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Stable CLTs and Rates for Power Variation of α-Stable Lévy Processes

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Abstract In a central limit type result it has been shown that the pth power variations of an α-stable Lévy process along sequences of equidistant partitions of a given time interval have $\frac{\alpha}{p}$-stable limits. In this paper we give precise orders of convergence for the distances of the approximate power variations computed for partitions with mesh of order $\frac{1}{n}$ and the limiting law, measured in terms of the Kolmogorov-Smirnov metric. In case $2\alpha < p$ the convergence rate is seen to be of order $\frac{1}{n}$, in case $\alpha < p < 2\alpha$ the order is $n^{1-\frac{p}{\alpha}}$.

Keywords Lévy process · Stable process · Power variation · Central limit theorem · Fourier transform · Tail probability · Rate of convergence · Empirical distribution function · Minimum distance estimator · Brownian bridge

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1 Introduction

In Ditlevsen (1999) it was suggested to model a paleo-climatic temperature time series representing measurements extending over the last glacial period, and found in ice cores obtained from a drilling in the Greenland ice shield, by *stochastic differential equations* (SDE) driven by additive Lévy processes. By an elementary correlation analysis it was argued that an α-stable Lévy noise with $\alpha = 1.75$ fits the data best. To make this claim accessible to a statistical analysis, in Hein et al.

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(2010) it was proposed to choose the power variations of the diffusion modeled by the time series as test statistics. For this purpose it was necessary to investigate the asymptotic behavior of variations of power p, taken along a partition of the time axis by equidistant intervals of length of the order $\frac{1}{n}$, of trajectories of α-stable processes, and more generally of trajectories of diffusion processes with additive α-stable noise. In a functional central limit type result, it was shown that as $n \to \infty$, the laws of the power variations taken over intervals $[0, t]$ and considered as processes in the variable t, weakly converge to the law of an $\frac{\alpha}{p}$-stable process.

In this paper, our goal is to prepare the construction of a statistical test with confidence bands for the right α in the paleo-climatic series considered by Ditlevsen (1999), by evaluating the rates of convergence in the previously sketched central limit theorem as the mesh of the partition sequence along which p-variations are calculated, tends to infinity. In fact, we compare the distance of the laws of the power variations taken on time grids of order $\frac{1}{n}$ with the laws of the limiting $\frac{\alpha}{p}$-stable process, in terms of the Kolmogorov-Smirnov metric. In our main result (Theorem 2) we prove that the convergence order is as fast as $\mathcal{O}(\frac{1}{n})$ in case $2\alpha < p$, of order $\mathcal{O}(\frac{\log n}{n})$ if $2\alpha = p$, of order $\mathcal{O}(n^{1-\frac{p}{\alpha}})$ as long as $\alpha < p < 2\alpha$, of order $\mathcal{O}(\frac{(\log n)^2}{n})$ if $p = \alpha$, and of order $\mathcal{O}(n^{1-2\frac{p}{\alpha}})$ in case $\frac{\alpha}{2} < p < \alpha$. Its proof is mainly based on a direct estimate of the Kolmogorov-Smirnov distance of two probability measures by integrals of their characteristic functions on finite intervals $[-T, T]$, up to an error term converging to 0 as $T \to \infty$. The exact convergence rates result from asymptotic expansions of the characteristic functions of appropriate functionals of stable processes, that in turn can be derived from a careful treatment of integrals of asymptotic expansions of tail probabilities for stable random variables, obtained in Uchaikin and Zolotarev (1999).

With a view towards the construction of a minimum-distance estimator we consider the empirical disitribution of an i.i.d. sample of approximate p-variations of α-stable processes. We show that their standardized Kolmogorov-Smirnov-distance from the α/p-stable limit fluctuates according to the supremum of a Brownian bridge as expected from empirical process theory.

The material of this paper is organized as follows. An introductory Section 2 explains the different types of stable distributions and processes we consider, and states the central limit theorem for p-variations of α-stable processes. In the short Section 3 we state our main result, which is carefully derived in Section 4. Its main ingredients, asymptotic expansions of tail probabilities, and the resulting asymptotic expansions of characteristic functions of stable laws, are treated in Sections 4.1 and 4.2.

2 Notation and Preliminaries

The model dynamics considered in Ditlevsen (1999) and Hein et al. (2010) are governed by an SDE of the form

$$X_t = x + \int_0^t f(s, X_s)\mathrm{d}s + L_t\,, \quad t \geq 0\,. \tag{1}$$

Here $f = -\frac{\partial}{\partial x}U$ can be thought of as the gradient of a pseudo potential describing simple features of the climate system. The noise process L is assumed to be an

α-stable Lévy process. Loosely speaking L is a time homogeneous stochastic process with independent increments and stochastically continuous trajectories. It is also standard to consider a version that has right continuous trajectories with left limits (see Sato 1999). Stable processes form the important subclass of *self similar* Lévy processes, i.e. they admit a scaling $L_{at} \overset{d}{=} a^{1/\alpha} L_t$ for any $a > 0$ with $\alpha \in (0, 2]$, where $\overset{d}{=}$ denotes equality in distribution. This, together with additivity, is the key to the validity of a central limit theorem. The one dimensional marginals of stable processes are α-stable random variables with characteristic function $\mathbb{E}e^{i\lambda L_t} = e^{t\psi(\lambda)}$ where ψ is the second characteristic of an α-stable law.

In general four parameters are required to fully describe a one dimensional stable law, a *location* parameter $\mu \in \mathbb{R}$, a *scale* parameter $c > 0$, a *skewness parameter* $\beta \in [-1, 1]$ and the *tail index* $\alpha \in (0, 2]$. Then ψ can be written as

$$\psi(\lambda) = \begin{cases} -c^\alpha |\lambda|^\alpha \left(1 - i\beta \operatorname{sign}(\lambda) \tan \dfrac{\pi\alpha}{2}\right) + i\mu\lambda\,, & \alpha \neq 1, \\ -c|\lambda| \left(1 + i\beta \dfrac{2}{\pi} \operatorname{sign}(\lambda) \log |\lambda|\right) + i\mu\lambda\,, & \alpha = 1, \end{cases} \qquad \lambda \in \mathbb{R}\,. \tag{A}$$

This is the most frequent parametrization, and we adopt the notation of Samorodnitsky and Taqqu (1994) in writing $L_1 \sim \mathcal{S}_\alpha(c, \beta, \mu)$. Throughout the work we will only consider *strictly stable* distributions which by definition requires $\mu = 0$. This is not a real restriction since we only exclude a linear drift in the noise process L.

We will also need a second parametrization for strictly stable laws to obtain a series representation of densities following Uchaikin and Zolotarev (1999). Here ψ is expressed as

$$\psi(\lambda) = -c^\alpha |\lambda|^\alpha e^{-i\frac{\pi}{2}\delta \operatorname{sign}(\lambda)}\,, \quad \lambda \in \mathbb{R}\,, \tag{B}$$

with skewness parameter

$$\delta = \begin{cases} \dfrac{2}{\pi} \arctan\left(\beta \tan\left(\dfrac{\pi}{2}\alpha\right)\right)\,, & \alpha \neq 1, \\ 0\,, & \alpha = 1, \end{cases} \qquad |\delta| \le \min\{\alpha, 2 - \alpha\} \le 1, \tag{2}$$

and scale

$$c_{(B)} = \begin{cases} c_{(A)} \cos\left(\dfrac{\pi}{2}\delta\right)^{-1/\alpha}\,, & \alpha \neq 1 \\ c_{(A)}\,, & \alpha = 1 \end{cases}.$$

To avoid confusion we sometimes distinguish the two parametrizations by capital letter indices $\mathcal{S}_\alpha^{(A)}(c, \beta, \mu)$, $\mathcal{S}_\alpha^{(B)}(1, \delta)$.

In Hein et al. (2010) it was observed that the variation of trajectories of the solution X of Eq. 1 is only determined by the α-stable noise L. More precisely for $p > 0$ the *equidistant power variation* or short *p-variation*

$$V_p^n(X)_t := \sum_{i=1}^{\lfloor nt \rfloor} |X_{i/n} - X_{(i-1)/n}|^p\,, \quad t \ge 0,$$

is considered there. They show now that in the limit the processes coincide in Skorokhod topology, i.e. $\lim_{n\to\infty} V_p^n(X) = \lim_{n\to\infty} V_p^n(L)$. The following functional limit theorem for α-stable Lévy processes from Hein et al. (2010) will be the starting point for our investigation of convergence rates for CLT for power variations of stable processes.

