space of stable curves $\overline{\mathcal{M}}_g$: first that the Harris-Morrison Slope Conjecture fails to hold on $\overline{\mathcal{M}}_{10}$ and second, that in order to compute the slope of $\overline{\mathcal{M}}_g$ for $g \leq 23$, one only has to look at the coefficients of the classes λ and δ_0 in the expansion of the relevant divisors. The proofs are based on a general result providing inequalities between the first few coefficients of effective divisors on $\overline{\mathcal{M}}_g$. We give the technical statements in what follows.

On $\overline{\mathcal{M}}_g$ we denote by λ the class of the Hodge line bundle, by $\delta_0, \ldots, \delta_{[g/2]}$ the boundary divisor classes corresponding to singular stable curves and by $\delta := \delta_0 + \cdots + \delta_{[g/2]}$ the total boundary. If $\mathbb{E} \subset \mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{R}$ is the effective cone, then following [HMo] we define the *slope function* $s : \mathbb{E} \to \mathbb{R} \cup \{\infty\}$ by the formula

$$s(D) := \inf \{\frac{a}{b} : a, b > 0 \text{ such that } a\lambda - b\delta - D \equiv \sum_{i=0}^{[g/2]} c_i \delta_i, \text{ where } c_i \geq 0\}.$$

From the definition it follows that $s(D) = \infty$ unless $D \equiv a\lambda - \sum_{i=0}^{[g/2]} b_i \delta_i$ with $a, b_i \geq 0$ for all i (and it is well-known that $s(D) < \infty$ for any D which is the closure of an effective divisor on $\mathcal{M}_g$). In the second case one has that $s(D) = a/\min_{i=0}^{[g/2]} b_i$. We denote by s_g the slope of the moduli space $\overline{\mathcal{M}}_g$, defined as $s_g := \inf \{s(D) : D \in \mathbb{E}\}$. The Slope Conjecture of Harris and Morrison predicts that $s_g \geq 6 + 12/(g+1)$ (cf. [HMo] Conjecture 0.1). This is known to hold for $g \leq 12, g \neq 10$ (cf. [HMo] and [Ta]).

Following [CU], we consider the divisor $\mathcal{K}$ on $\mathcal{M}_{10}$ consisting of smooth curves lying on a $K3$ surface, and we denote by $\overline{\mathcal{K}}$ its closure in $\overline{\mathcal{M}}_{10}$. For any $g \geq 20$, we look at the locus in $\overline{\mathcal{M}}_g$ of curves obtained by attaching a pointed curve of genus $g - 10$ to a curve in $\overline{\mathcal{K}}$ with a marked point. This gives a divisor in $\Delta_{10} \subset \overline{\mathcal{M}}_g$, which we denote by Z.

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The key point in what follows is that, based on the study of curves lying on $K3$ surfaces, one can establish inequalities involving a number of coefficients of any effective divisor coming from $\mathcal{M}_g$ in the expansion in terms of the generating classes.

Theorem 1.1. *Let $D \equiv a\lambda - \sum_{i=0}^{[g/2]} b_i\delta_i$ be the closure in $\overline{\mathcal{M}}_g$ of an effective divisor on $\mathcal{M}_g$.*
(a) For $2 \leq i \leq 9$ and $i = 11$ we have $b_i \geq (6i+18)b_0 - (i+1)a$. The same formula holds for $i = 10$ if D does not contain the divisor $Z \subset \Delta_{10}$.
(b) ($g \geq 20$) If D contains Z, then either $b_{10} \geq 78b_0 - 11a$ as above, or $b_{10} \geq (71.3866...) \cdot b_0 - (10.1980...) \cdot a$.
(c) We always have that $b_1 \geq 12b_0 - a$.

Part (a) (and (c)) of this theorem essentially only make more concrete results – and the technique of intersecting with special Lefschetz pencils– already existing in the literature mentioned above. Part (b) however is more involved: it requires pull-backs to $\overline{\mathcal{M}}_{10,1}$ and the intrinsic use of our partial knowledge about the divisor $\overline{\mathcal{K}}$, plus some facts about the Weierstrass divisor on $\overline{\mathcal{M}}_{g,1}$. Here we claim more originality.

