LaGO - A solver for mixed integer nonlinear programming

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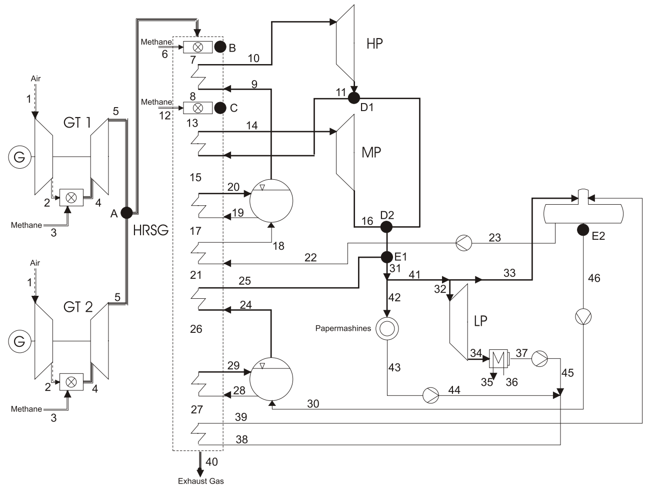
Problem formulation

MINLP:

$$\begin{array}{ll} \min & f(x,y) \\ \text{s.t.} & g(x,y) \leq 0 \\ & h(x,y) = 0 \\ & x \in [\underline{x}, \overline{x}] \\ & y \in [\underline{y}, \overline{y}] \quad \text{integer} \end{array}$$



MINLP:

- $n << 5.000$
- large problems are structured
- many possible applications
- in contrast to MIP, not used very much in practice

production planning, man power planing, scheduling, blending, refinery optimization, process design, engineering design, investment/de-investment, network design, financial optimization

Algorithmic overview of LaGO

- First branch-cut-and-price system for MINLP

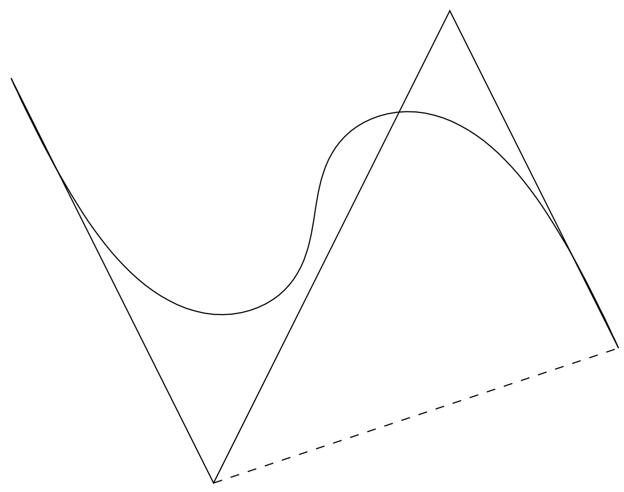
- Deformation, rounding and Lagrange heuristics

- Block-separable reformulation:
  $\min\{c^T x \mid Ax + b \leq 0,\, x_{J_k} \in G_k,\, k = 1, \ldots, p\}$

- Convex and polyhedral relaxations:
  replace G_k by a convex set or polyhedron $\hat{G}_k \supseteq G_k$
  Optimal relaxations by solving dual problems
  $|J_k|$ influences the quality and computational cost of a relaxation/lower bound

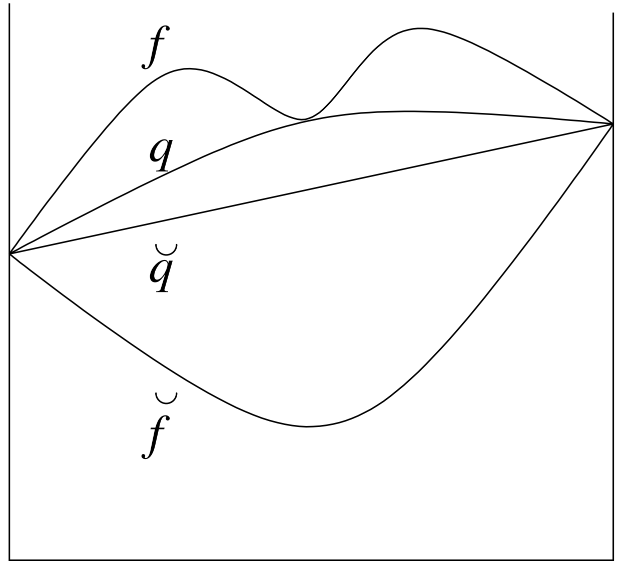
Convex relaxations using underestimators

Bézier-underestimators [No96]:



- Bézier representation:
  $p(x) = \sum_{i=0}^{l} a_i x^i = \sum_{i=0}^{l} b_i \cdot B_i(x)$
- Convex hull property:
  $p(x) \in \mathrm{conv}\{b_i\}$

α-underestimators (sampling-technique) [NoAlVi03]



- Underestimation by a nonconvex quadratic form q
- $\breve{q}(x) = q(x) + \alpha(x - \underline{x})^T(\overline{x} - x)$
- $\breve{q}$ often better than $\breve{f}$

Both methods produce consistent bounds.

