ON THE COMPACTNESS OF ISO-SPECTRAL POTENTIALS

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1. Introduction

Let $M$ be a compact Riemannian manifold and $\Delta$ its Laplace-Beltrami operator. For a potential $q \in C^0(M)$ the Schrödinger operator $H_q := -\Delta + q$ is essentially self-adjoint in $L^2(M)$ and its spectrum consists of a sequence $\lambda_1 < \ldots < \lambda_j \to \infty$ of eigenvalues with finite multiplicity. Thus we may call $(\lambda_j)_{j \in \mathbb{N}}$ the spectrum of $q$. It is natural to ask to what extent the spectrum determines $q$. This question has attracted much interest in recent years but it does not seem to have a simple
answer. First of all, if $\psi$ is an isometry of $M$ then $q$ and $q \circ \psi$ are isospectral. If the set $\text{Is}(q)$ of potentials isospectral with $q$ contains only functions of this form $q$ is called spectrally rigid. Then it is known that certain potentials on certain manifolds are spectrally rigid (cf. [2]) whereas the periodic solutions of the Korteweg - de Vries equation provide nontrivial isospectral deformations for potentials on $S^1$ (cf. [3]). It is therefore interesting to investigate general structural properties of the set $\text{Is}(q)$ for example compactness in various function spaces. If $M = S^1$ it is well known that $\text{Is}(q)$ is compact in $C^\infty(S^1)$. In general dimensions, Gilkey has proved that compactness in $C^\infty(M)$ reduces to boundedness in certain Sobolev spaces ([1] Theorem 2.4). In this note we improve Gilkey's result which enables us to show that $\text{Is}(q)$ is compact in $C^\infty(M)$ if $\dim M \leq 3$. The proof is based on the asymptotic expansion of the trace of the heat kernel of $H_q$:

$$\text{tr} \; e^{-sH_q} \sim (4\pi s)^{-n/2} \sum_{j \geq 0} s^j a_j(q), \quad n := \dim M.$$  

With respect to the Riemannian measure on $M$ the $a_j(q)$ are integrals of certain functions $u_j(q)$ over $M$ which
are universal polynomials in the covariant derivatives of \( q \) and of the curvature tensor of \( M \). It can be shown \((1)\) that for \( j \geq 2 \) \( a_j(q) \) equals the square of the Sobolev norm of \( q \) of order \( j - 2 \) plus integrals of products involving only lower order derivatives. This suggests the possibility of estimating the Sobolev norms of \( q \) in terms of the \( a_j(q) \) hence in terms of the spectrum. In fact, using the Sobolev and Gagliardo-Nirenberg inequalities we show that \( Is(q) \) is bounded in every Sobolev space if it is bounded in the Sobolev space of order \( 3n_o - 2 \) where

\[
n_o := \inf \{ m \in \mathbb{N} \mid m \geq n/2 \} \tag{1}
\]

To verify the latter condition, however, and thus to prove compactness we have to introduce a restriction on the dimension.

I wish to thank Victor Guillemin and Marty Schwarz for several stimulating discussions on this subject.

2. Heat invariants and Sobolev norms

We will use Gilkey's result in the following form.

**Theorem 1** a) There are functions \( w_1, w_2 \in C^\infty(M) \) depending only on the curvature of \( M \) such that
\[ a_0(q) = \text{vol } M, \quad a_1(q) = \int_M (q + w_1), \]
\[ a_2(q) = \frac{1}{2} \int_M (q^2 + w_2). \]

b) For \( j \geq 3 \) we have
\[ a_j(q) = \frac{(-1)^j(j-1)!}{(2j-1)!} \int_M |D^{j-2} q|^2 + \]
\[ + \sum_{k=1}^j \sum_{\alpha \in \mathbb{Z}^k_+} \int_M \prod_{\alpha_i} p^{\alpha_i}_k(q) \]
\[ \text{ord } p^{\alpha}_k \leq j - 3, \quad \sum_{i=1}^k \text{ord } p^{\alpha}_i \leq 2(j-3). \]

where \( D \) denotes covariant derivative and \( (p^{\alpha}_k)_{1 \leq i \leq k} \) is a family of differential operators with \( C^\infty \) coefficients depending only on the metric of \( M \). Moreover, the orders satisfy the inequalities