Theorem 1 *Let L be a strictly α-stable Lévy process with $L_1 \sim \mathcal{S}_\alpha(c, \beta, 0)$. If $p > \alpha/2$, then*

$$\left(V_p^n(L)_t - ntB_n(\alpha, p)\right)_{t\geq 0} \xrightarrow{\mathbb{D}} (L'_t)_{t\geq 0} \quad (n \to \infty)\ ,$$

in the Skorokhod topology. Here, L' is an α/p-stable process, where $L'_1 \sim \mathcal{S}_{\alpha/p}(c', 1, 0)$ with scale

$$c' = \begin{cases} c^p \left(\frac{2}{\pi}\Gamma(\alpha)\Gamma\left(1 - \frac{\alpha}{p}\right)\sin\left(\frac{\pi}{2}\alpha\right)\cos\left(\frac{\pi\alpha}{2p}\right)\right)^{p/\alpha} & ,\ \alpha \neq p\ , \\ c^p \left(\Gamma(\alpha)\sin\left(\frac{\pi}{2}\alpha\right)\right)^{p/\alpha} & ,\ \alpha = p\ . \end{cases} \tag{3}$$

The normalizing sequence $(B_n(\alpha, p))_{n\in\mathbb{N}}$ is deterministic and given by

$$B_n(\alpha, p) = \begin{cases} n^{-p/\alpha}\mathbb{E}|L_1|^p & ,\ \frac{\alpha}{2} < p < \alpha\ , \\ \mathbb{E}\sin(n^{-1}|L_1|^\alpha) & ,\ p = \alpha\ , \\ 0 & ,\ p > \alpha\ . \end{cases} \tag{4}$$

Remark 1 The scaling constant c' as a function of p is continuous at $p = \alpha$ because

$$\lim_{x\to 1}\Gamma(1 - x)\cos\left(\frac{\pi}{2}x\right) = \frac{\pi}{2}\ .$$

Also the c' given here differs from the one given in Hein et al. (2010). Yet using trigonometric identities and the property that

$$\Gamma(\alpha)\Gamma(1 - \alpha) = \frac{\pi}{\sin(\pi\alpha)}\ , \quad \alpha \neq 1\ ,$$

we verify that they coincide avoiding the singularity at $p = \alpha$ in Hein et al. (2010).

3 Convergence Rates for Power Variations

3.1 Convergence Rates

Let L be an α-stable Lévy process with $\alpha \in (0, 2)$, i.e. $L_1 \sim \mathcal{S}_\alpha(c, \beta, 0)$. Furthermore let $\left(V_p^n(L)_t\right)_{t\geq 0}$ be the associated realized power variation processes for some $p > \frac{\alpha}{2}$, and $n \in \mathbb{N}$. By convergence of the one dimensional marginals the random variables $V_p^n(L)_t$ converge to an α-stable random variable $V_{p,t} \sim \mathcal{S}_{\alpha/p}(t^{p/\alpha}c', 1, 0)$ for any fixed $t > 0$, with the limiting scale c' given in Theorem 1. The main goal of this article is to quantify this convergence in terms of the *Kolmogorov-Smirnov* or *uniform metric* given here for two real valued random variables X, Z by

$$\mathcal{D}(P, Q) = \sup_{x\in\mathbb{R}} |\mathbb{P}(X \leq x) - \mathbb{P}(Z \leq x)|\ .$$

Theorem 2 (Convergence rates) *Let $t > 0$ and $\frac{\alpha}{2} < p$. Then we have*

$$\mathcal{D}\big(V_n^p(L)_t, V_{p,t}\big) = \begin{cases} \mathcal{O}(n^{-1}), & \text{if } 2\alpha < p\ , \\ \mathcal{O}(\log(n)n^{-1}), & \text{if } p = 2\alpha\ , \\ \mathcal{O}\big(n^{1-\frac{p}{\alpha}}\big), & \text{if } \alpha < p < 2\alpha\ . \end{cases}$$

For $\frac{\alpha}{2} < p \leq \alpha$ we have

$$\mathcal{D}\big(\left(V_n^p(L)_t - ntB_n(\alpha, p)\right), V_{p,t}\big) = \begin{cases} \mathcal{O}\left(\log(n)^2 n^{-1}\right), & \text{if } p = \alpha\ , \\ \mathcal{O}\big(n^{1-2\frac{p}{\alpha}}\big), & \text{if } \frac{\alpha}{2} < p < \alpha\ . \end{cases}$$

Before we give the proofs in Section 4 let us investigate the consequences of these results to an estimation procedure.

3.2 Simple Minimum Distance Estimation

We have derived explicit bounds on the rate of convergence to the α/p-stable limit distribution of the power variation functionals $V_n^p(X)$ of sample paths. This knowledge enables us to set up a simple *minimum distance* fitting procedure for the parameters of the noise in the spirit of Hein et al. (2010).

Assume for the rest of this section that we observe an independent sequence $\{(X_t^{(i)})_{0\leq t\leq T}\}_{i\in\mathbb{N}}$ of copies of the process X, and for simplicity set $T = 1$. (Alternatively one could also observe the process X on disjoint intervals of length $T = 1$). For any m, n we obtain an i.d.d.sample $\{V_n^{p\,(i)}\}_{1\leq i\leq m}$ by computing power variations $V_n^p(X^{(i)})$ along equidistant partitions of length n. Denote by $\mu_m^n = \frac{1}{m}\sum_{i=1}^m \delta(\cdot - V_n^{p\,(i)})$ the empirical measure obtained from this sample and by $\mathcal{S}_{\alpha/p} = \mathcal{S}_{\alpha/p}(c', 1, 0)$ the stable limit distribution.

The aim is to derive the asymptotic distribution of the quantity $\mathcal{D}(\mu_m^n, \mathcal{S}_{\alpha/p})$ and under mild conditions on m and n as they tend to infinity.

In classical *empirical process theory* the Kolmogorov-Smirnov statistic $\Delta_m = \mathcal{D}(\mu_m, \mu)$ that measures the distance between the empirical distribution of a sample μ_m to the underlying distribution μ, is extensively studied. It is well known that Δ_m is *distribution free*—meaning that its distribution does not depend on μ, and that $\sqrt{m}\Delta_m$ is asymptotically distributed like $B^* := \max_{0\leq t\leq 1}|B_t^0|$, for a Brownian bridge B^0 independent of the sample (e.g. see Billingsley 1999).

The question arises under what conditions on m, n this limit behavior carries over to the statistic $\mathcal{D}(\mu_m^n, \mathcal{S}_{\alpha/p})$.

Theorem 3 *Let $(n_m)_{m\in\mathbb{N}}$ be a increasing sequence of integers with $\lim_{m\to\infty} n_m = \infty$ such that the limit of the following expressions is zero*

$$0 = \lim_{m\to\infty} \begin{cases} \sqrt{m}/n_m, & \text{if } 2\alpha < p\ , \\ \sqrt{m}\cdot\log(n_m)/n_m, & \text{if } p = 2\alpha\ , \\ \sqrt{m}\cdot(n_m)^{1-\frac{p}{\alpha}}, & \text{if } \alpha < p < 2\alpha\ . \end{cases}$$

Then there is a Brownian Bridge B^0 independent of X such that

$$\sqrt{m}\mathcal{D}\big(\mu_m^{n_m}, \mathcal{S}_{\alpha/p}\big) \xrightarrow{d} B^* \quad \text{as } m\to\infty\ ,$$

where $B^* := \max_{0\leq t\leq 1}|B_t^0|$.

