THE BIRATIONAL TYPE OF THE MODULI SPACE OF EVEN SPIN CURVES

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The moduli space $\mathcal{S}_g$ of smooth spin curves parameterizes pairs $[C, \eta]$, where $[C] \in \mathcal{M}_g$ is a curve of genus g and $\eta \in \text{Pic}^{g-1}(C)$ is a theta-characteristic. The finite forgetful map $\pi : \mathcal{S}_g \to \mathcal{M}_g$ has degree 2^{2g} and $\mathcal{S}_g$ is a disjoint union of two connected components $\mathcal{S}_g^+$ and $\mathcal{S}_g^-$ of relative degrees $2^{g-1}(2^g+1)$ and $2^{g-1}(2^g-1)$ corresponding to even and odd theta-characteristics respectively. A compactification $\overline{\mathcal{S}}_g$ of $\mathcal{S}_g$ over $\overline{\mathcal{M}}_g$ is obtained by considering the coarse moduli space of the stack of stable spin curves of genus g (cf. [C], [CCC] and [AJ]). The projection $\mathcal{S}_g \to \mathcal{M}_g$ extends to a finite branched covering $\pi : \overline{\mathcal{S}}_g \to \overline{\mathcal{M}}_g$. In this paper we determine the Kodaira dimension of $\overline{\mathcal{S}}_g^+$:

Theorem 0.1. *The moduli space $\overline{\mathcal{S}}_g^+$ of even spin curves is a variety of general type for $g > 8$ and it is uniruled for $g < 8$. The Kodaira dimension of $\overline{\mathcal{S}}_8^+$ is non-negative* [1].

It was classically known that $\overline{\mathcal{S}}_2^+$ is rational. The Scorza map establishes a birational isomorphism between $\overline{\mathcal{S}}_3^+$ and $\overline{\mathcal{M}}_3$, cf. [DK], hence $\overline{\mathcal{S}}_3^+$ is rational. Very recently, Takagi and Zucconi [TZ] showed that $\overline{\mathcal{S}}_4^+$ is rational as well. Theorem 0.1 can be compared to [FL] Theorem 0.3: The moduli space $\overline{\mathcal{R}}_g$ of Prym varieties of dimension $g-1$ (that is, non-trivial square roots of $\mathcal{O}_C$ for each $[C] \in \mathcal{M}_g$) is of general type when $g > 13$ and $g \neq 15$. On the other hand $\overline{\mathcal{R}}_g$ is unirational for $g < 8$. Surprisingly, the problem of determining the Kodaira dimension has a much shorter solution for $\overline{\mathcal{S}}_g^+$ than for $\overline{\mathcal{R}}_g$ and our results are complete.

We describe the strategy to prove that $\overline{\mathcal{S}}_g^+$ is of general type for a given g. We denote by $\lambda = \pi^*(\lambda) \in \text{Pic}(\overline{\mathcal{S}}_g^+)$ the pull-back of the Hodge class and by $\alpha_0, \beta_0 \in \text{Pic}(\overline{\mathcal{S}}_g^+)$ and $\alpha_i, \beta_i \in \text{Pic}(\overline{\mathcal{S}}_g^+)$ for $1 \leq i \leq [g/2]$ boundary divisor classes such that

$$\pi^*(\delta_0) = \alpha_0 + 2\beta_0 \text{ and } \pi^*(\delta_i) = \alpha_i + \beta_i \text{ for } 1 \leq i \leq [g/2]$$

(see Section 2 for precise definitions). Using Riemann-Hurwitz and [HM] we find that

$$K_{\overline{\mathcal{S}}_g^+} \equiv \pi^*(K_{\overline{\mathcal{M}}_g}) + \beta_0 \equiv 13\lambda - 2\alpha_0 - 3\beta_0 - 2\sum_{i=1}^{[g/2]}(\alpha_i + \beta_i) - (\alpha_1 + \beta_1).$$

We prove that $K_{\overline{\mathcal{S}}_g^+}$ is a big $\mathbb{Q}$-divisor class by comparing it against the class of the closure in $\overline{\mathcal{S}}_g^+$ of the divisor Θ_{null} on $\mathcal{S}_g^+$ of non-vanishing even theta characteristics:

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[1] Building on the results of this paper, we have proved quite recently in joint work with A. Verra, that $\kappa(\overline{\mathcal{S}}_8^+) = 0$. Details will appear later.

Theorem 0.2. *The closure in $\overline{\mathcal{S}}_g^+$ of the divisor $\Theta_{\mathrm{null}} := \{[C,\eta] \in \mathcal{S}_g^+ : H^0(C,\eta) \neq 0\}$ of non-vanishing even theta characteristics has class equal to*

$$\overline{\Theta}_{\mathrm{null}} \equiv \frac{1}{4}\lambda - \frac{1}{16}\alpha_0 - \frac{1}{2}\sum_{i=1}^{[g/2]}\beta_i \in \ \mathrm{Pic}(\overline{\mathcal{S}}_g^+).$$

Note that the coefficients of β_0 and α_i for $1 \leq i \leq [g/2]$ in the expansion of $[\overline{\Theta}_{\mathrm{null}}]$ are equal to 0. To prove Theorem 0.2, one can use test curves on $\overline{\mathcal{S}}_g^+$ or alternatively, realize $\overline{\Theta}_{\mathrm{null}}$ as the push-forward of the degeneracy locus of a map of vector bundles of the same rank defined over a certain Hurwitz scheme covering $\overline{\mathcal{S}}_g^+$ and use [F1] and [F2] to compute the class of this locus. Then we use [FP] Theorem 1.1, to construct for each genus $3 \leq g \leq 22$ an effective divisor class $D \equiv a\lambda - \sum_{i=0}^{[g/2]} b_i\delta_i \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ with coefficients satisfying the inequalities

