1. Introduction. In 1912 H. Weyl [24] determined the asymptotic behavior of the eigenvalues of the Laplacian for a compact domain in $\mathbb{R}^n$. Almost forty years later Minakshisundaram and Pleijel generalized this to a full asymptotic expansion for the corresponding Dirichlet series for a compact Riemannian manifold [20], giving rise to an extremely fruitful new development based on the identification of the coefficients in the expansion (see e.g. [16], [18], [4], [12], [2]) in singular spaces. In this paper we extend the Minakshisundaram–Pleijel expansion to the equivariant case. Even in the simplest nontrivial cases the structure of the coefficients becomes very complicated so we concentrate here on the existence of an expansion and the functions involved.

To describe the results let $\mathcal{M}$ be a compact $n$-dimensional Riemannian manifold, $G$ a compact group of isometries, and $\rho$ a finite dimensional irreducible representation of $G$ in a complex vector space $\mathcal{V}$. Denote by $E_\lambda$ the complexified eigenspace of the negative Laplacian $-\Delta$ with eigenvalue $\lambda, \lambda > 0$. Then $E_\lambda$ is $G$-invariant and we can consider the function

$$L_\rho(t) := \sum_{\lambda > 0} e^{-\lambda t} \dim \text{Hom}_G(\mathcal{V}, E_\lambda), \quad t > 0.$$  

Note that $\dim \text{Hom}_G(\mathcal{V}, E_\lambda)$ is the multiplicity of $\rho$ in $E_\lambda$ which is equal to $\dim E_\lambda^G$ in case $\rho$ is the trivial representation. Our main result (Theorem 4 below) states that $L_\rho$ has an asymptotic expansion as $t \to 0$ of the form

$$L_\rho(t) \sim (4\pi)^{-n/2} \sum_{0 < \lambda \leq K_\rho} a_\lambda t^{1/2}(\log t)^{n}$$

where $m := \dim M/\mathcal{O}$ and $K_\rho$ is bounded by the number of different dimensions of $G$-orbits in $\mathcal{M}$. Recall that the union of principal orbits, $M_0$, is open and dense in $\mathcal{M}$ and that $M_0$ is a manifold whose dimension is $\dim M/\mathcal{O}$ by definition. If $\mathcal{G}$ is trivial Theorem 4 gives the classical result of Minakshisundaram and Pleijel mentioned above. If $\mathcal{G}$ has no singular orbits (and hence $K_0 = 1$) it is contained

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in Donnelly [11]. The first order asymptotics have been obtained in [9] and [11] implying that $a_m = \text{vol}^M / \text{dim} \nu^M$. Here $\Gamma$ is a principal isometry group, and $a_m = 0$, $k > 0$. Our result has been announced in [7]. It generalizes to transversally elliptic pseudodifferential operators acting on bundles as will be shown in a subsequent paper.

We now sketch the main ideas leading to the proof of Theorem 4. For $t > 0$, the heat operator $e^{tA}$ has a smooth kernel $\Gamma_t$, and it is well known that
\[ \sum_{k \geq 0} e^{-k\dim E_k} \chi_k - \text{trace } e^{tA} = \int_\mathcal{M} \Gamma_t(p, p) \, dp. \] (1)

The fundamental result of Minakshisundaram and Pleijel states that in a neighborhood of the diagonal $\Gamma_t$ has an asymptotic expansion
\[ \Gamma_t(p, q) = -(4\pi t)^{-\frac{\dim G}{2}} e^{-\frac{1}{4} t^{-\frac{\dim G}{4}}} \sum_{\nu > 0} \frac{\nu}{\nu \cdot e^{i\nu \cdot \frac{\dim G}{4}}} \nu \cdot e^{i\nu \cdot \frac{\dim G}{4}} \sum_{\rho \geq 0} \frac{\rho}{\rho \cdot e^{i\rho \cdot \frac{\dim G}{4}}} \rho \cdot e^{i\rho \cdot \frac{\dim G}{4}}, \quad t \to 0. \] (2)