Proof a) This is proved in [1] Theorem 4.3.

b) This is contained in the proof of [1] Theorem 2.4. ■

For \( 1 \leq p \leq \infty \) and \( s \in \mathbb{R} \) we denote by \( W_{s,p}(\mathbb{R}^n) \) the Sobolev space of distributions \( u \in S'(\mathbb{R}^n) \) such that \( \hat{u} \) is a function with \( k_s u \in L^p(\mathbb{R}^n) \) where \( k_s(\xi) = (1 + |\xi|^2)^{s/2} \); \( W_{s,p}(\mathbb{R}^n) \) is a Banach space with the norm
\[ \|u\|_{W_{s,p}(\mathbb{R}^n)} := \|k_S u\|_{L^p(\mathbb{R}^n)}. \]

The space \( W_{s,p}(M) \) consists of all distributions \( u \) on \( M \) such that \((fu) \circ \psi^{-1} \in W_{s,p}(\mathbb{R}^n)\) for every coordinate system \((U, \psi)\) and every \( f \in C^\infty_0(U)\). Choosing a finite atlas \((U_i, \psi_i)_{1 \leq i \leq m}\) and a subordinate partition of unity \((f_i)_{1 \leq i \leq m}\) we define a norm on \( W_{s,p}(M) \) by

\[ \|u\|_{W_{s,p}(M)} := \sum_{i=1}^m \| (f_i u) \circ \psi_i^{-1} \|_{W_{s,p}(\mathbb{R}^n)}. \]  

(2)

This norm depends on the choices made but any two such norms are equivalent, and \( W_{s,p}(M) \) is a Banach space with the norm (2). With \( \eta_0 \) defined in (1) we have the following consequences of the Gagliardo-Nirenberg inequalities (see [4] pp. 124).

**Lemma 1** For \( 1 \leq p < \infty \) \( W_{\eta_0,2}(M) \) imbeds continuously into \( L^p(M) \). i.e.

\[ \|u\|_{L^p(M)} \leq C \|u\|_{W_{\eta_0,2}(M)}, \quad u \in W_{\eta_0,2}(M), \]  

(3)

where \( C \) depends on \( p \) and \( M \). If \( \eta_0 > n/2 \) then (3) also holds for \( p = \infty \).

**Lemma 2** Let \( n > 2, 0 \leq \alpha \leq 1 \), and put
\[
\frac{1}{p} := \alpha \left( \frac{1}{2} - \frac{1}{n} \right) + \frac{1-\alpha}{2}.
\]

Then \( W_{1,2}(M) \subset L^3(M) \) and
\[
\| u \|_{L^p(M)} \leq C \| u \|_{W_{1,2}(M)}^\alpha \| u \|_{L^2(M)}^{1-\alpha}, \quad u \in W_{1,2}(M),
\]
where \( C \) depends on \( M \) and \( \alpha \). If \( n = 2 \) then the result is true for \( 0 \leq \alpha < 1 \).

We are now able to reduce the compactness of \( Is(q) \) in \( C^\infty(M) \) to boundedness in \( W_{3n_0-2,2}(M) \). Gilkey ([1] Theorem 2.4) gives a reduction to boundedness in \( W_{n'}(M) \) where \( n' := n \) if \( n \) is even, \( n' := n-1 \) if \( n \) is odd. To get this in terms of \( W_{s,2}(M) \) norms one needs at least boundedness in \( W_{3n_0,2}(M) \) which is a stronger condition.

**Theorem 2** If \( Is(q) \) is bounded in \( W_{3n_0-2,2}(M) \) then it is compact in \( C^\infty(M) \).

**Proof** It is sufficient to show that \( Is(q) \) is bounded in \( W_j(M) \) for every \( j \geq 3n_0 \). This is true for \( j = 3n_0 \) by assumption so assume that \( Is(q) \) is bounded in \( W_j(M) \) for some \( j > 3n_0 \). For \( \tilde{q} \in Is(q) \) we obtain from Theorem 1 the estimate
\[
\| \tilde{q} \|_{W_j(M)}^2 \leq C(1 + \sum_{1 \leq k \leq j} \prod_{\alpha \in \mathbb{Z}_+^k} \int_M | p_k^{\alpha}(\tilde{q}) | \prod_{\alpha \in \mathbb{Z}_+^k} p_k^{\alpha}(\tilde{q}) |)
\]
where $C$ depends only on $M$ and the spectrum of $q$ and

$$\sum_{i=1}^{k} \text{ord } p_i^k \leq j - 3, \quad \sum_{i=1}^{k} \text{ord } p_i^k \leq 2(j - 3). \quad (4)$$

We estimate the terms in the sum.