$$\frac{a}{b_0} \leq \begin{cases} 6 + \frac{12}{g+1}, & \text{if } g+1 \text{ is composite} \\ 7, & \text{if } g = 10 \\ \frac{6k^2+k-6}{k(k-1)}, & \text{if } g = 2k-2 \geq 4 \end{cases}$$

and $b_i/b_0 \geq 4/3$ for $1 \leq i \leq [g/2]$. When $g+1$ is composite we choose for D the closure of the Brill-Noether divisor of curves with a $\mathfrak{g}^r_d$, that is, $\mathcal{M}^r_{g,d} := \{[C] \in \mathcal{M}_g : G^r_d(C) \neq \emptyset\}$ in case when the Brill-Noether number $\rho(g,r,d) = -1$, and then cf. [EH2]

$$\overline{\mathcal{M}}^r_{g,d} \equiv c_{g,d,r}\Big((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{[g/2]} i(g-i)\delta_i\Big) \in \mathrm{Pic}(\overline{\mathcal{M}}_g).$$

For $g = 10$ we take the closure of the divisor $\mathcal{K}_{10} := \{[C] \in \mathcal{M}_{10} : C \text{ lies on a } K3 \text{ surface}\}$ (cf. [FP] Theorem 1.6). In the remaining cases, when necessarily $g = 2k-2$, we choose for D the Gieseker-Petri divisor $\overline{\mathcal{GP}}^1_{g,k}$ consisting of those curves $[C] \in \mathcal{M}_g$ such that there exists a pencil $A \in W^1_k(C)$ such that the multiplication map

$$\mu_0(A) : H^0(C,A) \otimes H^0(C, K_C \otimes A^\vee) \to H^0(C, K_C)$$

is not an isomorphism, see [EH2], [F2]. Having chosen D, we form the $\mathbb{Q}$-linear combination of divisor classes

$$8 \cdot \overline{\Theta}_{\mathrm{null}} + \frac{3}{2b_0}\cdot\pi^*(D) = \big(2 + \frac{3a}{2b_0}\big)\lambda - 2\alpha_0 - 3\beta_0 - \sum_{i=1}^{[g/2]}\frac{3b_i}{2b_0}\alpha_i - \sum_{i=1}^{[g/2]}\big(4 + \frac{3b_i}{2}\big)\beta_i \in \mathrm{Pic}(\overline{\mathcal{S}}_g^+),$$

from which we can write

$$K_{\overline{\mathcal{S}}_g^+} = \nu_g\cdot\lambda + 8\overline{\Theta}_{\mathrm{null}} + \frac{3}{2b_0}\pi^*(D) + \sum_{i=1}^{[g/2]}\big(c_i\cdot\alpha_i + c_i'\cdot\beta_i\big),$$

where $c_i, c_i' \geq 0$. Moreover $\nu_g > 0$ precisely when $g \geq 9$, while $\nu_8 = 0$. Since the class $\lambda \in \mathrm{Pic}(\overline{\mathcal{S}}_g^+)$ is big and nef, we obtain that $K_{\overline{\mathcal{S}}_g^+}$ is a big $\mathbb{Q}$-divisor class on the normal variety $\overline{\mathcal{S}}_g^+$ as soon as $g > 8$. It is proved in [Lud] that for $g \geq 4$ pluricanonical forms defined on $\overline{\mathcal{S}}_{g,\mathrm{reg}}^+$ extend to any resolution of singularities $\widehat{\mathcal{S}_g^+} \to \overline{\mathcal{S}}_g^+$, which shows that $\overline{\mathcal{S}}_g^+$ is of general type whenever $\nu_g > 0$ and completes the proof of Theorem 0.1 for $g \geq 8$.

When $g \leq 7$ we show that $K_{\overline{\mathcal{S}}_g^+} \notin \overline{\mathrm{Eff}}(\overline{\mathcal{S}}_g^+)$ by constructing a covering curve $R \subset \overline{\mathcal{S}}_g^+$ such that $R \cdot K_{\overline{\mathcal{S}}_g^+} < 0$, cf. Theorem 1.2. We then use [BDPP] to conclude that $\overline{\mathcal{S}}_g^+$ is uniruled.

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1. THE STACK OF SPIN CURVES

We review a few facts about Cornalba's compactification $\pi : \overline{\mathcal{S}}_g \to \overline{\mathcal{M}}_g$, see [C]. If X is a nodal curve, a smooth rational component $E \subset X$ is said to be *exceptional* if $\#(E \cap \overline{X - E}) = 2$. The curve X is said to be *quasi-stable* if $\#(E \cap \overline{X - E}) \geq 2$ for any smooth rational component $E \subset X$, and moreover any two exceptional components of X are disjoint. A quasi-stable curve is obtained from a stable curve by blowing-up each node at most once. We denote by $[st(X)] \in \overline{\mathcal{M}}_g$ the stable model of X.

Definition 1.1. A *spin curve* of genus g consists of a triple (X, η, β), where X is a genus g quasi-stable curve, $\eta \in \mathrm{Pic}^{g-1}(X)$ is a line bundle of degree $g-1$ such that $\eta_E = \mathcal{O}_E(1)$ for every exceptional component $E \subset X$, and $\beta : \eta^{\otimes 2} \to \omega_X$ is a sheaf homomorphism which is generically non-zero along each non-exceptional component of X.

A *family of spin curves* over a base scheme S consists of a triple $(\mathcal{X} \xrightarrow{f} S, \eta, \beta)$, where $f : \mathcal{X} \to S$ is a flat family of quasi-stable curves, $\eta \in \mathrm{Pic}(\mathcal{X})$ is a line bundle and $\beta : \eta^{\otimes 2} \to \omega_{\mathcal{X}}$ is a sheaf homomorphism, such that for every point $s \in S$ the restriction $(X_s, \eta_{X_s}, \beta_{X_s} : \eta_{X_s}^{\otimes 2} \to \omega_{X_s})$ is a spin curve.