Proof If we denote by μ^{n_m} the distribution of $V^p_{n_m}(X_1)$, the triangle inequality implies

$$\sqrt{m}|\mathcal{D}\big(\mu^{n_m}_m, \mathcal{S}_{\alpha/p}\big) - \mathcal{D}(\mu^{n_m}_m, \mu^{n_m})| \le \sqrt{m}\mathcal{D}(\mu^{n_m}, \mathcal{S}_{\alpha/p}) \ .$$

Since the Kolmogorov-Smirnov statistic is distribution free, the distribution of the second summand on the left hand side is the one of Δ_m independent of n_m and asymptotically that of B^* as m tends to infinity. The right hand side is converging to zero under the conditions on m, n_m if we recall the orders given in Theorem 2. Hence convergence to B^* pertains as m goes to infinity. □

4 Proofs for the Uniform Distance

In this section we verify the rates claimed in Theorem 2. To simplify the notation, throughout the proof we introduce the "centered" variation

$$Y^p_n(L) := \big(V^p_n(L)_1 - nB_n(\alpha, p)\big) \ , \quad n \in \mathbb{N}, \ p > 0 \ .$$

We identify its characteristic function to be

$$\varphi_{Y^p_n}(\lambda) = \left(\varphi_{|L_1|^p}\left(\frac{\lambda}{n^{p/\alpha}}\right)\right)^n \mathrm{e}^{-\mathrm{i}\lambda n B_n(\alpha,p)}.$$

Since $(Y^p_n(L))_{n\in\mathbb{N}}$ converges to the value at time 1 of the α/p-stable $V_{p,\cdot}$ with characteristic function $\gamma(\lambda) = \exp(-\psi(\lambda))$, where $\Re\psi(\lambda) = \sigma|\lambda|^{\frac{\alpha}{p}}$ (and $\sigma = c^{\frac{\alpha}{p}}$), it is clear that the product

$$\eta = \varphi_{|L_1|^p}\left(\frac{\lambda}{n^{p/\alpha}}\right) \mathrm{e}^{\frac{\psi(\lambda)}{n} - \mathrm{i}\lambda B_n(\alpha,p)} \tag{5}$$

converges to 1 for any $\lambda \in \mathbb{R}$. The product is of the form $\eta = \eta(\xi)$ with $\xi = \frac{\lambda}{n^{p/\alpha}}$ except in case $\alpha = p$ which we will neglect for convenience of notation. This is due to Eq. A leading to the simple relationship $\frac{\psi(\lambda)}{n} = \psi(\frac{\lambda}{n^{p/\alpha}}) = \psi(\xi)$, and the equation $\lambda B_n(\alpha, p) = \frac{\lambda}{n^{p/\alpha}}\mathbb{E}(|L_1|^p) = \xi\,\mathbb{E}(|L_1|^p)$ which follows from Eq. 4.

We use a refined version of CLTs with stable limit laws (see Paulauskas 1974; Christoph and Wolf 1992), where the order is refined to regular variation at 0. A real valued function $H \ge 0$ is said to be *regularly varying* at 0 of index $\beta \ge 0$ (see e.g. Bingham et al. 1989) if for all $a > 0$

$$\lim_{x\to 0} \frac{H(ax)}{H(x)} = a^\beta \ .$$

Theorem 4 *Assume that we have* $|\eta(\xi) - 1| = \mathcal{O}(H(\xi))$ *in the framework just explained, with a function* $H \ge 0$ *regularily varying at zero of index* $\beta > \frac{\alpha}{p} \vee 1$. *Then*

$$\mathcal{D}\big(Y^p_n(L)_t, V_{p,t}\big) = \mathcal{O}(nH(n^{-p/\alpha})) + \mathcal{O}(n^{-p/\alpha}) \ .$$

Proof The well known smoothness inequality of Berry and Esséen (e.g. Feller (1971, lemma XVI.3.1, p. 537)) relates the distance of distribution functions to the distance of characteristic functions. Here it reads

$$\mathcal{D}\big(Y^p_n(L)_t, V_{p,t}\big) \le \frac{1}{\pi}\int_{-T}^{T}\left|\frac{\varphi_{Y^p_n}(\lambda) - \gamma(\lambda)}{\lambda}\right| \mathrm{d}\lambda + \frac{24m}{\pi T} \ , \tag{6}$$

for any $T > 0$ and $\gamma(\lambda) = \exp(-\psi(\lambda))$. If f denotes the density of $V_{p,t}$, m can be chosen according to

$$\sup_{x\in\mathbb{R}} f(x) \leq \frac{1}{\pi}\int_0^\infty \exp(-\sigma\lambda^{\frac{\alpha}{p}})\mathrm{d}\lambda = \sigma\frac{p}{\alpha}\frac{\Gamma(\frac{p}{\alpha})}{\pi} =: m < \infty\,.$$

We estimate as follows

$$\begin{aligned}
|\varphi_{Y_n^p}(\lambda) - \mathrm{e}^{-\psi(\lambda)}| &= \left|\left(\varphi_{|L_1|^p}\left(\frac{\lambda}{n^{p/\alpha}}\right)\right)^n \mathrm{e}^{-\mathrm{i}\lambda n B_n(\alpha,p)} - \mathrm{e}^{-\psi(\lambda)}\right| \\
&= \mathrm{e}^{-\sigma|\lambda|^{\frac{\alpha}{p}}}|\eta^n - 1| = \mathrm{e}^{-\sigma|\lambda|^{\frac{\alpha}{p}}}|(\eta-1+1)^n - 1| \\
&\leq \mathrm{e}^{-\sigma|\lambda|^{\frac{\alpha}{p}}} n|\eta-1|\exp(n|\eta-1|)\,,
\end{aligned}$$

where the last line results from an elementary estimation starting with the binomial formula. We will chose $T = \tau n^{p/\alpha}$ for a scaling constant τ to be determined. By this choice the second summand of Eq. 6 is $\mathcal{O}(n^{-p/\alpha})$. For the first term, we find the estimation

$$\begin{aligned}
&\int_{-T}^{T}\left|\frac{\varphi_{Y_n^p}(\lambda)-\gamma(\lambda)}{\lambda}\right|\mathrm{d}\lambda \\
&\leq 2\int_0^{\tau n^{p/\alpha}} \frac{n|\eta\left(\frac{\lambda}{n^{p/\alpha}}\right)-1|}{\lambda}\exp\left(n|\eta\left(\frac{\lambda}{n^{p/\alpha}}\right)-1| - \sigma\lambda^{\alpha/p}\right)\mathrm{d}\lambda \\
&\leq 2\int_0^{\tau n^{p/\alpha}} \frac{n}{\lambda}H\left(\frac{\lambda}{n^{p/\alpha}}\right)\exp\left(-\left(\sigma - h\left(\frac{\lambda}{n^{p/\alpha}}\right)\right)\lambda^{\alpha/p}\right)\mathrm{d}\lambda
\end{aligned}$$

Here for $\xi \in \mathbb{R}$ we express $H(\xi) = h(\xi)\xi^{\alpha/p}$ with a function h which is regularly varying at zero of index $\beta - \frac{\alpha}{p} > 0$. In particular $h(\xi) = \mathbf{o}(1)$. By choosing τ small enough we can therefore guarantee that $\sigma - h(\xi) \geq \kappa$ for some $\kappa > 0$ for $0 \leq \xi \leq \tau$. Hence we can further estimate

$$\int_{-T}^{T}\left|\frac{\varphi_{Y_n^p}(\lambda)-\gamma(\lambda)}{\lambda}\right|\mathrm{d}\xi \leq 2\int_0^\infty \frac{n}{\lambda}H\left(\frac{\lambda}{n^{p/\alpha}}\right)\exp\big(-\kappa\lambda^{\alpha/p}\big)\mathrm{d}\lambda.$$

In order to obtain the asymptotic equivalence we apply Fatou's lemma to the fraction

$$\begin{aligned}
&\limsup_{n\to\infty}\left[\left(nH\left(\frac{1}{n^{p/\alpha}}\right)\right)^{-1}\int_0^\infty \frac{n}{\lambda}H\left(\frac{\lambda}{n^{p/\alpha}}\right)\exp\big(-\kappa\lambda^{\alpha/p}\big)\mathrm{d}\lambda\right] \\
&\leq \int_0^\infty \limsup_{n\to\infty}\frac{1}{\lambda}H\left(\frac{\lambda}{n^{p/\alpha}}\right)\Big/ H\left(\frac{1}{n^{p/\alpha}}\right)\exp\big(-\kappa\lambda^{\alpha/p}\big)\mathrm{d}\lambda \\
&= \int_0^\infty \lambda^{\beta-1}\exp\big(-\kappa\lambda^{\alpha/p}\big)\mathrm{d}\lambda = \frac{p}{\alpha}\kappa^{-\frac{p}{\alpha}\beta}\Gamma\left(\frac{p}{\alpha}\beta\right) < \infty\,.
\end{aligned}$$

This completes the proof. □

Remark 2 If $|\eta(\xi)-1|$ is actually of polynomial order $\mathcal{O}(|\xi|^r)$ for $r > \alpha/p$ as considered in the general CLT in Paulauskas (1974) and Christoph and Wolf (1992), the condition $|\eta(\xi)-1| = \mathbf{o}(\xi^{\alpha/p})$ holds and we recover

$$\mathcal{D}\big(Y_n^p(L)_t, V_{p,t}\big) = \mathcal{O}(n^{-(rp/\alpha-1\wedge p/\alpha)}) ,$$

since then $|\eta(\xi)-1| = \mathcal{O}(n^{-rp/\alpha})$. This will be sufficient for all cases except $p=\alpha$, $p=2\alpha$ (c.f. Lemma 5).