1st case We have $\text{ord } p_i^k \leq j - 3 - n_0$ for all $i$. Then we deduce from Lemma 1 for $1 \leq p < \infty$, $1 \leq i \leq k$

$$\|p_i^k(q)\|_{L^p(M)} \leq C_p \|p_i^k(q)\|_{W_{n_0, 2}(M)}$$

$$\leq C_p \|q\|_{W_{j-3, 2}(M)},$$

hence the generalized Hölder inequality shows that the term considered is uniformly bounded on $Is(q)$.

2nd case We have $\text{ord } p_i^k \geq j - 2 - n_0$ for some $i$. But this can happen at most for two different values of $i$ since otherwise by (4)

$$3(j - 2 - n_0) \leq \sum_{i=1}^{k} \text{ord } p_i^k \leq 2(j - 3)$$

and $j \leq 3n_0$, a contradiction. If there is only one such value, say $i = 1$, then we apply the generalized
Hölder inequality with \( p_1 = 2 \), and since \( \text{ord} \ p_{\alpha_1}^k \leq j-3 \) the term is bounded as before.

Now assume \( \text{ord} \ p_{\alpha_1}^k \geq j-2-n_0 \) for \( i = 1, 2 \). We have to estimate

\[
\int_M |p_{\alpha_1}^k(\tilde{q}) p_{\alpha_2}^k(q) F(q)|
\]

(5)

where by the argument of case 1 any \( L^p \) norm of \( F(q) \), 
\( 1 \leq p < \infty \), is uniformly bounded on \( I_{s}(q) \). Now choose \( \varepsilon > 0 \) such that

\[
\frac{1}{2} - \frac{1}{n} < \frac{1}{2+\varepsilon} < \frac{1}{2}
\]

and \( 0 < \alpha < 1 \) such that

\[
\frac{1}{2+\varepsilon} = \alpha \left( \frac{1}{2} - \frac{1}{n} \right) + \frac{1-\alpha}{2}.
\]

We apply Lemma 2 and recall that \( \text{ord} \ p_{\alpha_1}^k \leq j-3 \) to obtain the estimate

\[
\|p_{\alpha_1}^k(\tilde{q})\|_{L^{2+\varepsilon}(M)} \leq C \|p_{\alpha_1}^k(q)\|_{W^{1,2}(M)} \|p_{\alpha_1}^k(q)\|^{1-\alpha}_{L^2(M)}
\]

\[
\leq C \|\tilde{q}\|_{W^{j-2,2}(M)} \|\tilde{q}\|^{1-\alpha}_{W^{j-3,2}(M)}
\]
hence by assumption
\[ \| p^k_{\alpha_1} (\tilde{q}) \|_{L^{2+\varepsilon}(M)} \leq C \| \tilde{q} \|_{W^{j-2,2}(M)}^{\alpha} \]  \tag{6}

with \( C \) depending only on \( M \) and the spectrum of \( q \).

Applying the generalized Hölder inequality with 
\( p_1 := 2+\varepsilon, \ p_2 := 2, \) and 
\( p_3 := (\frac{1}{2} + \frac{1}{2+\varepsilon})^{-1} \) we see that the integral (5) can be estimated by the right hand side of (6), possibly with a different constant.