To describe locally the map $\pi : \overline{\mathcal{S}}_g \to \overline{\mathcal{M}}_g$ we follow [C] Section 5. We fix $[X, \eta, \beta] \in \overline{\mathcal{S}}_g$ and set $C := st(X)$. We denote by $E_1, \ldots, E_r$ the exceptional components of X and by $p_1, \ldots, p_r \in C_{\mathrm{sing}}$ the nodes which are images of exceptional components. The automorphism group of (X, η, β) fits in the exact sequence of groups

$$1 \longrightarrow \mathrm{Aut}_0(X, \eta, \beta) \longrightarrow \mathrm{Aut}(X, \eta, \beta) \xrightarrow{\mathrm{res}_C} \mathrm{Aut}(C).$$

We denote by $\mathbb{C}_\tau^{3g-3}$ the versal deformation space of (X, η, β) where for $1 \leq i \leq r$ the locus $(\tau_i = 0) \subset \mathbb{C}_\tau^{3g-3}$ corresponds to spin curves in which the component $E_i \subset X$ persists. Similarly, we denote by $\mathbb{C}_t^{3g-3} = \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$ the versal deformation space of C and denote by $(t_i = 0) \subset \mathbb{C}_t^{3g-3}$ the locus where the node $p_i \in C$ is not smoothed. Then around the point $[X, \eta, \beta]$, the morphism $\pi : \overline{\mathcal{S}}_g \to \overline{\mathcal{M}}_g$ is locally given by the map

$$(1) \quad \frac{\mathbb{C}_\tau^{3g-3}}{\mathrm{Aut}(X, \eta, \beta)} \to \frac{\mathbb{C}_t^{3g-3}}{\mathrm{Aut}(C)}, \quad t_i = \tau_i^2 \ \ (1 \leq i \leq r) \ \text{ and } \ t_i = \tau_i \ \ (r+1 \leq i \leq 3g-3).$$

From now on we specialize to the case of even spin curves and describe the boundary of $\overline{\mathcal{S}}_g^+$. In the process we determine the ramification of the finite covering $\pi : \overline{\mathcal{S}}_g^+ \to \overline{\mathcal{M}}_g$.

1.1. The boundary divisors of $\overline{\mathcal{S}}_g^+$.

If $[X, \eta, \beta] \in \pi^{-1}([C \cup_y D])$ where $[C, y] \in \mathcal{M}_{i,1}$ and $[D, y] \in \mathcal{M}_{g-i,1}$, then necessarily $X := C \cup_{y_1} E \cup_{y_2} D$, where E is an exceptional component such that $C \cap E = \{y_1\}$ and $D \cap E = \{y_2\}$. Moreover

$$\eta = \big(\eta_C, \eta_D, \eta_E = \mathcal{O}_E(1)\big) \in \mathrm{Pic}^{g-1}(X),$$

where $\eta_C^{\otimes 2} = K_C, \eta_D^{\otimes 2} = K_D$. The condition $h^0(X, \eta) \equiv 0 \bmod 2$, implies that the theta-characteristics η_C and η_D have the same parity. We denote by $A_i \subset \overline{\mathcal{S}}_g^+$ the closure of the locus corresponding to pairs $([C, y, \eta_C], [D, y, \eta_D]) \in \mathcal{S}_{i,1}^+ \times \mathcal{S}_{g-i,1}^+$ and by $B_i \subset \overline{\mathcal{S}}_g^+$ the closure of the locus corresponding to pairs $([C, y, \eta_C], [D, y, \eta_D]) \in \mathcal{S}_{i,1}^- \times \mathcal{S}_{g-i,1}^-$.

For a general point $[X, \eta, \beta] \in A_i \cup B_i$ we have that $\mathrm{Aut}_0(X, \eta, \beta) = \mathrm{Aut}(X, \eta, \beta) = \mathbb{Z}_2$. Using (1), the map $\mathbb{C}_\tau^{3g-3} \to \mathbb{C}_t^{3g-3}$ is given by $t_1 = \tau_1^2$ and $t_i = \tau_i$ for $i \geq 2$. Furthermore, $\mathrm{Aut}_0(X, \eta, \beta)$ acts on $\mathbb{C}_\tau^{3g-3}$ via $(\tau_1, \tau_2, \ldots, \tau_{3g-3}) \mapsto (-\tau_1, \tau_2, \ldots, \tau_{3g-3})$. It follows that $\Delta_i \subset \overline{\mathcal{M}}_g$ is not a branch divisor for $\pi : \overline{\mathcal{S}}_g^+ \to \overline{\mathcal{M}}_g$ and if $\alpha_i = [A_i] \in \mathrm{Pic}(\overline{\mathcal{S}}_g^+)$ and $\beta_i = [B_i] \in \mathrm{Pic}(\overline{\mathcal{S}}_g^+)$, then for $1 \leq i \leq [g/2]$ we have the relation

$$\pi^*(\delta_i) = \alpha_i + \beta_i. \tag{2}$$

Moreover, $\pi_*(\alpha_i) = 2^{g-2}(2^i + 1)(2^{g-i} + 1)\delta_i$ and $\pi_*(\beta_i) = 2^{g-2}(2^i - 1)(2^{g-i} - 1)\delta_i$.

For a point $[X, \eta, \beta]$ such that $st(X) = C_{yq} := C/y \sim q$, with $[C, y, q] \in \mathcal{M}_{g-1,2}$, there are two possibilities depending on whether X possesses an exceptional component or not. If $X = C_{yq}$ and $\eta_C := \nu^*(\eta)$ where $\nu : C \to X$ denotes the normalization map, then $\eta_C^{\otimes 2} = K_C(y + q)$. For each choice of $\eta_C \in \mathrm{Pic}^{g-1}(C)$ as above, there is precisely one choice of gluing the fibres $\eta_C(y)$ and $\eta_C(q)$ such that $h^0(X, \eta) \equiv 0 \bmod 2$. We denote by A_0 the closure in $\overline{\mathcal{S}}_g^+$ of the locus of points $[C_{yq}, \eta_C \in \sqrt{K_C(y + q)}]$ as above and clearly $\deg(A_0/\Delta_0) = 2^{2g-2}$.