Optimal convex relaxation of block-separable quadratic problems
[No03]

- Reformulation (Q) of the original problem (P) by elimination of linear terms

- Formulation of dual problem to (Q) as an eigenvalue optimization problem: $\max_\mu \min_x L(x; \mu) = \max_\mu \sum_k \lambda_1(A_k(\mu)) + c(\mu)$

- Solution of the eigenvalue optimization problem using a subgradient or bundle method

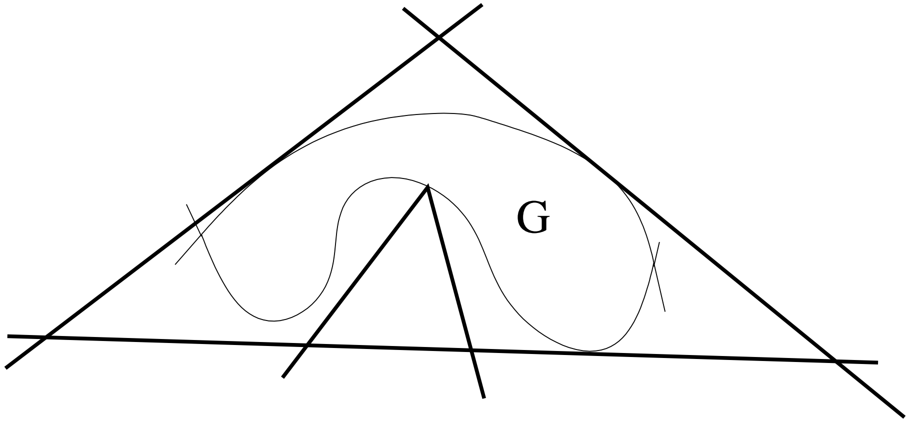
- Proof of the dual equivalence of (P) and (Q)

Polyhedral relaxation I

- Linearization (if g_i is convex):
  $g_i(\hat{x}) + \nabla g_i(\hat{x})^T (x - \hat{x}) \leq 0$

- Knapsack-cut (by solving a separation problem):
  $b^T x \geq \underline{b} = \min_{x \in Z} b^T x,$
  $Z = G$ or
  $Z = \{x \in X \mid g_i(x) \leq 0\}$

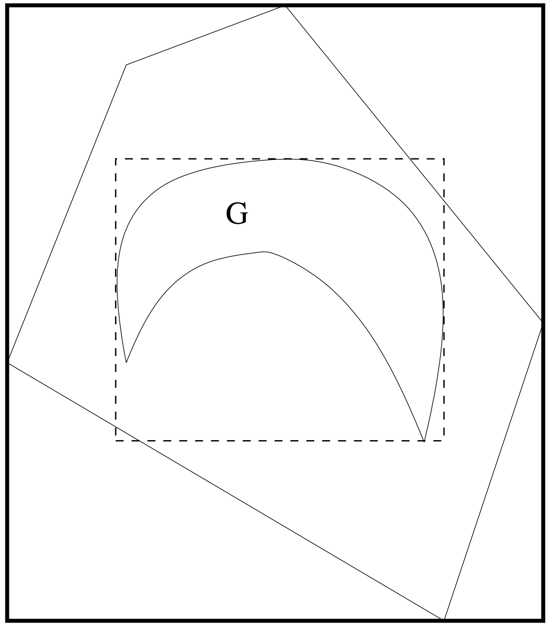
- Interval gradient cut (new):
  $g_i(\hat{x}) + \min_{d \in [\underline{d}, \overline{d}]} d^T (x - \hat{x}) \leq 0$
  where $\nabla g_i(x) \in [\underline{d}, \overline{d}]$ für alle $x \in [\underline{x}, \overline{x}]$



Nonconvex polyhedral outer approximation $\hat{G} \supset G$

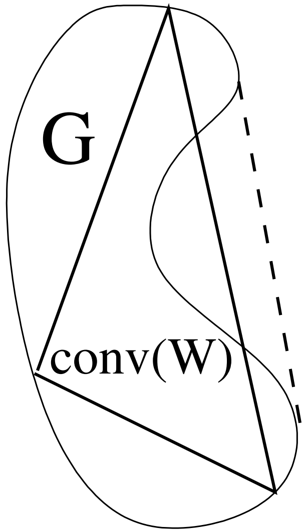
Polyhedral relaxation II

- Level-cut: $c^T x \leq \overline{v}$, where $\overline{v} \geq \mathrm{val}(\mathrm{MINLP})$
- Box-reduction (constraint propagation) improves the quality of cuts