Summing up we arrive at the inequality
\[ \| \tilde{q} \|_{W^{j-2,2}(M)}^{2} \leq C (1 + \| \tilde{q} \|_{W^{j-2,2}(M)}^{\alpha}) \]

for \( \tilde{q} \in Is(q) \) where \( 0 < \alpha < 1 \) and \( C \) depends only on \( M \) and the spectrum of \( q \). But this implies that \( Is(q) \) is bounded in \( W^{j-2,2}(M) \). \( \Box \)

3. Compactness of \( Is(q) \)

To prove compactness of \( Is(q) \) in \( C^0(\overline{M}) \) we have to verify the condition of Theorem 2. The difficulty of doing this increases with the dimension of \( M \). Our argument breaks down in dimensions greater than 3 since then we do not control the \( L^p \) norms on \( Is(q) \) for \( p > 4 \).
Theorem 3 Is(q) is compact in $C^\infty(M)$ if $\dim M \leq 3$.

Proof Suppose first that $\dim M \leq 2$ such that $n_0 = 1$. By Theorem 2 we have to prove the boundedness of $Is(q)$ in $W_{1,2}(M)$. It is clear from Theorem 1, a) that $Is(q)$ is bounded in $L^2(M)$. From Lemma 2 we find $0 < \alpha \leq 1/3$ such that

$$
\|\tilde{q}\|_{L^3(M)} \leq C \|\tilde{q}\|_{W_{1,2}(M)}^\alpha \|\tilde{q}\|_{L^2(M)}^{1-\alpha}
$$

(7)

and from Theorem 1, b) with $j = 3$ we derive

$$
\|\tilde{q}\|_{W_{1,2}(M)}^2 \leq C (1 + \|\tilde{q}\|_{L^3(M)}^3)
$$

(8)

where $C$ depends only on $M$ and the spectrum of $q$. Combining (7) and (8) we see that $Is(q)$ is bounded in $W_{1,2}(M)$.

Now let $\dim M = 3$ implying $n_0 = 2$. Thus we have to prove boundedness in $W_{4,2}(M)$. Arguing as before (with $\alpha = 1/2$ in (7)) we obtain the boundedness of $Is(q)$ in $W_{1,2}(M)$. Using the full range of $p$ in Lemma 2 with $n = 3$ we have for $2 \leq p \leq 6$ with $\alpha_p := 3/2 - 3/p$

$$
\|\tilde{q}\|_{L^p(M)} \leq C_{M,p} \|\tilde{q}\|_{W_{1,2}(M)}^\alpha \|\tilde{q}\|_{L^2(M)}^{1-\alpha}
$$

(9)

$\tilde{q} \in C^\infty(M)$,
saying that $I_s(q)$ is bounded in $L^p(M)$ for $2 \leq p \leq 6$, too. Consider now a term in $a_4(q)$ (Theorem 1,b)) which can be bounded by an integral

$$J := \int_M |p_1(q) p_2(q) \tilde{q}^k|, \quad \text{ord } p_1 = 1, i = 1, 2, \quad k \leq 2.$$  

Applying the generalized Hölder inequality with $P_1 = P_2 = P_3 = 3$ and using the inequalities (7) (with $\tilde{q}$ replaced by $p(q)$ and $\alpha = 1/2$) and (9) it follows that $J$ is bounded on $I_s(q)$. The remaining terms can be bounded by integrals

$$\int_M |p(q) \tilde{q}^k|, \quad \text{ord } P = 1, k \leq 3,$$

or

$$\int_M |q^k|, \quad k \leq 4,$$

with obvious bounds in view of (9). Thus $I_s(q)$ is bounded in $W_{2,2}(M)$. Moreover, since $2 > 3/2 \; ||\tilde{q}||_{L^\infty(M)}$ is bounded on $I_s(q)$, and by (9) the same is true for $||p(q)||_{L^p(M)}$ if $2 \leq p \leq 6$ and $P$ is any differential operator of order $\leq 1$ with smooth coefficients.

Examining the terms in $a_5(q)$ it is then easy to show
that $I_s(q)$ is bounded in $W_{3,2}(M)$. Observing that this implies the boundedness of $\|P(\tilde q)\|_{L^\infty(M)}$ and $\|P(\tilde q)\|_{L^p(M)}$, $2 \leq p \leq 6$, for differential operators of order 1 and 2, respectively, on $I_s(q)$, a similar study of $a_6(\tilde q)$ leads to the conclusion that $I_s(q)$ is bounded in $W_{4,2}(M)$. The theorem is proved. ☐

REFERENCES


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