If $X = C \cup_{\{y,q\}} E$ where E is an exceptional component, then $\eta_C := \eta \otimes \mathcal{O}_C$ is a theta-characteristic on C. Since $H^0(X, \omega) \cong H^0(C, \omega_C)$, it follows that $[C, \eta_C] \in \mathcal{S}_{g-1}^+$. For $[C, y, q] \in \mathcal{M}_{g-1,2}$ sufficiently generic we have that $\mathrm{Aut}(X, \eta, \beta) = \mathrm{Aut}(C) = \{\mathrm{Id}_C\}$, and then from (1) it follows that π is simply branched over such points. We denote by $B_0 \subset \overline{\mathcal{S}}_g^+$ the closure of the locus of points $[C \cup_{\{y,q\}} E, \eta_C \in \sqrt{K_C}, \eta_E = \mathcal{O}_E(1)]$. If $\alpha_0 = [A_0] \in \mathrm{Pic}(\overline{\mathcal{S}}_g^+)$ and $\beta_0 = [B_0] \in \mathrm{Pic}(\overline{\mathcal{S}}_g^+)$, we then have the relation

$$\pi^*(\delta_0) = \alpha_0 + 2\beta_0. \tag{3}$$

Note that $\pi_*(\alpha_0) = 2^{2g-2}\delta_0$ and $\pi_*(\beta_0) = 2^{g-2}(2^{g-1} + 1)\delta_0$.

1.2. The uniruledness of $\overline{\mathcal{S}}_g^+$ for small g.

We employ a simple negativity argument to determine $\kappa(\overline{\mathcal{S}}_g^+)$ for small genus. Using an analogous idea we showed that similarly, for the moduli space of Prym curves, one has that $\kappa(\overline{\mathcal{R}}_g) = -\infty$ for $g < 8$, cf. [FL] Theorem 0.7.

Theorem 1.2. *For $g < 8$, the space $\overline{\mathcal{S}}_g^+$ is uniruled.*

Proof. We start with a fixed $K3$ surface S carrying a Lefschetz pencil of curves of genus g. This induces a fibration $f : \mathrm{Bl}_{g^2}(S) \to \mathbf{P}^1$ and then we set $B := (m_f)_*(\mathbf{P}^1) \subset \overline{\mathcal{M}}_g$, where $m_f : \mathbf{P}^1 \to \overline{\mathcal{M}}_g$ is the moduli map $m_f(t) := [f^{-1}(t)]$. We have the following well-known formulas on $\overline{\mathcal{M}}_g$ (cf. [FP] Lemma 2.4):

$$B \cdot \lambda = g + 1,\ B \cdot \delta_0 = 6g + 18, \quad \text{and } B \cdot \delta_i = 0 \ \text{ for } i \geq 1.$$

We lift B to a pencil $R \subset \overline{\mathcal{S}}_g^+$ of spin curves by taking

$$R := B \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{S}}_g^+ = \{[C_t,\ \eta_{C_t}] \in \overline{\mathcal{S}}_g^+ : [C_t] \in B, \eta_{C_t} \in \overline{\mathrm{Pic}}^{g-1}(C_t), t \in \mathbf{P}^1\} \subset \overline{\mathcal{S}}_g^+.$$

Using (3) one computes the intersection numbers with the generators of $\mathrm{Pic}(\overline{\mathcal{S}}_g^+)$:

$R\cdot\lambda=(g+1)2^{g-1}(2^g+1),\ R\cdot\alpha_0=(6g+18)2^{2g-2}$ and $R\cdot\beta_0=(6g+18)2^{g-2}(2^{g-1}+1)$.

Furthermore, R is disjoint from all the remaining boundary classes of $\overline{\mathcal{S}}_g^+$, that is, $R\cdot\alpha_i=R\cdot\beta_i=0$ for $1\leq i\leq [g/2]$. One verifies that $R\cdot K_{\overline{\mathcal{S}}_g^+}<0$ precisely when $g\leq 7$. Since R is a covering curve for $\overline{\mathcal{S}}_g^+$ in the range $g\leq 7$, we find that $K_{\overline{\mathcal{S}}_g^+}$ is not pseudo-effective, that is, $K_{\overline{\mathcal{S}}_g^+}\in\overline{\mathrm{Eff}}(\overline{\mathcal{S}}_g^+)^c$. Pseudo-effectiveness of the canonical bundle is a birational property for normal varieties, therefore the canonical bundle of any smooth model of $\overline{\mathcal{S}}_g^+$ lies outside the pseudo-effective cone as well. One can apply [BDPP] Corollary 0.3, to conclude that $\overline{\mathcal{S}}_g^+$ is uniruled for $g\leq 7$. □

2. The geometry of the divisor $\overline{\Theta}_{\mathrm{null}}$

We compute the class of the divisor $\overline{\Theta}_{\mathrm{null}}$ using test curves. The same calculation can be carried out using techniques developed in [F1], [F2] to calculate push-forwards of tautological classes from stacks of limit linear series $\mathfrak{g}^r_d$ (see also Remark 2.1).