Recalling the definition of η in Eq. 5 we can write

$$|\eta-1| = \mathrm{e}^{\sigma\frac{|\lambda|}{n}^{\alpha/p}} |\varphi_{|L_1|^p}\left(\frac{\lambda}{n^{p/\alpha}}\right) - \mathrm{e}^{-\frac{\psi(\lambda)}{n}+\mathrm{i}\lambda B_n(\alpha,p)}| .$$

Since we estimate this quantity on the interval defined by $\frac{|\lambda|}{n^{p/\alpha}} \le 1$, it is enough to bound

$$|\varphi_{|L_1|^p}\left(\frac{\lambda}{n^{p/\alpha}}\right) - \mathrm{e}^{-\frac{\psi(\lambda)}{n}+\mathrm{i}\lambda B_n(\alpha,p)}| \le |\varphi_{|L_1|^p}\left(\frac{\lambda}{n^{p/\alpha}}\right) - 1 + \frac{\psi(\lambda)}{n} - \mathrm{i}\lambda B_n(\alpha,p)| + |\frac{\psi(\lambda)}{n} - \mathrm{i}\lambda B_n(\alpha,p)|^2 . \tag{7}$$

In the last inequality we use the fact that the real part in the exponent is negative. The second summand of the right hand side is known as it consists of the second characteristic of the stable limit and deterministic normalization constants. In order to control the behavior of the first summand we establish an asymptotic expansion for the characteristic function $\varphi_{|L_1|^p}$ in the following lemma, normalized by the numbers $B_n(\alpha,p), n \in \mathbb{N}$.

Lemma 5 *Let $\varphi_{|L_1|^p}$ be the characteristic function of the random variable $|L_1|^p$ and $(B_n(\alpha,p))_{n\in\mathbb{N}}$ the normalizing sequence of Theorem 1. If λ and n are such that $\xi = \frac{|\lambda|}{n^{p/\alpha}} \le 1$ we have the following representation*

$$\varphi_{|L_1|^p}(\xi) - 1 + \psi(\xi) - \mathrm{i}\xi\mathbb{E}(|L_1|^p) = 1 - \psi(\xi) + R(\xi) ,$$

with the convention that the expectation is set to zero if $p > \alpha$. The remainder term R is of order

$$R(\xi) = \begin{cases} \mathcal{O}\big(|\xi|^{2\frac{\alpha}{p}}\big), & \textit{if} \quad p > 2\alpha , \\ \mathcal{O}\big(|\xi|\log\big(|\xi|^{-1}\big)\big), & \textit{if} \quad p = 2\alpha , \\ \mathcal{O}\big(|\xi|^1\big), & \textit{if} \quad \alpha < p < 2\alpha , \\ \mathcal{O}\big(|\xi|^2\big), & \textit{if} \quad \frac{\alpha}{2} < p < \alpha . \end{cases}$$

If $p=\alpha$, we have

$$\varphi_{|L_1|^\alpha}\left(\frac{\lambda}{n}\right) - 1 + \psi\left(\frac{\lambda}{n}\right) - \mathrm{i}\lambda\mathbb{E}\sin\left(n|L_1|^\alpha\right) = 1 - \psi\left(\frac{\lambda}{n}\right) + R(\lambda,n) ,$$

with $R(\lambda,n) = \mathcal{O}\big(\log(\frac{n}{\lambda})\frac{\lambda^2\vee|\lambda|}{n^2}\big)$.

The preceding lemma provides all the necessary tools needed to prove our main result. Since the lemma just collects the explicit error bounds stated in the Lemmas 7, 8 and 9 a formal proof is omitted.

Proof (Proof of Theorem 2) First consider the case $p \neq \alpha$. We have seen that it suffices to bound Eq. 7. Recalling our convention for the expectation to be interpreted as zero in case $p > \alpha$, we see that the second term asymptotically satisfies

$$|\psi\left(\frac{\lambda}{n^{p/\alpha}}\right) - \mathrm{i}\frac{\lambda}{n^{p/\alpha}}\mathbb{E}(|L_1|^p)|^2 = \begin{cases} \mathcal{O}\left(\left(\frac{|\lambda|}{n^{p/\alpha}}\right)^{2\frac{\alpha}{p}}\right), & \textit{if} \quad p > \alpha\,, \\ \mathcal{O}\left(\left(\frac{|\lambda|}{n^{p/\alpha}}\right)^{2}\right), & \textit{if} \quad \frac{\alpha}{2} < p < \alpha\,. \end{cases}$$

Writing $\xi_n = \frac{\lambda}{n^{p/\alpha}}$, we observe that the resulting convergence orders for $n \to \infty$ are non-slower than the ones given in Lemma 5. Thus the right hand side of Eq. 7 is of the order stated in Lemma 5. We can then apply Theorem 4 with $H = R$. Retaining only the dependence on n to verify the hypotheses of Theorem 4, we obtain

$$|\eta(\xi_n) - 1| = \begin{cases} \mathcal{O}(n^{-2}), & \textit{if} \quad p > 2\alpha\,, \\ \mathcal{O}(n^{-2}\log n), & \textit{if} \quad p = 2\alpha\,, \\ \mathcal{O}(n^{-\frac{\alpha}{p}}), & \textit{if} \quad \alpha < p < 2\alpha\,, \\ \mathcal{O}(n^{-2\frac{\alpha}{p}}), & \textit{if} \quad \frac{\alpha}{2} < p < \alpha\,. \end{cases}$$

This completes the proof for $p \neq \alpha$. Let now $p = \alpha$. Here Lemma 10 assures that

$$|\Im\frac{\psi(\lambda)}{n} - \lambda\mathbb{E}\sin(n^{-1}|L_1|^{\alpha})| = \mathcal{O}\left(\frac{\lambda}{n}\log\left(\frac{n}{\lambda}\right)\right)\,.$$

Taking the square we see that the convergence order for $n \to \infty$ is actually slower than the rate resulting from Lemma 5 since the logarithm appears squared. Thus we obtain the convergence order $\mathcal{O}(\frac{\lambda^2\vee\lambda}{n^2}\log\left(\frac{n}{\lambda}\right)^2)$. Since only regular variation at zero determines the order, Theorem 4 applies with $H(\xi) = \xi^2\log(\xi^{-1})^2$. The proof is complete. □

4.1 Tail Probabilities

In this subsection we obtain a series expansion of the tail probabilities of $|X|^p$, $p > 0$, with a strictly stable random variable $X \sim \mathcal{S}_{\alpha}^{(A)}(c, \beta, 0)$. We look for an asymptotic expansion in $x > 0$ for the quantity

$$\mathbb{P}[|X|^p \geq x]\,, \quad p > 0\,,\ x > 0\,.$$

The tail probabilities determine the characteristic function of $|L_\alpha|^p$ which we investigate in the following subsection. It is possible to derive series expansions by inverting Fourier transforms. For the density of a strictly stable distribution in parametrization (B), Uchaikin and Zolotarev (1999, p. 115) give a series expansion with negative powers. For any $N \in \mathbb{N}$ the density is given by

$$f_\alpha^\delta(x) = \pi^{-1}\sum_{k=1}^{N}(-1)^{k-1}\frac{\Gamma(\alpha k+1)}{k!}\sin(k\rho^+\pi)x^{-\alpha k-1}$$
$$+\pi^{-1}\frac{\Gamma(\alpha(N+1)+1)}{(N+1)!}\theta(x)x^{-\alpha(N+1)-1}\,, \quad |\theta(x)|\le 1,\ x>0\,,$$

where $f_\alpha^\delta(-x) = f_\alpha^{-\delta}(x)$, and

$$\rho^+ := \frac{\alpha+\delta}{2}\,, \quad \rho^- := \frac{\alpha-\delta}{2}\,.$$

The quantities θ as well as $\Theta^{(\pm)}$ will be used to denote parameter dependent small quantities bounded by one.