This yields the asymptotic expansion of (1) immediately. In the equivariant case (1) generalizes to
\[ L_\rho(t) = \int_{G \times N} \chi_{\rho} \Gamma_t(p, g \cdot \xi) \chi_{\rho}(g) \, dg \, dp, \]
where $\chi_{\rho}$ denotes the character of $\rho$. Using the expansion (2) it is therefore enough to determine the asymptotic behavior of the integrals
\[ \int_{G \times N} e^{-\nu \cdot f((p, g) \cdot \xi)} \, dg \, dp \] (3)
with $f((p, g) \cdot \xi) = f(p, g) \cdot \xi$ and $f((p, g) \cdot \xi) = f(p, g) \cdot \xi$. A very instructive special case arises if we consider an isometric action of the $k$-torus $T^k$ on Euclidean space $R^{2n}$ (without trivial part). Splitting the representation of $T^k$ into irreducibles the integral (3) becomes
\[ \int_{T^k \times R^{2n}} e^{-i\sum_{\alpha \in \mathcal{A}} \alpha \cdot w_\alpha((p, \xi))} f((p, \xi)) \, d\theta, \quad f \in \mathcal{C}_0^\infty(T^k \times R^{2n}), \] (4)
where each $w_\alpha$ is a character of $T^k$. It is clear from the Weyl integration formula that the treatment of these integrals is decisive in understanding the general case. But it turns out that it is already a very difficult problem to determine the explicit asymptotic expansion of (4) in general. Leaving this open we attack the problem using the following more qualitative argument.

Obviously (3) depends only on the behavior of $\nu$ and $f$ in an arbitrary neighborhood of the set
\[ \mathcal{A} := \varphi^{-1}(0) = \{(p, g) \in G \times M | g \cdot \xi = \xi \}. \]
If $\mathcal{A}$ is a submanifold it is necessarily a nondegenerate critical submanifold of $\varphi$. Thus $\varphi$ can be replaced locally by $x_\chi^1$ in appropriate coordinates $(x_1, \ldots, x_n)$

and the asymptotic expansion follows readily. Unfortunately this happens only if the $C^*$ action has no singular orbits. More generally, if $\varphi$ is real analytic (locally in appropriate coordinates) we can apply Hironaka's theorem on the resolution of singularities (cf. Atiyah [1] and Bernstein–Gelfand [5]) to replace $\varphi$ by a monomial. Thus we only have to deal with integrals of the type
\[ \int_{R^k} e^{-r \cdot \varphi_0(x)} \, dx, \quad \varphi_0 \in N, \quad k \in \mathcal{C}_0^\infty(R^k), \quad \text{where} \]
\[ R^k := \{ x \in R^k | x_i > 0, 1 \leq i \leq k \}. \] (5)

These have asymptotic expansions of the form
\[ \sum_{0 \leq k \leq 1} a_k(\varphi_0^k)(\log t)^k \]
(see e.g. i.c.m. (a_1, \ldots, a_n) and the $a_k$ are distributions with support in the set \{ $x \in R^k | x_i > 0, 1 \leq i \leq k$ \}. Thus we would get an asymptotic expansion of $L_\rho$ in case of an analytic action on an analytic manifold without information on $N$ and $l$, howeover. This has also been observed by Shafii–Dehahab [22]. To handle the general case we construct inductively and quite explicitly what we call a "weak resolution" of $\varphi(p, g) := \varphi_0(p, g(p))$. A weak resolution is more general than a resolution but its definition (given in section 2) is somewhat technical. However, it is excellently suited for the present purpose and might be useful in other situations, too. Our explicit construction leads again to integrals of the type (5) without any analyticity assumptions. It also has the advantage of giving very precise information on $N$ and $l$ which can hardly be deduced from the general Hironaka theorem.

Thus, we have to expect logarithmic terms if $a_k > 1$. Our first results in this direction were negative: there are no logarithmic terms for $S^1$ actions or more generally for actions of rank 1 groups (Theorem 7) and also not for arbitrary isometric group actions on the standard sphere (Theorem 5). This is surprising since for these actions the local structure of $\mathcal{A}$ can be as complicated as in the general case. However, since this paper has been written the first named author has obtained an example of a $T^2$-action on a (nonstandard) $S^2$ producing logarithmic terms. The example is based on the analysis of certain integrals of type (4), underlining their importance. Thus, our result is precise concerning the functions of $t$ involved in the expansion whereas the nature of the coefficients needs further clarification.

2. Weak resolutions and asymptotic expansions. We begin with the notion of resolution (cf. [1]). In the following a manifold is always assumed to be $C^\infty$ and paracompact.