Corollary 1.2. *If $a/b_0 \leq 71/10$, then $b_i \geq b_0$ for all $1 \leq i \leq 9$. The same conclusion holds for $i = 10$ if $a/b_0 \leq 6.906\ldots$, and for $i = 11$ if $a/b_0 \leq 83/12$.*

Based on this we obtain that the divisor $\overline{\mathcal{K}} \subset \overline{\mathcal{M}}_{10}$ provides a counterexample to the Slope Conjecture. Its class can be written as

$$\overline{\mathcal{K}} \equiv a\lambda - b_0\delta_0 - \ldots - b_5\delta_5,$$

and by [CU] Proposition 3.5, we have $a = 7$ and $b_0 = 1$. In view of Corollary 1.2, this information is sufficent to show that the slope of $\overline{\mathcal{K}}$ is smaller than the one expected based on the Slope Conjecture.

Corollary 1.3. *The slope of $\overline{\mathcal{K}}$ is equal to $a/b_0 = 7$, so strictly smaller than the bound $78/11$ predicted by the Slope Conjecture. In particular $s_{10} = 7$ (since by [Ta] $s_{10} \geq 7$).*

Theorem 1.1 also allows us to formulate (at least up to genus 23, and conjecturally beyond that) the following principle: *the slope s_g of $\overline{\mathcal{M}}_g$ is computed by the quotient a/b_0 of the relevant divisors*. We have more generally:

Theorem 1.4. *For any $g \leq 23$, there exists $\epsilon_g > 0$ such that for any effective divisor D on $\overline{\mathcal{M}}_g$ with $s_g \leq s(D) \leq s_g + \epsilon_g$ we have $s(D) = a/b_0$, i.e. $b_0 \leq b_i$ for all $i \geq 1$.*

Conjecture 1.5. *The statement of the theorem holds in arbitrary genus.*

Theorem 1.1 is proved in §2. Theorem 1.4 is proved in §3, where we also remark that the methods of the present paper give a very quick proof of the fact that the Kodaira dimension of the universal curve $\overline{\mathcal{M}}_{g,1}$ is $-\infty$ for $g \leq 15$, $g \neq 13, 14$.

2. Inequalities between coefficients of divisors

Let $\mathcal{F}_g$ be the moduli space of canonically polarized $K3$ surfaces (S, H) of genus g. We consider the $\mathbf{P}^g$-bundle $\mathcal{P}_g = \{(S, C) : C \in |H|\}$ over $\mathcal{F}_g$ which comes with a natural rational map $\phi_g : \mathcal{P}_g - - > \mathcal{M}_g$. By Mukai's results [Mu1] and [MM], this map is dominant if and only if $2 \leq g \leq 9$ or $g = 11$. In this range $\mathcal{M}_g$ can be covered by curves corresponding to Lefschetz pencils of curves on $K3$ surfaces (cf. [Ta]). This is not true any more when $g = 10$: in this case $\mathrm{Im}(\phi_{10})$ is a divisor $\mathcal{K}$ in $\mathcal{M}_{10}$ (cf. [CU] Proposition 2.2).

Note that any such Lefschetz pencil, considered as a family of curves over $\mathbf{P}^1$, has at least one section, since its base locus is nonempty.

Given $2 \leq i \leq 11$, consider as above a Lefschetz pencil of curves of genus i lying on a general $K3$ surface of degree $2i-2$ in $\mathbf{P}^i$. This gives rise to a curve B in the moduli space $\overline{\mathcal{M}}_i$.

Lemma 2.1. *We have the formulas* $B \cdot \lambda = i + 1$, $B \cdot \delta_0 = 6i + 18$ *and* $B \cdot \delta_j = 0$ *for* $j \neq 0$.

Proof. The first two numbers are computed e.g. in [CU] Proposition 3.1, based on the formulas in [GH] pp. 508–509. The last assertion is obvious since there are no reducible curves in a Lefschetz pencil. □

For each $g \geq i+1$, starting with the pencil B in $\overline{\mathcal{M}}_i$ we can construct a new pencil B_i in $\overline{\mathcal{M}}_g$ in the following way: we fix a general pointed curve (C, p) genus $g - i$. We then glue the curves in the pencil B with C at p, along one of the sections corresponding to the base points of the pencil. We have that all such B_i fill up $\Delta_i \subset \overline{\mathcal{M}}_g$ for $i \neq 10$, and the divisor $Z \subset \Delta_{10}$ when $i = 10$.

Lemma 2.2. *We have* $B_i \cdot \lambda = i + 1$, $B_i \cdot \delta_0 = 6i + 18$, $B_i \cdot \delta_i = -1$ *and* $B_i \cdot \delta_j = 0$ *for* $j \neq 0, i$.