A polyhedral relaxation can also be used for sensitivity analysis and multicriterial optimization

Optimal polyhedral relaxations using column-generation



Find inner approximation points
$W = \{w_1, w_2, \dots\} \subset \mathrm{conv}(G)$ such that

$$\min\{c^T x \mid Ax + b \le 0,\ x \in \mathrm{conv}(W)\} =$$
$$\min\{c^T x \mid Ax + b \le 0,\ x \in \mathrm{conv}(G)\}$$

Algorithm:

1. Solve the restricted master problem (RMP):
   $\hat{\mu} = \mathrm{argmax}_{\mu} \min_x \{c^T x + \mu^T (Ax + b) \mid x \in \mathrm{conv}(W)\}$

2. Solve the Lagrange problem: $w = \mathrm{argmin}\{(c^T + \hat{\mu}^T A)x \mid x \in G\}$,
   set $W = W \cup \{w\}$ and add new columns to RMP.

Equivalent to dual cutting plane method and therefore convergent

Remarks

- Lagrange problem decomposes into sub-problems, which can be solved by an arbitrary global solver (branch+cut, populations heuristic).

- Each solution of a sub-problem generates a Lagrangian cut, which is added to the outer approximation

- By comparing the outer and inner approximation it can be determined how good the approximation of a sub-problem is.

- In contrast to bundle methods, relaxations of stochastic programs and optimal control problems can be updated efficiently after refining szenarios or grids respectively.

Heuristics

1. Deformation heuristic by successively solvin:

$$\min\{(1-t_k)\breve{f}(x)+t_kf(x) \mid (1-t_k)\breve{g}(x)+t_kg(x)\leq 0\}$$

2. Rounding heuristic by rounding some components of the solution of an inner or outer approximation and backtracking

3. Lagrange heuristic by combining several inner approximation points $x_{J_k} \in W_k$, where $c^Tx+\delta\|Ax+b\|_+$ is small, and projection onto $\{x \mid Ax+b\leq 0, x_B\,\mathrm{binary}\}$

Branch-Cut-and-Price

Lower bounds:

(B1) dual bounds: column generation (BCP) or eigenvalue optimization

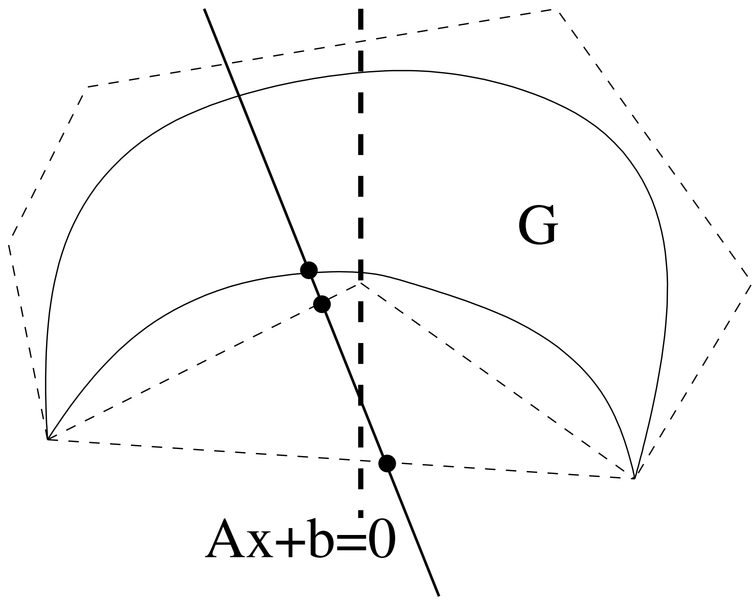
(B2) LP-bounds (with Knapsack and linearization cuts)