Definition 1. Let $M$ and $\tilde{M}$ be manifolds of dimension $n$ and $\varphi: \tilde{M} \to M$ a $C^\infty$ map.
(1) \( \psi \) is called a resolution if \( \psi \) is proper and there is a closed subset \( X \subseteq M \) of measure 0 such that \( \psi^{-1}(X) \) has measure 0 and \( \psi \) restricts to a diffeomorphism \( \tilde{M} \to \psi^{-1}(0) \to M \setminus X \).

(2) \( \psi \) is called a resolution of \( \psi \in C^\infty(M) \) (more precisely: of the set of zeros of \( \psi \)) if:
   (a) \( \psi \) is a resolution with \( X := \psi^{-1}(0) \),
   (b) for each \( p \in M \) there is a coordinate system \( (x_1, \ldots, x_n) \) centered at \( p \) such that:
   \[
   \psi \circ \phi(x) = \phi(x) + \sum_{i=1}^{n} k_i x_i \quad \text{near } p \quad \text{where} \quad k_i \in \mathbb{Z}
   \]
   and \( k \) is smooth and nowhere zero.

A typical example of a resolution is the following. Let:
\[
\tilde{M} := \left\{ (u, x) \in \mathbb{R}^p \times \mathbb{R}^n \mid x \in u \right\}
\]
and define \( \psi : \tilde{M} \to \mathbb{R}^n \) as the restriction of the natural projection. \( \tilde{M} \) results from "blowing up the origin" in \( \mathbb{R}^n \) and coincides with the total space of the canonical line bundle on \( \mathbb{R}^n \).

As pointed out by Atiyah [1] the existence of resolutions has important applications in analysis. Our interest stems from the following essentially well known theorem on asymptotic expansions (cf. [6], [15]).

**Theorem 1.** Let \( M \) be a manifold of dimension \( n \) and \( \psi \in C^\infty(M) \), \( \psi > 0 \). If there is a resolution \( \psi : M \to \psi \) of \( \psi \) then the integral:
\[
I(f, \psi) := \int_{\psi^{-1}(0)} e^{-\psi/\psi} f
\]
which is well defined for any smooth density \( f \) with compact support has an asymptotic expansion as \( t \to 0^+ \) of the form:
\[
I(f, \psi) \sim \sum_{\alpha \in \mathbb{C} \times \mathbb{K}} \alpha_\psi(f) t^{1/\ell} / \log t \quad (7)
\]
for some integers \( J > 1 \) and \( K > 0 \). The \( \alpha_\psi \) are distributions on \( M \) with support in \( \psi^{-1}(0) \).

**Proof.** Denote by \( (K),_{\alpha}^J \) and \( (K),_{\alpha}^J \), the components of \( M \setminus \psi^{-1}(0) \) and \( M \setminus \psi^{-1}(0) \) respectively. Then we have:
\[
\int_{\psi^{-1}(0)} e^{-\psi/\psi} = \sum_{\ell J = \rho} \int_{K,_{\alpha}^J} e^{-\psi/\psi} = \sum_{\ell J = \rho} e^{-\psi/\psi} = \psi^J
\]
By condition (3)(b) we see that each point in \( \tilde{k} \) has a neighborhood \( \tilde{U} \) in \( \tilde{M} \) such that \( \tilde{U} \cap \tilde{k} \) is diffeomorphic to a (relatively) open subset of \( \mathbb{R}^n := \{ x \in \mathbb{R}^n \mid x_j > 0, 1 < j < n \} \). Since \( \psi \) is proper the same condition shows that \( \sup k^J \tilde{k} \neq 0 \)

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only for finitely many \( I \). Thus we see that \( I(f, \psi) \) can be written as a finite sum of integrals of the form:
\[
\int_{\psi^{-1}(0)} e^{-\psi/\psi} f \psi^J \psi
\]
where \( \lambda \in C^\infty(\mathbb{R}^n), \psi \in C^\infty(\mathbb{R}^n), \) and \( k(\psi) \neq 0 \) for \( x \in \supp g \). But the asymptotic expansion of the integrals \( (8) \) is known to exist and to be of the form (7) [15] [8]. In particular, we can choose:
\[
J := \text{l.c.m.} \{ \alpha \psi(\psi) \neq 0 \} \quad \text{and} \quad \mathcal{K} := \# \{ \alpha \psi(\psi) \neq 0 \}
\]
The existence of resolutions follows from the celebrated theorem of Hironaka in the analytic case. To get a result analogous to Theorem 1 in the \( C^\infty \) case we will propose the notion of "weak resolution" which is still sufficient to yield the asymptotic expansion. In addition, it turns out to be explicitly constructable for the functions we are interested in. In particular, we will have good control on the integers \( J \) and \( K \) above.