We recall that we have $\mathcal{S}_\alpha^{(B)}(1,\delta) = \mathcal{S}_\alpha(\cos(\frac{\pi}{2}\delta)^{1/\alpha},\beta,0)$. This implies that for $X \sim \mathcal{S}_\alpha^{(A)}(c,\beta,0)$ we have

$$b\cdot X \sim \mathcal{S}_\alpha^{(B)}(1,\delta) \quad \text{with} \quad b = \begin{cases} c^{-1}\cos\left(\frac{\pi}{2}\delta\right)^{1/\alpha}\,, & \alpha\neq 1\,,\\ c^{-1}\,, & \alpha = 1\,,\end{cases}$$

and the parameter δ chosen according to Eq. 2. Then we have for $x>0$

$$\mathbb{P}[|X|^p \ge x] = \mathbb{P}[X\le -x^{1/p}, X\ge x^{1/p}] = \int_{-\infty}^{-bx^{1/p}} f_\alpha^{-\delta}(|y|)\mathrm{d}y + \int_{bx^{1/p}}^{\infty} f_\alpha^\delta(y)\mathrm{d}y,$$

where

$$\int_{bx^{1/p}}^{\infty} f_\alpha^\delta(y)\mathrm{d}y = \pi^{-1}\sum_{k=1}^{N}(-1)^{k-1}\frac{\Gamma(\alpha k+1)}{k!}\sin(k\rho^+\pi)\int_{bx^{1/p}}^{\infty} y^{-\alpha k-1}\mathrm{d}y$$
$$+\pi^{-1}\frac{\Gamma(\alpha(N+1)+1)}{(N+1)!}\int_{bx^{1/p}}^{\infty}\theta(y)y^{-\alpha(N+1)-1}\mathrm{d}y$$
$$= \pi^{-1}\sum_{k=1}^{N}(-1)^{k-1}\frac{\Gamma(\alpha k+1)}{k!\alpha k}\sin(k\rho^+\pi)b^{-k\alpha}x^{-k\alpha/p}$$
$$+\pi^{-1}\frac{\Gamma(\alpha(N+1)+1)}{(N+1)!\alpha(N+1)}b^{-\alpha(N+1)}\Theta^+(x)x^{-(N+1)\alpha/p},$$

and $\Theta^+(x)$ is determined by

$$\Theta^+(x) = \alpha(N+1)\int_1^\infty \theta(bx^{1/p}z)z^{-\alpha(N+1)-1}\mathrm{d}z,$$

thus $|\Theta^+(x)| \leq 1$. And similarly

$$\int_{-\infty}^{-bx^{1/p}} f_\alpha^{-\delta}(|y|)\mathrm{d}y = \int_{bx^{1/p}}^{\infty} f_\alpha^{-\delta}(y)\mathrm{d}y$$

$$= \pi^{-1}\sum_{k=1}^{N}(-1)^{k-1}\frac{\Gamma(\alpha k+1)}{k!\alpha k}\sin(k\rho^-\pi)b^{-k\alpha}x^{-k\alpha/p}$$

$$+\pi^{-1}\frac{\Gamma(\alpha(N+1)+1)}{(N+1)!\alpha(N+1)}b^{-\alpha(N+1)}\Theta^-(x)x^{-(N+1)\alpha/p}.$$

Re-substituting b we obtain for $x > 0$

$$\mathbb{P}[|X|^p \geq x] = \pi^{-1}\sum_{k=1}^{N}(-1)^{k-1}\frac{\Gamma(\alpha k+1)}{k!\alpha k}\left\{\sin(k\rho^+\pi)+\sin(k\rho^-\pi)\right\}b^{-k\alpha}x^{-k\alpha/p}$$

$$+\pi^{-1}\frac{\Gamma(\alpha(N+1)+1)}{(N+1)!\alpha(N+1)}b^{-\alpha(N+1)}\left\{\Theta^+(x)+\Theta^-(x)\right\}x^{-(N+1)\alpha/p}$$

$$= \frac{2}{\pi}\sum_{k=1}^{N}(-1)^{k-1}\frac{\Gamma(\alpha k)}{k!}c^{\alpha k}\sin(\frac{\pi}{2}\alpha k)\cos(\frac{\pi}{2}\delta k)\cos(\frac{\pi}{2}\delta)^{-k}x^{-k\alpha/p}$$

$$+\frac{2}{\pi}\frac{\Gamma(\alpha(N+1))}{(N+1)!}c^{\alpha(N+1)}\cos(\frac{\pi}{2}\delta)^{-(N+1)}\Theta(x)x^{-(N+1)\alpha/p}.$$

Here $\Theta = \frac{1}{2}(\Theta^+(x)+\Theta^-(x))$, with $|\Theta| \leq 1$, and we made use of the trigonometric identity

$$\sin(k\rho^+\pi)+\sin(k\rho^-\pi) = 2\sin\left(\frac{\alpha}{2}k\pi\right)\cos\left(\frac{\delta}{2}k\pi\right).$$

With these arguments we have proven the following lemma.

Lemma 6 (Tail probabilities) *Let $X \sim \mathcal{S}_\alpha^{(A)}(c,\beta,0)$ be a random variable with a strictly stable distribution, and let δ be given by Eq. 2. For $p > 0$ and any $N \in \mathbb{N}$ we then have*

$$\mathbb{P}[|X|^p \geq x] = \sum_{k=1}^{N} A_k x^{-k\alpha/p} + \bar{A}_{N+1}\Theta(x)x^{-(N+1)\alpha/p}, \quad x > 0, \tag{8}$$

where $|\Theta(x)| \leq 1$, and for $1 \leq k \leq N+1$ we have

$$A_k = \frac{2}{\pi}(-1)^{k-1}\frac{\Gamma(\alpha k)}{k!}c^{\alpha k}\sin\left(\frac{\pi}{2}\alpha k\right)\cos\left(\frac{\pi}{2}\delta k\right)\cos\left(\frac{\pi}{2}\delta\right)^{-k},$$

$$\bar{A}_k = \frac{2}{\pi}\frac{\Gamma(\alpha k)}{k!}c^{\alpha k}\cos\left(\frac{\pi}{2}\delta\right)^{-k}.$$

4.2 Expansion of the Characteristic Function

Let L be a strictly stable Lévy process. Then $L_1 \sim \mathcal{S}_\alpha(c,\beta,0)$ is a strictly stable random variable. We want to calculate the first terms in a power series expansion for the characteristic function $\varphi_{|L_1|^p}$ of $|L_1|^p$, $p > 0$, and give an estimate for the remainder. We will establish an equation of the following form

$$\varphi_{|L_1|^p}\left(\frac{\lambda}{n^{p/\alpha}}\right) - \mathrm{i}\lambda B_n(\alpha, p) = 1 - \frac{\psi(\lambda)}{n} + R$$

for λ in a certain neighborhood of zero. Let $\exp(-\psi(\lambda))$ be the characteristic function of the $\frac{\alpha}{p}$-stable limit distribution as given in Theorem 1. If $p \neq \alpha$ we recall that $\psi(\lambda)$ is of the form $-\sigma|\lambda|^{\frac{\alpha}{p}}, \sigma > 0$ so that we can simplify the expansion using $\xi = \frac{\lambda}{n^{p/\alpha}}$.