(B3) by a convex relaxation

All bounds are consistent and
we have $(B1) \geq (B2) \geq (B3)$

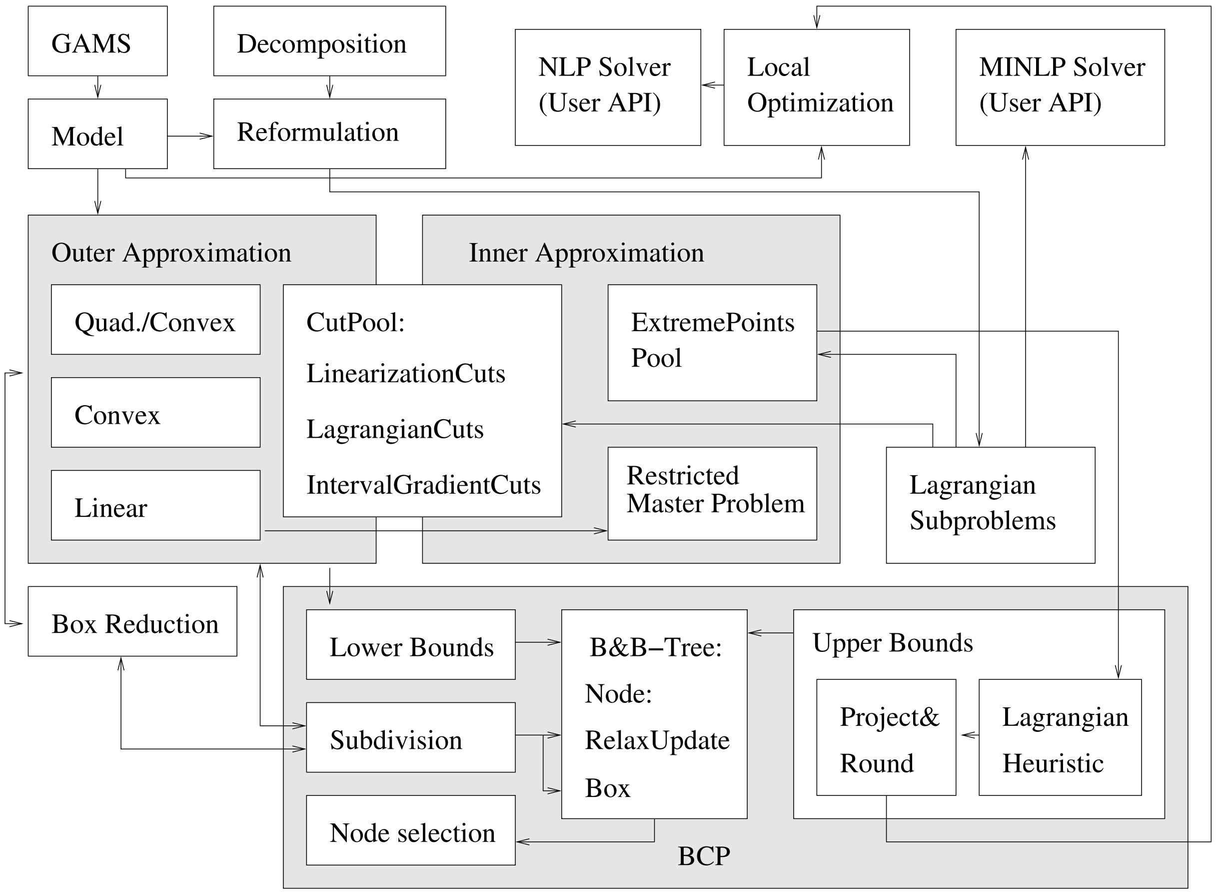
Upper bounds by heuristics

Several branching strategies



C++ Library

Together with Stefan Vigerske, more than 40000 lines of code
Object oriented design similar to COIN/BCP and ABACUS
Interfaces: GAMS, AMPL, COIN, SNOPT, CPLEX, ARPACK, NOA, FILIB, TNT, METIS



Numerical experiments

- MaxCut experiments, $n \leq 1000$: performance similar to specialed solvers
- GAMS-MinlpLIB experiments, $n \leq 500$, Comparison with BARON, November 2004, BC-Algorithmus of LaGO using LP-bounds and BARON with default parameters:

	total number	obj LaGO better	obj equal	obj BARON better
LaGO much faster :	3	3	-	-
LaGO faster :	1	-	1	-
Both solvers same performance :	5	1	3	1
BARON faster :	9	-	8	1
BARON much faster :	28	5	17	6
Only BARON solution :	5	-	-	5
Both solvers no solution :	1	-	1	-
Number of problems :	52	9	30	13

DFG-Project

Design of Complex Energy Conversion Systems,
Technical University of Berlin (Institute for Energy Engineering) and
Humboldt-University Berlin (Department of Mathematics)

[Ahadi-Oskui, Alperin, Cziesla, Nowak, Tsatsaronis, 2001-2004]

MINLP, $n = 1300$: BCP and specialized heuristic

Found acceptable solution in reasonable time

BARON and SBB were not able to find a solution

Final Remarks

- `http://www.mathematik.hu-berlin.de/~eopt/LaGO/documentation/` .

- Book: Relaxation and Decomposition Methods for Mixed-Integer Nonlinear Optimization
  Birkhäuser Verlag, to appear

- Possible improvements:
  - Reduction of duality gap by nonconconvex polyhedral inner approximation (MIP master problem)
    Li, D., Sun, X. L., Wang, J., and McKinnon, K. (2002). A convergent lagrangian and domain cut method for nonlinear knapsack problems. Technical report, SEEM2002-10, Department of Systems Engineering & Engineering Management, The Chinese University of Hong Kong.
  - consisten bounds and branching

- New DFG project planned

- Integration into GAMS planned