We start with weakening Definition 1(1).

**Definition 2.** Let \( M \) and \( N \) be manifolds. A differentiable map \( \psi : M \to N \) is called an almost diffeomorphism (more precisely: an almost diffeomorphism onto \( \psi(M) \)) if there is a closed subset \( X \subseteq M \) of measure 0 such that \( \psi \) restricts to a diffeomorphism \( M \setminus X \to \psi(M \setminus X) \).

Certainly, every resolution is an almost diffeomorphism. Another typical example is:
\[
\tilde{\psi} : \tilde{M} \times S^J_{\psi} \to \mathbb{R}^n \quad \text{where} \quad S^J_{\psi} = \{ (x_1, \ldots, x_n) \in S^J_{\psi} \mid x_j > 0 \}
\]
where \( \psi \neq 0 \). Note that \( \tilde{\psi} \) is not proper. We collect some obvious properties of almost diffeomorphisms.

**Lemma 1.** Let \( \psi : M \to N \) be an almost diffeomorphism.

(1) If \( K \subseteq N \) has measure 0 so has \( \psi^{-1}(K) \).

(2) The restriction of \( \psi \) to any open subset of \( M \) is again an almost diffeomorphism.

(3) If \( \psi : M \to N \) is an almost diffeomorphism then also \( \tilde{\psi} \circ \psi : M \to N \).

(4) If \( \psi, \tilde{\psi} : M \to N \) are almost diffeomorphism so is \( \psi \times \psi : M \times \tilde{M} \to N \times N \).

The weak analogue of Definition 1 now reads as follows.

**Definition 3.** Let \( M \) be a manifold of dimension \( n \).

(1) A family \( (V, \psi_{\alpha}) \) of almost diffeomorphisms \( \psi : V \to M \) is called a weak resolution of \( M \) if for each \( \psi \in M \) there are a finite subset \( I' \subseteq I \) and functions \( f_i \in C^\infty(V), i \in I' \), such that:
\[
\sum_{i \in I'} f_i \psi = 1
\]
a.e. in a neighborhood of $p$. Here the functions $f_j \circ \phi^{-1}$ (which are defined a.e. in $\phi_1(V_f)$) are extended to $0$ to all of $M$.

(2) A family $(V_i, \phi_i)_{i \in I}$ of almost diffeomorphisms $\phi_i : V_i \to M$ is called a weak resolution of $\varphi \in C^m(M)$ (more precisely: of the set of zeros of $\varphi$) if

(a) $(V_i, \phi_i)_{i \in I}$ is a weak resolution with $V_i$ open in $\mathbb{R}^n$,

(b) $\varphi = \phi_0(x) - \sum_{j=1}^n a_j(x) \prod_{i=1}^n \phi_i^{a_i}$

where $a_j \in \mathbb{Z}_+$ and $k$ is smooth and nowhere zero in $V_i$. We say that this weak resolution has order at most $L$ and degrees contained in $A \subset \mathbb{Z}_+$ if

$$\# \{ j \mid a_j \neq 0 \} \leq L \quad \text{for all } i \in I$$

and

$$a_j \in A \quad \text{for all } i \in I \text{ and } 1 \leq j \leq n.$$

Before discussing this notion in more detail we want to show how a weak resolution of $\varphi$ leads to an asymptotic expansion of the integrals (6). All we need is the following simple fact the proof of which we omit.

**Lemma 2.** Let $(V_i, \phi_i)_{i \in I}$ be a weak resolution of $M$. Then for any compact subset $K$ of $M$ there are a finite subset $I' \subset I$ and functions $f_i \in C^\infty(\tilde{V}_i)$, $i \in I'$, such that

$$\sum_{i \in I'} f_i \circ \phi_i^{-1} = 1$$

a.e. in a neighborhood of $K$.

Using property (1) of Definition 3 we can now repeat the proof of Theorem 1 to obtain the following result.