For $g\geq 9$, Harer [H] has showed that $H^2(\mathcal{S}_g^+,\mathbb{Q})\cong\mathbb{Q}$. The range for which this result holds has been recently improved to $g\geq 5$ in [P]. In particular, it follows that $\mathrm{Pic}(\overline{\mathcal{S}}_g^+)_{\mathbb{Q}}$ is generated by the classes λ, α_i,β_i for $i=0,\ldots,[g/2]$. Thus we can expand the divisor class $\overline{\Theta}_{\mathrm{null}}$ in terms of the generators of the Picard group

$$\overline{\Theta}_{\mathrm{null}}\equiv\bar{\lambda}\cdot\lambda-\bar{\alpha}_0\cdot\alpha_0-\bar{\beta}_0\cdot\beta_0-\sum_{i=1}^{[g/2]}\bigl(\bar{\alpha}_i\cdot\alpha_i+\bar{\beta}_i\cdot\beta_i\bigr)\in\mathrm{Pic}(\overline{\mathcal{S}}_g^+)_{\mathbb{Q}},\tag{4}$$

and determine the coefficients $\bar{\lambda},\bar{\alpha}_0,\bar{\beta}_0,\bar{\alpha}_i$ and $\bar{\beta}_i\in\mathbb{Q}$ for $1\leq i\leq [g/2]$.

Remark 2.1. To show that the class $[\Theta_{\mathrm{null}}]\in\mathrm{Pic}(\mathcal{S}_g^+)_{\mathbb{Q}}$ is a multiple of λ and thus, the expansion (4) makes sense for all $g\geq 3$, one does not need to know that $\mathrm{Pic}(\mathcal{S}_g^+)_{\mathbb{Q}}$ is infinite cyclic. For instance, for even $g=2k-2\geq 4$, we note that, via the base point free pencil trick, $[C,\eta]\in\Theta_{\mathrm{null}}$ if and only if the multiplication map

$$\mu_C(A,\eta):H^0(C,A)\otimes H^0(C,A\otimes\eta)\to H^0(C,A^{\otimes 2}\otimes\eta)$$

is not an isomorphism for a base point free pencil $A\in W^1_k(C)$. We set $\widetilde{\mathcal{M}}_g$ to be the open subvariety consisting of curves $[C]\in\mathcal{M}_g$ such that $W^1_{k-1}(C)=\emptyset$ and denote by $\sigma:\mathfrak{G}^1_k\to\widetilde{\mathcal{M}}_g$ the Hurwitz scheme of pencils $\mathfrak{g}^1_k$ and by

$$\tau:\mathfrak{G}^1_k\times_{\widetilde{\mathcal{M}}_g}\mathcal{S}_g^+\to\mathcal{S}_g^+,\ \ u:\mathfrak{G}^1_k\times_{\widetilde{\mathcal{M}}_g}\mathcal{S}_g^+\to\mathfrak{G}^1_k$$

the (generically finite) projections. Then $\Theta_{\mathrm{null}}=\tau_*(\mathcal{Z})$, where

$$\mathcal{Z}=\{[A,C,\eta]\in\mathfrak{G}^1_k\times_{\widetilde{\mathcal{M}}_g}\mathcal{S}_g^+:\mu_C(A,\eta)\text{ is not injective}\}.$$

Via this determinantal presentation, the class of the divisor $\mathcal{Z}$ is expressible as a combination of $\tau^*(\lambda),u^*(\mathfrak{a}),u^*(\mathfrak{b})$, where $\mathfrak{a},\mathfrak{b}\in\mathrm{Pic}(\mathfrak{G}^1_k)_{\mathbb{Q}}$ are the tautological classes defined in e.g. [FL] p.15. Since $\tau_*(u^*(\mathfrak{a}))=\pi^*(\sigma_*(\mathfrak{a}))$ (and similarly for the class $\mathfrak{b}$), the conclusion follows. For odd genus $g=2k-1$, one uses a similar argument replacing $\mathfrak{G}^1_k$ with any generically finite covering of $\mathcal{M}_g$ given by a Hurwitz scheme (for instance, we take the space of pencils $\mathfrak{g}^1_{k+1}$ with a triple ramification point).

We start the proof of Theorem 0.2 by determining the coefficients of α_i and β_i ($i \geq 1$) in the expansion of $[\overline{\Theta}_{\mathrm{null}}]$.

Theorem 2.2. *We fix integers $g \geq 3$ and $1 \leq i \leq [g/2]$. The coefficient of α_i in the expansion of $[\overline{\Theta}_{\mathrm{null}}]$ equals 0, while the coefficient of β_i equals $-1/2$. That is, $\bar{\alpha}_i = 0$ and $\bar{\beta}_i = 1/2$.*

Proof. For each integer $2 \leq i \leq g-1$, we fix general curves $[C] \in \mathcal{M}_i$ and $[D, q] \in \mathcal{M}_{g-i,1}$ and consider the test curve $C^i := \{C \cup_{y \sim q} D\}_{y \in C} \subset \Delta_i \subset \overline{\mathcal{M}}_g$. We lift C^i to test curves $F_i \subset A_i$ and $G_i \subset B_i$ inside $\overline{\mathcal{S}}_g^+$ constructed as follows. We fix even (resp. odd) theta-characteristics $\eta_C^+ \in \mathrm{Pic}^{i-1}(C)$ and $\eta_D^+ \in \mathrm{Pic}^{g-i-1}(D)$ (resp. $\eta_C^- \in \mathrm{Pic}^{i-1}(C)$ and $\eta_D^- \in \mathrm{Pic}^{g-i-1}(D)$).

If $E \cong \mathbf{P}^1$ is an exceptional component, we define the family F_i (resp. G_i) as consisting of spin curves

$$F_i := \big\{t := [C \cup_y E \cup_q D,\ \eta_C = \eta_C^+, \eta_E = \mathcal{O}_E(1), \eta_D = \eta_D^+] \in \overline{\mathcal{S}}_g^+ : y \in C\big\}$$

and

$$G_i := \big\{t := [C \cup_y E \cup_q D,\ \eta_C = \eta_C^-, \eta_E = \mathcal{O}_E(1), \eta_D = \eta_D^-] \in \overline{\mathcal{S}}_g^+ : y \in C\big\}.$$

Since $\pi_*(F_i) = \pi_*(G_i) = C^i$, clearly $F_i \cdot \alpha_i = C^i \cdot \delta_i = 2 - 2i, F_i \cdot \beta_i = 0$ and F_i has intersection number 0 with all other generators of $\mathrm{Pic}(\overline{\mathcal{S}}_g^+)$. Similarly

$$G_i \cdot \beta_i = 2 - 2i,\ G_i \cdot \alpha_i = 0,\ G_i \cdot \lambda = 0,$$

and G_i does not intersect the remaining boundary classes in $\overline{\mathcal{S}}_g^+$.