This will be done in three lemmas that cover the cases $p > \alpha$, $p = \alpha$ and $\alpha > p > \frac{\alpha}{2}$ separately. To calculate moments, we shall use the following equation relating them to tail probabilities. For $X \geq a$ and $g \in C^1(\mathbb{R}_+)$ we may write

$$\mathbb{E}(g(X)) = \int_a^\infty g(x)\mathrm{d}\mathbb{P}(X \leq x) = g(a) + \int_a^\infty g'(x)\mathbb{P}(X > x)\mathrm{d}x \,. \tag{9}$$

If $p > \alpha$, the normalizing sequence $(B_n(\alpha, p))_{n\in\mathbb{N}}$ is zero and the desired expansion is stated in the following first approximation lemma.

Lemma 7 *For $p > \alpha$ and $|\xi| \leq 1$ we have*

$$\varphi_{|L_1|^p}(\xi) = 1 - A_1|\xi|^{\frac{\alpha}{p}}\Gamma\left(1 - \frac{\alpha}{p}\right)\cos\left(\frac{\pi\alpha}{2p}\right)\left(1 - \mathrm{i}\,\mathrm{sign}(\xi)\tan\left(\frac{\pi\alpha}{2p}\right)\right) + R(\xi)$$

For the remainder term R we have

$$|R(\xi)| = \begin{cases} \mathcal{O}\left(|\xi|^{1\wedge 2\frac{\alpha}{p}}\right), & \text{if } p \neq 2\alpha \,, \\ \mathcal{O}\left(|\xi|\log\left(|\xi^{-1}|\right)\right), & \text{if } p = 2\alpha \,. \end{cases}$$

Proof Throughout the proof we only consider $\xi > 0$. The case $\xi < 0$ just amounts to considering the complex conjugate. Integration by parts and the series expansion (8) yield for $N \in \mathbb{N}$

$$\begin{aligned} \varphi_{|L_1|^p}(\xi) =& \ \mathbb{E}\mathrm{e}^{\mathrm{i}\xi|L_1|^p} = 1 + \int_0^\infty (\mathrm{i}\xi)\mathrm{e}^{\mathrm{i}\xi x}\mathbb{P}(|L_1| > x^{1/p})\mathrm{d}x \\ =& \ 1 + \sum_{k=1}^N A_k\xi\int_0^\infty (\mathrm{i}\mathrm{e}^{\mathrm{i}\xi x})x^{-k\frac{\alpha}{p}}\mathrm{d}x + \bar{A}_{N+1}\xi\int_0^\infty (\mathrm{i}\mathrm{e}^{\mathrm{i}\xi x})\Theta(x)x^{-(N+1)\frac{\alpha}{p}}\mathrm{d}x \\ =& \ 1 + \sum_{k=1}^N A_k\xi^{k\frac{\alpha}{p}}\int_0^\infty (\mathrm{i}\mathrm{e}^{\mathrm{i}y})y^{-k\frac{\alpha}{p}}\mathrm{d}y + \bar{A}_{N+1}\xi\int_0^\infty (\mathrm{i}\mathrm{e}^{\mathrm{i}\xi x})\Theta(x)x^{-(N+1)\frac{\alpha}{p}}\mathrm{d}x. \end{aligned}$$

We choose $N := \max(k \in \mathbb{N} : k\frac{\alpha}{p} < 1) \geq 1$ such that the integrals in our asymptotic expansion are of the form

$$\int_0^\infty \left(\mathrm{i}\mathrm{e}^{\mathrm{i}y}\right)y^{-b}\mathrm{d}y = -\Gamma(1-b)\cos\left(\frac{\pi}{2}b\right)\left(1 - \mathrm{i}\tan\left(\frac{\pi}{2}b\right)\right),$$

for exponents satisfying $0 < b < 1$. If we exclude $(N+1)\frac{\alpha}{p} = 1$, the last integral is finite and the lemma is proven. For $p < 2\alpha$ observe that $N = 1$ and thus no error term of order $\mathcal{O}(\xi^{2\frac{\alpha}{p}})$ exists. It remains to consider the case $(N+1)\frac{\alpha}{p} = 1$. The last integral takes the form

$$\int_0^\infty \mathrm{i}\xi \mathrm{e}^{\mathrm{i}\xi x} \bar{A}_{N+1}\Theta(x)x^{-1}\mathrm{d}x \,.$$

Since $\Theta(x)$ is not known explicitly we divide the integral into

$$\begin{aligned}
&\left|\int_0^1 \mathrm{i}\xi \mathrm{e}^{\mathrm{i}\xi x} \bar{A}_{N+1}\Theta(x)x^{-1}\mathrm{d}x\right| \\
&\quad = \left|\int_0^1 (\mathrm{i}\xi)\mathrm{e}^{\mathrm{i}\xi x}\left(\mathbb{P}(|L_1| > x^{1/p}) - \sum_{k=1}^N A_k x^{-k\frac{\alpha}{p}}\right)\mathrm{d}x\right| \\
&\quad \le \left(1 + \sum_{k=1}^N \frac{p}{k\alpha}|A_k|\right)\xi \,,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\bar{A}_{N+1}\int_1^\infty \mathrm{i}\xi \mathrm{e}^{\mathrm{i}\xi x}x^{-1}\mathrm{d}x| = |\bar{A}_{N+1}\int_\xi^\infty \mathrm{i}\xi \mathrm{e}^{\mathrm{i}y}y^{-1}\mathrm{d}y\right| \\
&\quad \le |\bar{A}_{N+1}|\xi\left|\log(1) - \log(\xi) + \int_1^\infty \frac{\mathrm{i}\cos(y) - \sin(y)}{y}\mathrm{d}y\right| \\
&\quad = |\bar{A}_{N+1}|\xi\left(\log(\xi^{-1}) + |\mathrm{si}(1)| + |\mathrm{ci}(1)|\right) \,.
\end{aligned}$$

The trigonometric integrals ci(1) and si(1) are finite constants, given by the *sine-* (resp. *cosine-*) *integral function*. Hence we obtain the desired logarithmic bound. □

For $\frac{\alpha}{2} < p < \alpha$ we normalize by $B_n(\alpha, p) = n^{-p/\alpha}\mathbb{E}|L_1|^p, n \in \mathbb{N}$. The corresponding expansion is given next.

Lemma 8 *Let $\frac{\alpha}{2} < p < \alpha$. Then we have for $|\xi| \le 1$*

$$\varphi_{|L_1|^p}(\xi) - \mathrm{i}\xi\mathbb{E}|L_1|^p = 1 - \Gamma\left(1 - \frac{\alpha}{p}\right)\cos\left(\frac{\pi\alpha}{2p}\right)\left(1 - \mathrm{i}\,\mathrm{sign}(\xi)\tan\left(\frac{\pi\alpha}{2p}\right)\right)A_1\xi^{\alpha/p} + R(\xi)$$

with a remainder term R bounded according to $|R(\xi)| = \mathcal{O}(\xi^2)$.

Proof Again is suffices to consider $\xi > 0$. By integration by parts and the series expansion of Lemma 6 for $N = 1$ we get the following expansion

$$\begin{aligned}
\varphi_{|L_1|^p}(\xi) - \mathrm{i}\xi\mathbb{E}|L_1|^p &= \int_0^\infty \mathrm{e}^{\mathrm{i}\xi x}\mathrm{d}\mathbb{P}(|L_1|^p \le x) - \mathrm{i}\xi\int_0^\infty x\,\mathrm{d}\mathbb{P}(|L_1|^p \le x) \\
&= 1 + \mathrm{i}\xi\int_0^\infty \left(\mathrm{e}^{\mathrm{i}\xi x} - 1\right)\mathbb{P}(|L_1|^p > x)\mathrm{d}x \\
&= 1 + \mathrm{i}\xi^{\frac{\alpha}{p}}\int_0^\infty \left(\mathrm{e}^{\mathrm{i}y} - 1\right)A_1 y^{-\frac{\alpha}{p}}\mathrm{d}y \\
&\quad + \mathrm{i}\xi\int_0^\infty \left(\mathrm{e}^{\mathrm{i}\xi x} - 1\right)\bar{A}_2\Theta(x)x^{-2\frac{\alpha}{p}}\mathrm{d}x \,.
\end{aligned}$$