**Theorem 2.** Let $M$ be a manifold of dimension $n$ and $\varphi \in C^\infty(M)$, $\varphi > 0$. If there is a weak resolution $(V_i, \phi_i)_{i \in I}$ for $\varphi$ then we have for any smooth density with compact support the following asymptotic expansion as $t \to 0^+$

$$\int_{M} e^{-t\varphi} \sum_{a \in A} a_0(\varphi) e^{t\varphi_i} \sum_{a \in A} a_0(\varphi) e^{t\varphi_i} \sum_{a \in A} a_0(\varphi) e^{t\varphi_i}$$

for some integers $J > 1$ and $K > 0$. The $a_a(\varphi)$ are distributions on $M$ with support in $\varphi^{-1}(0)$.

If in addition the weak resolution has order at most $L$ and degrees contained in $A$, a finite, then we can choose

$$K = L - 1 \quad \text{and} \quad J = i.e.m \{ a \mid a \in A \cap \{0\} \}.$$
LEMMA 5. Let $M$ be a manifold and let $\varphi \in C^{\infty}(R^+ \times M)$ be homogeneous of degree $k$ in the space variable. If $\tilde{\varphi} := \varphi \cdot S^{-1} \times M$ has a weak resolution of order at most $L$ with degrees contained in $A$, then $\varphi$ has a weak resolution of order at most $L + 1$ and with degrees contained in $A \cup \{k\}$.

Proof. Put $\varphi : R \times S^{n-1} \times M \ni (r, u, p) \rightarrow (ru, p) \in R^* \times M$. Then $\varphi + \tilde{\varphi}$ has a weak resolution of order at most $L + 1$ and with degrees contained in $A \cup \{k\}$ by Remark 4 above. But $\tilde{\varphi}$ is invariant under the natural $Z_2$ action $(r', u, p) \rightarrow (-r', -u, p)$. This action is free and the quotient is diffeomorphic to $R^* \times M$ with $R^*$ described above. Hence we have a factorization $\varphi = \tilde{\varphi} \circ \varphi$ where $\tilde{\varphi}$ is a resolution and $\varphi$ a covering. Thus the lemma follows from Remarks (2) and (3).

Though simple the next lemma is a powerful tool for our constructions. Note that—somewhat unexpectedly—we don't require $\varphi - \tilde{\varphi} \leq 0$ to be smooth.

LEMMA 6. Let $\varphi, \tilde{\varphi} \in C^{\infty}(M)$ be nonnegative and satisfy the inequality

$$
\frac{1}{C} \varphi \leq \tilde{\varphi} \leq C \varphi
$$

for some positive constant $C$. If $\varphi$ has a weak resolution of order at most $L$ and with degrees contained in $A$ so has $\tilde{\varphi}$.

Proof. Let $(V, \varphi_{\delta})_{\delta \in A}$ be a weak resolution for $\varphi$ and let

$$
\tilde{\varphi} + \varphi_{\delta}(x) = \tilde{k}(x) \prod_{j=1}^{n} x_j^y, \quad x \in V,
$$

with $\tilde{k}$ smooth and $\tilde{k}(x) > 0$. The right hand inequality in (9) implies that $\varphi + \varphi_{\delta}$ is $C^{-}\text{divisible}$ by $\prod_{j=1}^{n} x_j^y$. In fact this is obvious if $\beta_{\delta} := \sum_{j=1}^{n} y_j \delta$ is equal to zero or one and follows in general by induction on $\beta_{\delta}$. Hence

$$
\varphi + \varphi_{\delta}(x) = \tilde{k}(x) \prod_{j=1}^{n} x_j^y, \quad x \in V,
$$

for some $k \in C^{\infty}(V)$. But the left hand inequality in (9) implies that $k_1$ is nonzero.

Finally we can "sum" weak resolutions under special circumstances.

LEMMA 7. Suppose that the nonnegative functions $\varphi, \tilde{\varphi} \in C^{\infty}(M)$ and $\varphi \in C^{\infty}(M)$ have weak resolutions of order at most $L$ and $L$ respectively but both with degrees contained in $(0, d)$ for some $d \in N$. Then $\varphi + \tilde{\varphi} \in C^{\infty}(M \times M)$ has a weak resolution of order at most $L + \tilde{\varphi}$ and with degrees contained in $A \cup \{d\}$.