Proof. This follows immediately from Lemma 2.1 and from general principles, as explained in [CR] pp.271. □

Proof. (*of Theorem 1.1 (a), (c)*) (a) Let us fix $2 \leq i \leq 11, i \neq 10$. Since D is the closure of a divisor coming from $\mathcal{M}_g$, it cannot contain the whole boundary Δ_i. Thus we must have a pencil B_i as above such that $B_i \cdot D \geq 0$. The same thing holds true for $i = 10$ if we know that Z is not contained in D. But by Lemma 2.2 this is precisely the statement of this part.

(c) We follow the same procedure, but this time we produce a pencil B_1 in $\Delta_1 \subset \overline{\mathcal{M}}_g$ by gluing a fixed pointed curve (C, p) of genus $g - 1$ to a generic pencil of plane cubics along one of its 9 sections. We have the well-known relations:

$$B_1 \cdot \lambda = 1, B_1 \cdot \delta_0 = 12, B_1 \cdot \delta_1 = -1 \text{ and } B_1 \cdot \delta_j = 0 \text{ for } j \neq 0, 1.$$

The conclusion follows similarly, since we can find a B_1 such that $B_1 \cdot D \geq 0$. $\square$

The study of the coefficient b_{10} is more involved, since in $\mathcal{M}_{10}$ the Lefschetz pencils of curves on $K3$ surfaces only fill up a divisor. We need some preliminaries on divisors on the universal curve $\overline{\mathcal{M}}_{g,1}$. Let $\pi : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ be the forgetful morphism. The generators of $\text{Pic}(\overline{\mathcal{M}}_{g,1}) \otimes \mathbb{Q}$ are the tautological class $\psi = c_1(\omega_\pi)$, the boundary δ_0, the Hodge class λ, and for $1 \leq i \leq g-1$ the class δ_i corresponding to the locus of pointed curves consisting of two components of genus i and $g-i$ respectively with the marked point being on the genus i component.

Lemma 2.3. *(cf.* [AC] *§1) One has the following relations:*

$$\pi_*(\lambda^2) = \pi_*(\lambda \cdot \delta_i) = \pi_*(\delta_0 \cdot \delta_i) = 0 \text{ for all } i = 0, \ldots, g-1,\ \pi_*(\psi^2) = 12\lambda - \delta,$$

$$\pi_*(\lambda \cdot \psi) = (2g-2)\lambda,\ \pi_*(\psi \cdot \delta_0) = (2g-2)\delta_0,\ \pi_*(\psi \cdot \delta_i) = (2i-1)\delta_i \text{ for } i \geq 1,$$

$$\pi_*(\delta_i^2) = -\delta_i \text{ for } 1 \leq i \leq g-1,\ \pi_*(\delta_i \cdot \delta_{g-i}) = \delta_i, \text{ for } 1 \leq i < g/2, \text{ and}$$

$$\pi_*(\delta_i \cdot \delta_j) = 0 \text{ for all } i, j \geq 0 \text{ with } i \neq j, g-j.$$

We consider the *Weierstrass divisor* in $\mathcal{M}_{g,1}$

$$\mathcal{W} := \{[C,p] \in \mathcal{M}_{g,1} : p \in C \text{ is a Weierstrass point}\},$$

and denote by $\overline{\mathcal{W}}$ its closure in $\overline{\mathcal{M}}_{g,1}$. Its class has been computed by Cukierman [Ck]:

$$\overline{\mathcal{W}} \equiv -\lambda + \frac{g(g+1)}{2}\psi - \sum_{i=1}^{g-1} \binom{g-i+1}{2} \delta_i.$$

Proposition 2.4. *If $\pi : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ is the forgetful morphism, then $\pi_*(\overline{\mathcal{W}}^2)$ is an effective divisor class on $\overline{\mathcal{M}}_g$.*

Proof. From the previous Lemma we have that

$$\pi_*(\overline{\mathcal{W}}^2) \equiv a\lambda - \sum_{i=0}^{[g/2]} b_i \delta_i,$$

where $a = g(g+1)(3g^2+g+2), b_0 = g^2(g+1)^2/4$ while for $1 \leq i < g/2$ we have $b_i = i(g-i)(g^3+3g^2+g-1)$. When g is even $b_{g/2} = (8g^5 + 28g^3 + 33g^4 + 4g^2)/64$. On the other hand we have expressions for the classes of distinguished geometric divisors on $\overline{\mathcal{M}}_g$: when $g+1$ is composite, by looking at Brill-Noether divisors one sees that the class

$$(g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{[g/2]} i(g-i)\delta_i$$

is effective (cf. [EH] Theorem 1). When $g+1$ is prime one has to use the class of the Petri divisor, which gives a slightly worse estimate (cf. [EH] Theorem 2). In either case, by comparing the coefficients a, b_i above with those of these explicit effective classes, one obtains an effective representative for $\pi_*(\overline{\mathcal{W}}^2)$. For instance when $g+1$ is composite it is enough to check that $b_0/a \leq (g+1)/(6g+18)$ and that $b_i/a \leq i(g-i)/(g+3)$ for $i = 1, \dots, [g/2]$, which is immediate. □