Next we determine $F_i \cap \overline{\Theta}_{\mathrm{null}}$. Assume that a point $t \in F_i$ lies in $\overline{\Theta}_{\mathrm{null}}$. Then there exists a family of even spin curves $(f : \mathcal{X} \to S, \eta, \beta)$, where $S = \mathrm{Spec}(R)$, with R being a discrete valuation ring and $\mathcal{X}$ is a smooth surface, such that, if $0, \xi \in S$ denote the special and the generic point of S respectively and X_ξ is the generic fibre of f, then

$$h^0(X_\xi, \eta_\xi) \geq 2,\ h^0(X_\xi, \eta_\xi) \equiv 0 \bmod 2,\ \ \eta_\xi^{\otimes 2} \cong \omega_{X_\xi} \text{ and } \big(f^{-1}(0), \eta_{f^{-1}(0)}\big) = t \in \overline{\mathcal{S}}_g^+.$$

Following the procedure described in [EH1] p. 347-351, this data produces a limit linear series $\mathfrak{g}^1_{g-1}$ on $C \cup D$, say

$$l := \Big(l_C = (L_C, V_C), l_D = (L_D, V_D)\Big) \in G^1_{g-1}(C) \times G^1_{g-1}(D),$$

such that the underlying line bundles L_C and L_D respectively, are obtained from the line bundle $(\eta_C^+, \eta_E, \eta_D^+)$ by dropping the E-aspect and then tensoring the line bundles η_C^+ and η_D^+ by line bundles supported at the points $y \in C$ and $q \in D$ respectively. For degree reasons, it follows that $L_C = \eta_C^+ \otimes \mathcal{O}_C((g-i)y)$ and $L_D = \eta_D^+ \otimes \mathcal{O}_D(iq)$. Since both C and D are general in their respective moduli spaces, we have that $H^0(C, \eta_C^+) = 0$ and $H^0(D, \eta_D^+) = 0$. In particular $a_1^{l_C}(y) \leq g - i - 1$ and $a_0^{l_D}(q) < a_1^{l_D}(q) \leq i - 1$, hence $a_1^{l_C}(y) + a_0^{l_D}(q) \leq g - 2$, which contradicts the definition of a limit $\mathfrak{g}^1_{g-1}$. Thus $F_i \cap \overline{\Theta}_{\mathrm{null}} = \emptyset$. This implies that $\bar{\alpha}_i = 0$, for all $1 \leq i \leq [g/2]$ (for $i = 1$, one uses instead the curve $F_{g-1} \subset A_1$ to reach the same conclusion).

Assume that $t \in G_i \cap \overline{\Theta}_{\mathrm{null}}$. By the same argument as above, retaining also the notation, there is an induced limit linear series on $C \cup D$,

$$(l_C, l_D) \in G^1_{g-1}(C) \times G^1_{g-1}(D),$$

where $L_C = \eta_C^- \otimes \mathcal{O}_C((g-i)y)$ and $L_D = \eta_D^- \otimes \mathcal{O}_D(iq)$. Since $[C] \in \mathcal{M}_i$ and $[D,q] \in \mathcal{M}_{g-i,1}$ are both general, we may assume that $h^0(D,\eta_D^-) = h^0(C,\eta_C^-) = 1$, $q \notin \mathrm{supp}(\eta_D^-)$ and that $\mathrm{supp}(\eta_C^-)$ consists of $i-1$ distinct points. In particular $a_1^{l_D}(q) \leq i$, hence $a_0^{l_C}(y) \geq g-1-a_1^{l_D}(q) \geq g-i-1$. Since $h^0(C,\eta_C^-) = 1$, it follows that one has in fact equality, that is, $a_0^{l_C}(y) = g-i-1$ and then necessarily $a_1^{l_D}(q) = i$.

Similarly, $a_1^{l_C}(y) \leq g-i+1$ (otherwise $\mathrm{div}(\eta_C^-) \geq 2y$, that is, $\mathrm{supp}(\eta_C^-)$ would be non-reduced, a contradiction), thus $a_0^{l_D}(q) \geq i-2$, and the last two inequalities must be equalities as well (one uses that $h^0\big(D, L_D \otimes \mathcal{O}_D(-(i-1)q)\big) = h^0(D, \eta_D^- \otimes \mathcal{O}_D(q)) = 1$, that is, $a_0^{l_D}(q) < i-1$). Since $a_1^{l_C}(y) = g-i+1$, we find that $y \in \mathrm{supp}(\eta_C^-)$.

To sum up, we have showed that (l_C, l_D) is a refined limit $\mathfrak{g}^1_{g-1}$ and in fact

$$(5)\ \ l_D = |\eta_D^- \otimes \mathcal{O}_D(2q)| + (i-2)\cdot q \in G^1_{g-1}(D),\ \ l_C = |\eta_C^- \otimes \mathcal{O}_C(y)| + (g-i-1)\cdot y \in G^1_{g-1}(C),$$

hence $a^{l_D}(q) = (i-2, i)$ and $a^{l_C}(y) = (g-i-1, g-i+1)$.