Again we change the variable writing $y = \xi x$. The first integral is evaluated for $1 < b = \frac{\alpha}{p} < 2$ by

$$\int_0^\infty \left(\mathrm{e}^{\mathrm{i}y} - 1\right) y^{-b}\,\mathrm{d}y = \Gamma(1-b)\left(\sin\left(\frac{\pi}{2}b\right) + \mathrm{i}\cos\left(\frac{\pi}{2}b\right)\right)\,,$$
$$= -\Gamma(1-b)\cos\left(\frac{\pi}{2}b\right)\left(1 - \mathrm{i}\tan\left(\frac{\pi}{2}b\right)\right) A_1\,.$$

The remainder term R consists of the second integral. We make use of $|\mathrm{e}^{\mathrm{i}y} - 1| \le |y|$ and a re-substitution of the tail expansion. Integrating on [0, 1] we then have

$$|\mathrm{i}\xi \int_0^1 \left(\mathrm{e}^{\mathrm{i}\xi x} - 1\right) \bar{A}_2 \Theta(x) x^{-2\frac{\alpha}{p}}\mathrm{d}x| \le \xi \int_0^1 \xi x |\mathbb{P}\left(|L_1| > x^{1/p}\right) - A_1 x^{-\frac{\alpha}{p}}|\mathrm{d}x$$
$$\le \xi^2 \int_0^1 x\big(1 - |A_1| x^{-\frac{\alpha}{p}}\big)\mathrm{d}x$$
$$\le \left(\frac{1}{2} + \frac{|A_1|}{2 - \frac{\alpha}{p}}\right)\xi^2\,.$$

Let $0 < a < 1 < b < 2$ and calculate

$$\left|\int_a^\infty \left(\mathrm{e}^{\mathrm{i}y} - 1\right) y^{-2b}\,\mathrm{d}y\right| \le \int_a^1 y^{1-2b}\,\mathrm{d}y + \int_1^\infty 2y^{-2b}\,\mathrm{d}y$$
$$\le \frac{1 - a^{2(1-b)}}{2(1-b)} + \frac{2}{(2b-1)}\,.$$

If we set $y = \xi x$, $b = \frac{\alpha}{p}$ and $a = \xi$, the above estimate yields

$$\left|\mathrm{i}\xi \int_1^\infty \left(\mathrm{e}^{\mathrm{i}\xi x} - 1\right) \bar{A}_2 \Theta(x) x^{-2\frac{\alpha}{p}}\mathrm{d}x\right| = \left|\xi^{2\frac{\alpha}{p}} \int_\xi^\infty \left(\mathrm{e}^{\mathrm{i}y} - 1\right) \bar{A}_2 \Theta\left(\xi^{-1}y\right) y^{-2\frac{\alpha}{p}}\mathrm{d}y\right|$$
$$\le |\bar{A}_2| \xi^{2\frac{\alpha}{p}} \left(\frac{1 - \xi^{2(1-\frac{\alpha}{p})}}{\left(\frac{\alpha}{p} - 1\right)} + \frac{2}{\left(2\frac{\alpha}{p} - 1\right)}\right)$$
$$\le 2|\bar{A}_2| \left(\frac{1}{\left(\frac{\alpha}{p} - 1\right)} + \frac{1}{\left(2\frac{\alpha}{p} - 1\right)}\right)\xi^2\,.$$

In total we get a remainder term of the required form. □

For $p = \alpha$ a normalization by $B_n(\alpha, p) = \mathbb{E}\sin(n^{-1}|L_1|^\alpha)$, $n \in \mathbb{N}$, is required. In this case it is not possible to state the lemma for a general argument ξ because the normalization does not depend explicitly on a parameter $\xi = \frac{\lambda}{n}$, but implicitly as n appears inside the sine.

Lemma 9 *Let $p = \alpha$, $n \in \mathbb{N}$. Then we have for $0 < \lambda < n$*

$$\varphi_{|L_1|^\alpha}\left(\frac{\lambda}{n}\right) - \mathrm{i}\lambda\mathbb{E}\sin(n^{-1}|L_1|^\alpha) = 1 - \frac{\pi}{2}A_1\frac{\lambda}{n}\left(1 - \mathrm{i}\frac{2}{\pi}\log(\lambda)\right) + R\left(\frac{\lambda}{n}\right)$$

with a remainder term R of order $\mathcal{O}\left(\frac{\lambda^2 \vee |\lambda|}{n^2}\log(\frac{n}{\lambda})\right)$.

Proof As in the previous proofs, the integration by parts formula (9) and the series expansion of Lemma 6 for $N = 1$ yield the following expansion

$$\begin{aligned}
\varphi_{|L_1|^\alpha}\left(\frac{\lambda}{n}\right) - \mathrm{i}\lambda\mathbb{E}\sin(n^{-1}|L_1|^\alpha) &= \int_0^\infty \left(\mathrm{e}^{\mathrm{i}\frac{\lambda}{n}x} - \mathrm{i}\lambda\sin\left(\frac{x}{n}\right)\right)\mathrm{d}\mathbb{P}(|L_1|^\alpha \le x) \\
&= 1 + \int_0^\infty \left(\mathrm{i}\frac{\lambda}{n}\mathrm{e}^{\mathrm{i}\frac{\lambda}{n}x} - \mathrm{i}\frac{\lambda}{n}\cos\left(\frac{x}{n}\right)\right)\mathbb{P}(|L_1|^\alpha > x)\mathrm{d}x \\
&= 1 + \frac{\lambda}{n}\int_0^\infty \left(-\sin\left(\frac{\lambda}{n}x\right)\left(\cos\left(\frac{\lambda}{n}x\right)\right.\right. \\
&\qquad \left.\left. + \ \mathrm{i} - \cos\left(\frac{x}{n}\right)\right)\right) A_1 x^{-1}\mathrm{d}x \\
&\quad + \frac{\lambda}{n}\int_0^\infty \left(-\sin\left(\frac{\lambda}{n}x\right) + \mathrm{i}\left(\cos\left(\frac{\lambda}{n}x\right)\right.\right. \\
&\qquad \left.\left. - \cos\left(\frac{x}{n}\right)\right)\right)\bar{A}_2\Theta(x)x^{-2}\mathrm{d}x\,.
\end{aligned}$$

Let us consider the real part of the first integral. This time set $y = \frac{\lambda}{n}x$ to get

$$\int_0^\infty \frac{\lambda}{n}\sin\left(\frac{\lambda}{n}x\right)A_1x^{-1}\mathrm{d}x = \frac{\lambda}{n}A_1\int_0^\infty \frac{\sin(y)}{y}\mathrm{d}y = \frac{\lambda}{n}A_1\mathrm{si}(0) = \frac{\lambda}{n}\frac{\pi}{2}A_1\,.$$

We identify this number as the real part of the desired constant. We evaluate the imaginary part, to get

$$\begin{aligned}
\int_0^\infty \frac{\lambda}{n}\left(\cos\left(\frac{\lambda}{n}x\right) - \cos\left(\frac{x}{n}\right)\right)A_1x^{-1}\mathrm{d}x &= \frac{\lambda}{n}A_1\int_0^\infty \frac{\cos(\lambda y) - \cos(y)}{y}\mathrm{d}y \\
&= \frac{\lambda}{n}A_1\log(\lambda)\,.
\end{aligned}$$