Corollary 2.5. *Let D be any effective divisor class on $\overline{\mathcal{M}}_{g,1}$. Then $\pi_*(\overline{\mathcal{W}} \cdot D)$ is an effective class on $\overline{\mathcal{M}}_g$.*

Proof. Since $\overline{\mathcal{W}}$ is irreducible we can write $D = m\overline{\mathcal{W}} + E$, where E is an effective divisor not containing $\overline{\mathcal{W}}$ and $m \in \mathbb{Z}_{\geq 0}$. Then we use the previous Proposition. □

Proof. (*of Theorem 1.1 (b)*) Assume that $b_{10} < 78b_0 - 11a$. We consider the map

$$j : \overline{\mathcal{M}}_{10,1} \longrightarrow \overline{\mathcal{M}}_g$$

obtained by attaching a fixed general pointed curve of genus $g-10$ to any curve of genus 10 with a marked point. Our assumption says that $R \cdot j^*(D) < 0$, where $R \subset \overline{\mathcal{M}}_{10,1}$ denotes the curve in the moduli space coming from a Lefschetz pencil of pointed curves of genus 10 on a general K3 surface. We can write $j^*(D) = m\pi^*(\overline{\mathcal{K}}) + E$, where E is an effective divisor not containing $\pi^*(\overline{\mathcal{K}})$ and $m \in \mathbb{Z}$ is such that

$$m \geq -R \cdot j^*(D) = -11a + 78b_0 - b_{10} > 0. \tag{1}$$

Note that we have the standard formulas $j^*(\lambda) = \lambda, j^*(\delta_0) = \delta_0$ and $j^*(\delta_{10}) = -\psi$ (cf. [AC]), while $\overline{\mathcal{K}} \equiv 7\lambda - \delta_0 - \cdots$, hence

$$E \equiv (a - 7m)\lambda + b_{10}\psi + (m - b_0)\delta_0 + \text{ (other boundaries)}$$

is an effective class on $\overline{\mathcal{M}}_{10,1}$. By applying Corollary 2.5 it follows that $\pi_*(\overline{\mathcal{W}} \cdot E)$ is an effective class on $\overline{\mathcal{M}}_{10}$. An easy calculation using Lemma 2.3 shows that

$$\pi_*(\overline{\mathcal{W}} \cdot E) \equiv \big(642b_{10} + 990(a - 7m)\big)\lambda - 55\big(b_{10} + 18(b_0 - m)\big)\delta_0 - \cdots .$$

We now use the fact that for every effective divisor on $\overline{\mathcal{M}}_g$ the coefficient a of λ is nonnegative [1]. From the previous formula we get an inequality which combined with (1) yields, after a simple computation

$$b_{10} \geq (71.3866...) \cdot b_0 - (10.1980...) \cdot a.$$

□

Question 2.6. For reasons of uniformity, it is natural to ask the following: does the second situation in Theorem 1.1(b) actually occur, or do we always have even for b_{10} the same inequality as in part (a)?

We conclude with some examples where these inequalities can be checked directly and are sometimes sharp.

[1]For the reader's convenience we recall that this follows immediately from the fact that $B \cdot \lambda > 0$ for any curve $B \subset \overline{\mathcal{M}}_g$ such that $B \cap \mathcal{M}_g \neq \emptyset$, while there is always a complete curve in $\mathcal{M}_g$ passing through a general point.

Example 2.7. When $g+1$ is composite, if $r, d > 0$ are such that $g+1 = (r+1)(g-d+r)$, then the locus of curves of genus g carrying a $\mathfrak{g}^r_d$ is a divisor with class (cf. [EH], Theorem 1):

$$\overline{\mathcal{M}}^d_{g,r} \equiv c\Big((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{[g/2]} i(g-i)\delta_i\Big),$$

where c is a positive constant depending on g, r and d. A simple calculation shows that the inequalities in Theorem 1.1 are satisfied. Moreover, they are sharp for $i = 1$ and $i = 2$, for any genus g.