To prove that the intersection between G_i and $\overline{\Theta}_{\mathrm{null}}$ is transversal, we follow closely [EH3] Lemma 3.4 (see especially the *Remark* on p. 45): The restriction $\overline{\Theta}_{\mathrm{null}\ |G_i}$ is isomorphic, as a scheme, to the variety $\tau : \mathfrak{T}^1_{g-1}(G_i) \to G_i$ of limit linear series $\mathfrak{g}^1_{g-1}$ on the curves of compact type $\{C \cup_{y\sim q} D : y \in C\}$, whose C and D-aspects are obtained by twisting suitably at $y \in C$ and $q \in D$ the fixed theta-characteristics η_C^- and η_D^- respectively. Following the description of the scheme structure of this moduli space given in [EH1] Theorem 3.3 over an arbitrary base, we find that because G_i consists entirely of singular spin curves of compact type, the scheme $\mathfrak{T}^1_{g-1}(G_i)$ splits as a product of the corresponding moduli spaces of C and D-aspects respectively of the limits $\mathfrak{g}^1_{g-1}$. By direct calculation we have showed that $\mathfrak{T}^1_{g-1}(G_i) \cong \mathrm{supp}(\eta_C^-) \times \{l_D\}$. Since $\mathrm{supp}(\eta_C^-)$ is a reduced 0-dimensional scheme, we obtain that $\overline{\Theta}_{\mathrm{null}\ |G_i}$ is everywhere reduced. It follows that $G_i \cdot \overline{\Theta}_{\mathrm{null}} = \#\mathrm{supp}(\eta_C^-) = i-1$ and then $\bar{\beta}_i = (G_i \cdot \overline{\Theta}_{\mathrm{null}})/(2i-2)$. This argument does not work for $i=1$, when one uses instead the intersection of $\overline{\Theta}_{\mathrm{null}}$ with G_{g-1}, and this finishes the proof. □

Next we construct two pencils in $\overline{\mathcal{S}}_g^+$ which are lifts of the standard degree 12 pencil of elliptic tails in $\overline{\mathcal{M}}_g$. We fix a general pointed curve $[C,q] \in \mathcal{M}_{g-1,1}$ and a pencil $f : \mathrm{Bl}_9(\mathbf{P}^2) \to \mathbf{P}^1$ of plane cubics together with a section $\sigma : \mathbf{P}^1 \to \mathrm{Bl}_9(\mathbf{P}^2)$ induced by one of the base points. We then consider the pencil $R := \{[C \cup_{q\sim\sigma(\lambda)} f^{-1}(\lambda)]\}_{\lambda\in\mathbf{P}^1} \subset \overline{\mathcal{M}}_g$.

We fix an odd theta-characteristic $\eta_C^- \in \mathrm{Pic}^{g-2}(C)$ such that $q \notin \mathrm{supp}(\eta_C^-)$ and $E \cong \mathbf{P}^1$ will again denote an exceptional component. We define the family

$$F_0 := \{[C \cup_q E \cup_{\sigma(\lambda)} f^{-1}(\lambda),\ \ \eta_C = \eta_C^-,\ \eta_E = \mathcal{O}_E(1),\ \ \eta_{f^{-1}(\lambda)} = \mathcal{O}_{f^{-1}(\lambda)}] : \lambda \in \mathbf{P}^1\} \subset \overline{\mathcal{S}}_g^+.$$

Since $F_0 \cap A_1 = \emptyset$, we find that $F_0 \cdot \beta_1 = \pi_*(F_0)\cdot\delta_1 = -1$. Similarly, $F_0 \cdot \lambda = \pi_*(F_0)\cdot\lambda = 1$ and obviously $F_0 \cdot \alpha_i = F_0 \cdot \beta_i = 0$ for $2 \leq i \leq [g/2]$. For each of the 12 points $\lambda_\infty \in \mathbf{P}^1$ corresponding to singular fibres of R, the associated $\eta_{\lambda_\infty} \in \overline{\mathrm{Pic}}^{g-1}(C \cup E \cup f^{-1}(\lambda_\infty))$ are actual line bundles on $C \cup E \cup f^{-1}(\lambda_\infty)$ (that is, we do not have to blow-up the extra node). Thus we obtain that $F_0 \cdot \beta_0 = 0$, therefore $F_0 \cdot \alpha_0 = \pi_*(F_0)\cdot\delta_0 = 12$.

We also fix an even theta-characteristic $\eta_C^+ \in \mathrm{Pic}^{g-2}(C)$ and consider the degree 3 branched covering $\gamma : \overline{\mathcal{S}}_{1,1}^+ \to \overline{\mathcal{M}}_{1,1}$ forgetting the spin structure. We define the pencil

$$G_0 := \{ \big[C\cup_q E\cup_{\sigma(\lambda)} f^{-1}(\lambda),\ \eta_C = \eta_C^+,\ \eta_E = \mathcal{O}_E(1), \eta_{f^{-1}(\lambda)} \in \gamma^{-1}[f^{-1}(\lambda)]\big] : \lambda \in \mathbf{P}^1\} \subset \overline{\mathcal{S}}_g^+.$$

Since $\pi_*(G_0) = 3R$, we have that $G_0 \cdot \lambda = 3$. Obviously $G_0 \cdot \beta_0 = G_0 \cdot \beta_1 = 0$, hence $G_0 \cdot \alpha_1 = \pi_*(G_0) \cdot \delta_1 = -3$. The map $\gamma : \overline{\mathcal{S}}_{1,1}^+ \to \overline{\mathcal{M}}_{1,1}$ is simply ramified over the point corresponding to j-invariant ∞. Hence, $G_0 \cdot \alpha_0 = 12$ and $G_0 \cdot \beta_0 = 12$, which is consistent with formula (3).