Now combing the real and the imaginary part we obtain the constant stated in the lemma. It remains to give a bound on R corresponding to the second integral. Again we will consider the integration domains $[0, 1]$ and $]1, \infty[$ separately. For the former we have

$$\begin{aligned}
\left|\int_0^1 \frac{\lambda}{n}\sin\left(\frac{\lambda}{n}x\right)\bar{A}_2\Theta(x)x^{-2}\mathrm{d}x\right| &\le \frac{\lambda}{n}\int_0^1 \sin\left(\frac{\lambda}{n}x\right)\left|\mathbb{P}(|L_1|^\alpha > x) - A_1x^{-1}\right|\mathrm{d}x \\
&\le \frac{\lambda}{n}\left(\int_0^1 \frac{\lambda}{n}x\mathrm{d}x + |A_1|\int_0^1 \frac{\lambda}{n}\mathrm{d}x\right) = \frac{\lambda^2}{n^2}\left(\frac{1}{2} + |A_1|\right)\,,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_0^1 \frac{\lambda}{n}\left(\cos\left(\frac{\lambda}{n}x\right) - \cos\left(\frac{x}{n}\right)\right)\bar{A}_2\Theta(x)x^{-2}\mathrm{d}x\right| \\
&\quad \le \frac{\lambda}{n}\int_0^1 \left|\cos\left(\frac{\lambda}{n}x\right) - \cos\left(\frac{x}{n}\right)\right|\left|\mathbb{P}(|L_1|^\alpha > x) - A_1x^{-1}\right|\mathrm{d}x
\end{aligned}$$

$$\leq \frac{\lambda}{n}\left(\int_0^1 \left|\frac{\lambda}{n}x - \frac{x}{n}\right| \mathrm{d}x + |A_1| \int_0^1 \left|\frac{\lambda}{n}x - \frac{x}{n}\right| x^{-1}\mathrm{d}x\right)$$

$$= \frac{\lambda|\lambda-1|}{n^2}\left(\frac{1}{2} + |A_1|\right) .$$

Here to estimate the integrals involving the cosine we use $|\cos(x) - \cos(y)| \leq |x-y|$ for all $x, y \in \mathbb{R}$. Now let us argue for the latter integral on $]1, \infty[$. Substituting $y = \frac{\lambda}{n}x$, we can bound the real part by

$$\frac{\lambda}{n}\int_1^\infty \left|\sin\left(\frac{\lambda}{n}x\right)\bar{A}_2 x^{-2}\right| \mathrm{d}x = \frac{\lambda^2}{n^2}|\bar{A}_2| \int_{\frac{\lambda}{n}}^\infty |\sin(y)y^{-2}|\mathrm{d}y$$

$$\leq \frac{\lambda^2}{n^2}|\bar{A}_2| \int_{\frac{\lambda}{n}}^\infty \frac{1 \wedge y}{y^2}\mathrm{d}y = \frac{\lambda^2}{n^2}|\bar{A}_2|\left(1 - \log\left(\frac{\lambda}{n}\right)\right) .$$

Similarly we can give a bound for the imaginary part, given by

$$\frac{\lambda}{n}\int_1^\infty \left|\left(\cos\left(\frac{\lambda}{n}x\right) - \cos\left(\frac{x}{n}\right)\right)\bar{A}_2 x^{-2}\right| \mathrm{d}x = \frac{\lambda^2}{n^2}|\bar{A}_2| \int_{\lambda/n}^\infty \left|\cos(y) - \cos\left(\frac{y}{\lambda}\right)\right| y^{-2}\mathrm{d}y$$

$$\leq \frac{\lambda^2}{n^2}|\bar{A}_2|\left(\int_{\lambda/n}^1 \left|1 - \frac{1}{\lambda}\right| y^{-1}\mathrm{d}y + \int_1^\infty 2y^{-2}\mathrm{d}y\right)$$

$$\leq \frac{\lambda}{n^2}|\bar{A}_2|\left(-|\lambda-1|\log\left(\frac{\lambda}{n}\right) + 2\lambda\right) .$$

Summing everything up we finally obtain

$$|R\left(\frac{\lambda}{n}\right)| \leq \left(1 + 2|A_1| + \left(3 - 2\log\left(\frac{\lambda}{n}\right)\right)|\bar{A}_2|\right)\frac{\lambda^2 \vee |\lambda|}{n^2} .$$

□

Lemma 10 *Let* $n \in \mathbb{N}$. *We have for* $0 < \lambda < n$

$$\left|A_1\frac{\lambda}{n}\log(\lambda) + \lambda\mathbb{E}\sin(n^{-1}|L_1|^\alpha)\right| \leq \frac{\lambda}{n}\left(|A_1|\log\left(\frac{n}{\lambda}\right) + |\mathbb{E}\sin(|L_1|^\alpha)| + 2 + \vartheta\right) .$$

We have $\vartheta := \sup_{0<x\leq 1}\Theta(x)/x < \infty$, *where* Θ *is the function that appears in Lemma* 6 *in the second order series expansion of* $\mathbb{P}(|L_1|^\alpha > x)$.

Proof As in the previous proofs we use integration by parts, with the result

$$A_1\frac{\lambda}{n}\log(\lambda) + \lambda\mathbb{E}\sin(n^{-1}|L_1|^\alpha)$$

$$= A_1\frac{\lambda}{n}\int_0^\infty \frac{\cos(\lambda x) - \cos(x)}{x}\mathrm{d}x + \frac{\lambda}{n}\int_0^\infty \cos(n^{-1}|L_1|^\alpha)\mathbb{P}(|L_1|^\alpha > x)\mathrm{d}x$$

$$= \frac{\lambda}{n}\int_0^\infty A_1\frac{\cos(\lambda x) - \cos(x)}{x} + \cos(n^{-1}x)\left\{\frac{A_1}{x} + \bar{A}_2\Theta(x)x^{-2}\right\}\mathrm{d}x$$

$$= \frac{\lambda}{n}\int_0^\infty A_1\frac{\cos(\lambda x) + \cos(n^{-1}x) - 2\cos(x)}{x}\mathrm{d}x$$

$$
\begin{aligned}
&+ \int_0^\infty A_1 \frac{\cos(x)}{x} + \bar{A}_2 \frac{\cos(n^{-1}x)}{x^2} \Theta(x) \mathrm{d}x \\
&= A_1 \frac{\lambda}{n} \log(\lambda/n) + \frac{\lambda}{n} \int_0^\infty A_1 \frac{\cos(x)}{x} + \bar{A}_2 \frac{\cos(x)}{x^2} \Theta(x) \mathrm{d}x \\
&+ \bar{A}_2 \frac{\lambda}{n} \int_0^\infty \frac{\cos(n^{-1}x) - \cos(x)}{x^2} \Theta(x) \mathrm{d}x \\
&= A_1 \frac{\lambda}{n} \log(\lambda/n) + \frac{\lambda}{n} \mathbb{E} \sin(|L_1|^\alpha) \\
&+ \bar{A}_2 \frac{\lambda}{n} \int_0^\infty \frac{\cos(n^{-1}x) - \cos(x)}{x^2} \Theta(x) \mathrm{d}x .
\end{aligned}
$$

The first two terms are easily controlled. We have to focus on the last integral. We will show that it is finite and bounded. For this aim we need to analyze the behavior of $\Theta(x)$ near zero. For big x we can simply estimate

$$
\left| \int_1^\infty \frac{\cos(n^{-1}x) - \cos(x)}{x^2} \Theta(x) \mathrm{d}x \right| \le \int_1^\infty \frac{2}{x^2} \mathrm{d}x = 2 \,.
$$

And for small ones we get

$$
\begin{aligned}
\left| \int_0^1 \frac{\cos(n^{-1}x) - \cos(x)}{x^2} \Theta(x) \mathrm{d}x \right| &= \left| \int_0^1 \frac{\cos(n^{-1}x) - \cos(x)}{x} \cdot \frac{\Theta(x)}{x} \mathrm{d}x \right| \\
&\le \vartheta \int_0^1 \frac{x|n^{-1} - 1|}{x} \mathrm{d}x = \vartheta \,,
\end{aligned}
$$

where we assume that $|\Theta(x)/x|$ is bounded by a constant $\vartheta > 0$.

This will finally be justified. Because $\mathbb{P}(|L_1|^\alpha > 0) = 1$ and by the continuity of the distribution function of L_1 we have

$$
1 = \lim_{x \to 0} \mathbb{P}(|L_1|^\alpha > 0) = \lim_{x \to 0} \left\{ A_1 x^{-1} + \bar{A}_2 \Theta(x) x^{-2} \right\} \,.
$$

We multiply both sides by x and take the limit as x tends to zero to see

$$
0 = \lim_{x \to 0} \left\{ A_1 + \bar{A}_2 \Theta(x) x^{-1} \right\} \,.
$$

And hence $\lim_{x \to 0} \Theta(x)/x = -A_1/\bar{A}_2$. As Θ is continuous and bounded by one, $\Theta(x)/x$ is also continuous and therefore is bounded on the interval [0, 1] by a constant we call ϑ. □

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