Example 2.8. A similar behavior is exhibited by the divisor of curves on $K3$ surfaces $\overline{\mathcal{K}} \subset \overline{\mathcal{M}}_{10}$, where we have $b_1 = 5$ and $b_2 = 9$ which again gives equality in the first two inequalities in Theorem 1.1. Note that this follows by the method of [EH] §2 if we show that the pull-back of $\overline{\mathcal{K}}$ to $\overline{\mathcal{M}}_{2,1}$ is supported on the Weierstrass divisor. We will obtain this in the forthcoming paper [FP], based on results of Voisin [V], and as a special case of a more general study of degenerations of spaces of sections of rank two vector bundles on curves. The same study will show a striking difference between the geometry of $\overline{\mathcal{K}}$ and that of the Brill-Noether divisors, namely that the image of the natural map from $\overline{\mathcal{M}}_{0,g}$ to $\overline{\mathcal{M}}_g$ is contained in the $K3$-locus for any g. Thus one cannot use the method of [EH] §3 in order to determine more coefficents of $\overline{\mathcal{K}}$.

3. **Slopes of divisors and further remarks**

The inequalities established in Theorem 1.1 allow us to show that, at least up to genus 23, if the slope of an effective divisor is sufficiently small, then it is computed by the ratio a/b_0.

Proof. (*of Theorem 1.4*) When g is such that $g+1$ is composite, we have that $s_g \leq 6 + 12/(g+1)$ (this being the slope of any Brill-Noether divisor). When g is even, one has the estimate $s_g \leq \frac{2(3g^2+13g+2)}{g(g+2)}$ (this being the slope of the Petri divisor, cf. [EH] Theorem 2). It follows that for any $g \leq 23$ there exists a positive number ϵ_g such that

$$s_g + \epsilon_g \leq 6 + \frac{11}{i+1} \text{ for all } i \leq [g/2] (\leq 11).$$

Assume first that $2 \leq i \leq 9$ or $i = 11$. Then by Theorem 1.1(a) we know that $b_i \geq (6i+18)b_0 - (i+1)a$, and so certainly $b_i \geq b_0$ if $s(D) \leq 6 + \frac{11}{i+1}$. For $i = 10$ we apply 1.1(b): if the inequality $b_{10} \geq 78b_0 - 11a$ holds, then the argument is identical. If not, we have the inequality $b_{10} \geq (71.3866...) \cdot b_0 - (10.1980...) \cdot a$. Thus $b_{10} \geq b_0$ as soon as the inequality $a/b_0 \leq 6.9$ is satisfied. But for $g \geq 20$ the inequality $s_g < 6.9$ holds, based on the same estimates as above.

For $i = 1$, the condition is even weaker because of the formula $b_1 \geq 12b_0 - a$ in 1.1(c). Thus the slope of D is computed by a/b_0. □

Remark 3.1. Let us consider again the curve $B \subset \overline{\mathcal{M}}_g$ corresponding to a Lefschetz pencil of curves of genus g on a general $K3$ surface (cf. Lemma 2.1). Since $B\cdot\delta_0/B\cdot\lambda = 6+12/(g+1)$ which is the conjectured value of s_g, it follows that the nefness [2] of B would be a sufficient condition for the Slope Conjecture to hold in genus g. Moreover, Theorem 1.4 and Corollary 1.2 imply that for $g \leq 23$ the Slope Conjecture in genus g is equivalent to B being a nef curve. The conjecture fails for $g = 10$ because $B \cdot \overline{\mathcal{K}} = -1$.

Remark 3.2. An amusing consequence of Proposition 2.5 is that the Kodaira dimension of the universal curve $\mathcal{M}_{g,1}$ is $-\infty$ for all $g \leq 15$, with $g \neq 13, 14$ (of course this can be proved directly when $g \leq 11$). Indeed, if we assume that the canonical class $K_{\overline{\mathcal{M}}_{g,1}} \equiv 13\lambda + \psi - 3(\delta_1+\delta_{g-1}) - 2\sum_{i=2}^{g-2}\delta_i$ is effective on $\overline{\mathcal{M}}_{g,1}$, then by Proposition 2.5 the class $D := \pi_*(K_{\overline{\mathcal{M}}_{g,1}} \cdot \overline{\mathcal{W}})$ is effective on $\overline{\mathcal{M}}_g$. It turns out that $s(D) = \frac{2(13g^3+6g^2-9g+2)}{g(g+1)(4g+3)}$, and from the definition of the slope of $\mathcal{M}_g$ we have that $s(D) \geq s_g$. But this contradicts the estimates on s_g from [Ta] and [CR].

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[2] Recall that, slightly abusively, a curve B on a projective variety X is called *nef* if $B \cdot D \geq 0$ for every efective divisor D on X.