The last pencil we construct lies in the boundary divisor $B_0 \subset \overline{\mathcal{S}}_g^+$: Setting $E \cong \mathbf{P}^1$ for an exceptional component, we define

$$H_0 := \{[C \cup_{\{y,q\}} E,\ \eta_C = \eta_C^+,\ \eta_E = \mathcal{O}_E(1)] : y \in C\} \subset \overline{\mathcal{S}}_g^+.$$

The fibre of H_0 over the point $y = q \in C$ is the even spin curve

$$\big[C \cup_q E' \cup_{q'} E'' \cup_{\{q'',y''\}} E,\ \eta_C = \eta_C^+, \eta_{E'} = \mathcal{O}_{E'}(1), \eta_E = \mathcal{O}_E(1), \eta_{E''} = \mathcal{O}_{E''}(-1)\big],$$

having as stable model $[C \cup_q E_\infty]$, where $E_\infty := E''/y'' \sim q''$ is the rational nodal curve corresponding to $j = \infty$. Here E', E'' are rational curves, $E' \cap E'' = \{q'\}$, $E \cap E'' = \{q'', y''\}$ and the stabilization map for $C \cup E \cup E' \cup E''$ contracts the components E' and E, while identifying q'' and y''.

We find that $H_0 \cdot \lambda = 0, H_0 \cdot \alpha_i = H_0 \cdot \beta_i = 0$ for $2 \leq i \leq [g/2]$. Moreover $H_0 \cdot \alpha_0 = 0$, hence $H_0 \cdot \beta_0 = \frac{1}{2}\pi_*(H_0) \cdot \delta_0 = 1 - g$. Finally, $H_0 \cdot \alpha_1 = 1$ and $H_0 \cdot \beta_1 = 0$.

Theorem 2.3. *If $F_0, G_0, H_0 \subset \overline{\mathcal{S}}_g^+$ are the families of spin curves defined above, then*

$$F_0 \cdot \overline{\Theta}_{\mathrm{null}} = G_0 \cdot \overline{\Theta}_{\mathrm{null}} = H_0 \cdot \overline{\Theta}_{\mathrm{null}} = 0.$$

Proof. From the limit linear series argument in the proof of Theorem 2.2 we get that the assumption $F_0 \cap \overline{\Theta}_{\mathrm{null}} \neq \emptyset$ implies that $q \in \mathrm{supp}(\eta_C^-)$, a contradiction. Similarly, we have that $G_0 \cap \overline{\Theta}_{\mathrm{null}} = \emptyset$ because $[C] \in \mathcal{M}_{g-1}$ can be assumed to have no even theta-characteristics $\eta_C^+ \in \mathrm{Pic}^{g-2}(C)$ with $h^0(C, \eta_C^+) \geq 2$, that is $[C, \eta_C^+] \notin \overline{\Theta}_{\mathrm{null}} \subset \overline{\mathcal{S}}_{g-1}^+$. Finally, we assume that there exists a point $[X := C \cup_{\{y,q\}} E, \eta_C = \eta_C^+, \eta_E = \mathcal{O}_E(1)] \in H_0 \cap \overline{\Theta}_{\mathrm{null}}$. Then certainly $h^0(X, \eta_X) \geq 2$ and from the Mayer-Vietoris sequence on X we find that

$$H^0(X, \eta_X) = \mathrm{Ker}\{H^0(C, \eta_C) \oplus H^0(E, \mathcal{O}_E(1)) \to \mathbb{C}^2_{y,q}\},$$

hence $h^0(C, \eta_C) = h^0(X, \eta_X) \geq 2$. This contradicts the assumption that $[C] \in \mathcal{M}_{g-1}$ is general. A similar argument works for the special point in $H_0 \cap \pi^{-1}(\Delta_1)$, hence $H_0 \cdot \overline{\Theta}_{\mathrm{null}} = 0$. □

Proof of Theorem 0.2. Looking at the expansion of $[\overline{\Theta}_{\mathrm{null}}]$, Theorem 2.3 gives the relations

$$F_0 \cdot \overline{\Theta}_{\mathrm{null}} = \bar{\lambda} - 12\bar{\alpha}_0 + \bar{\beta}_1 = 0,\ \ G_0 \cdot \overline{\Theta}_{\mathrm{null}} = 3\bar{\lambda} - 12\bar{\alpha}_0 - 12\bar{\beta}_0 + 3\bar{\alpha}_1 = 0$$

$$\text{and}\ \ H_0 \cdot \overline{\Theta}_{\mathrm{null}} = (g-1)\bar{\beta}_0 - \bar{\alpha}_1 = 0.$$

Since we have already computed $\bar{\alpha}_i = 0$ and $\bar{\beta}_i = 1/2$ for $1 \leq i \leq [g/2]$, (cf. Theorem 2.2), we obtain that $\bar{\lambda} = 1/4, \bar{\alpha}_0 = 1/16$ and $\bar{\beta}_0 = 0$. This completes the proof. □

A consequence of Theorem 0.2 is a new proof of the main result from [T]:

Theorem 2.4. *If $\mathcal{M}_g^1$ is the locus of curves $[C] \in \mathcal{M}_g$ with a vanishing theta-null then its closure has class equal to*

$$\overline{\mathcal{M}}_g^1 \equiv 2^{g-3}\Big((2^g+1)\lambda - 2^{g-3}\delta_0 - \sum_{i=1}^{[g/2]}(2^{g-i}-1)(2^i-1)\delta_i\Big) \in \mathrm{Pic}(\overline{\mathcal{M}}_g).$$

Proof. We use the scheme-theoretic equality $\pi_*(\overline{\Theta}_{\mathrm{null}}) = \overline{\mathcal{M}}_g^1$ as well as the formulas $\pi_*(\lambda) = 2^{g-1}(2^g+1)\lambda,\ \pi_*(\alpha_0) = 2^{2g-2}\delta_0,\ \pi_*(\beta_0) = 2^{g-2}(2^{g-1}+1)\delta_0,\ \pi_*(\alpha_i) = 2^{g-2}(2^i+1)(2^{g-i}+1)\delta_i$ and $\pi_*(\beta_i) = 2^{g-2}(2^i-1)(2^{g-i}-1)\delta_i$ valid for $1 \le i \le [g/2]$. □

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