On a Conjecture of Phadke and Thakare

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ABSTRACT

We prove the connectedness of the set of all nonzero bounded linear operators on a complex Hilbert space having a generalized inverse.

In a recent paper [3] S. V. Phadke and N. K. Thakare conjectured that in a complex Hilbert space $H$ the set of operators having a generalized inverse is not connected. The purpose of this note is to disprove this conjecture. We recall that a bounded linear operator $A \neq 0$ on $H$ is said to have a generalized inverse if there is a bounded linear operator $B$ on $H$ such that

\[ ABA = A. \]  

(1)

As usual we write $|A| := (A^*A)^{1/2}$ and denote by $s(|A|)$ the support of $|A|$. Then (1) is easily seen to be equivalent to the following condition: there is $C > 0$ such that

\[ A^*A \geq C|A|. \]  

(2)

The set of all operators with generalized inverse will be denoted by $GI(H)$. 

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THEOREM. GI(H) is pathwise connected.

Proof. Let A \neq 0 be a bounded linear operator on H with generalized inverse, and let U|A| = A be the polar decomposition of A. Then
\[ t \mapsto U((1 - \xi)|A| + ts(|A|)), \quad t \in [0, 1], \]
is a path in GI(H) in view of (2), connecting A and U. The operators
\[ P := UU^* \text{ and } Q := U^*U \]
amre orthogonal projections on H, and we may assume that \( \dim(1_H - P)(H) \leq \dim(1_H - Q)(H) \). Now if P is finite, then these dimensions are equal. Consequently, there exists a partial isometry V on H with \( VV^* = 1_H - P \), \( V^*V = 1_H - Q \). But then \( U + V \) is unitary and can be connected with U through a path in GI(H), namely
\[ t \mapsto U + tV, \quad t \in [0, 1]. \]

Next we assume that P is infinite. Then we can find a partial isometry V on H with \( VV^* = 1_H - P \) and \( V^*V < 1_H - Q \). As before, U can be connected with \( U + V \) in GI(H), so we may assume \( P = 1_H \) from now on. We pick projections \( P_1, P_2 \) on H with \( P_1 P_2 = 0 \), \( P_1 + P_2 = 1_H \), and \( \dim P_i(H) = \dim P_i(H) = \dim H \), \( i = 1, 2 \). Then also \( \dim(1_H - Q_i)(H) = \dim H \), implying that there is a partial isometry W on H with \( WW^* = P_2 \) and \( W^*W = 1_H - Q_1 \). We now define
\[ U(t) := \begin{cases} UQ_1 + (1 - t)UQ_2, & t \in [0, 1], \\ UQ_1 + (t - 1)W, & t \in [1, 2]. \end{cases} \]

Then \( U(0) = U \), and \( U(2) \) is again unitary. Moreover, using (2), it follows that \( U(t) \in GI(H) \) for \( t \in [0, 1] \). Since the set of all invertible bounded linear operators on H is connected [2, p. 70], U can be connected with \( 1_H \) and the theorem is proved. \( \blacksquare \)

We remark that (1) makes sense in an arbitrary \( W^* \)-algebra. The above statement holds also in this more general case; the details of the proof can be found in [1].

REFERENCES


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