Proof. By the functional properties of weak resolutions described above it is sufficient to construct a weak resolution of the functions

$$
\psi(x, y) := \prod_{j=1}^{n} x_j^y, \quad (x, y) \in R^{n+n},
$$
where \( \alpha, \beta \in (0, d) \) hence \( \sum \alpha = kd, \sum \beta = ld \) for some \( k, l \in \mathbb{Z}^+ \). To do so we use induction on \( k + l \). If \( k + l < 1 \) (or \( k = 0 \) or \( l = 0 \)) there is nothing to prove. Now suppose \( \alpha_n = \beta_n = d \) and consider the resolution \( \phi : \mathbb{R}^2 \times \mathbb{R}^{n-2} \to \mathbb{R}^{n+2} \) obtained by blowing up the origin in the \((x_\gamma, y_\beta)\) plane while leaving all other coordinates unchanged. On \( \mathbb{R}^2 \) we can locally introduce coordinates \( x_\beta, y_\beta \) with \( y_\beta = \beta_\beta x_\beta \) or \( y_\beta = \beta_\beta x_\beta \) with \( x_\beta = \beta_\beta y_\beta \). Thus we can use the induction hypothesis to get a weak resolution of \( \phi \times \phi \) which gives a weak resolution of \( \phi \), too. Obviously this resolution has order at most \( L + L' - 1 \) and degrees contained in \((0, d)\).

3. Weak resolution of the square of the Riemannian distance. We will now apply the techniques of the previous section to construct a weak resolution of the function \( \psi_0 \in C(M \times G) \) given by \( \psi(p, g) := d^2(p, g(p)) \) where \( d \) denotes Riemannian distance. More precisely we are going to show that every point \( (p, g_0) \in \mathcal{L} \) has a neighborhood \( U \subset M \times G \) such that \( \psi_0 | U \subset C^0 \) and has a weak resolution thus defining a weak resolution of \( \psi_0 \) in some neighborhood of \( \mathcal{L} \).

To do so we equip \( G \) with a biinvariant metric by choosing an Ad \( G \)-invariant scalar product on the Lie algebra \( \mathfrak{g} \) of \( G \). For \( p \in M \) we denote by \( G_p \) the isotropy group and by \( \mathfrak{m}_p \) an orthogonal complement to its Lie algebra \( \mathfrak{g}_p \). \( G_p \) denotes the \( G \)-orbit of \( p \) and \( N_p \) the orthogonal complement of its tangent space in \( T_p M \). Finally, let \( \exp \) and \( \exp_p \) be the exponential maps on \( \mathfrak{g} \) and \( T_p M \) respectively. With these notations we define a map \( \phi : N_p \times \mathfrak{m}_p \times \mathfrak{m}_p \times G_p \to M \times G \) by

\[
\phi(x, y, z, t) := \left( \exp x \exp_p x, \exp y \exp_p y \right) \cdot t,
\]

and a function \( f : N_p \times \mathfrak{m}_p \times \mathfrak{m}_p \times G_p \to \mathbb{R}^+ \) by

\[
f(x, y, z, t) := |x - d_{G_p}(x)x|^2 + |y|^2.
\]

Lemma 8. For every \( g_0 \in G_p \) there is a neighborhood \( W \) of \((0, 0, 0, 0)\) in \( N_p \times \mathfrak{m}_p \times \mathfrak{m}_p \times G_p \) such that

\( \phi | W \) is a diffeomorphism

and

\[
\frac{1}{C} f < \psi_0 \phi < Cf
\]

on \( W \) for some \( C > 0 \).

Proof. The image of \( d_{\phi}(0, 0, 0, g_0) \) clearly contains \( N_p \cdot d_{G_p}(g) \) where \( d_{G_p} \) denotes right translation by \( g \). Therefore, the map \( \phi \) is surjective hence also injective since the dimensions are equal. Hence \( \phi \) is a local diffeomorphism.

Since \( G \) acts by isometries we have

\[
\psi_0 \psi(x, y, z, t) := d^2(\exp x \exp_p x \exp y \exp_p y, d_{G_p}(x)) = d^2(\phi(x, 0, \phi(y, d_{G_p}(x)), y))
\]

where \( \phi : N_p \times \mathfrak{m}_p \to M \) is given by \( \phi(x, y) = \exp y \exp_p x \). Now \( \phi \) is a diffeomorphism in a neighborhood of \((0, 0)\) hence the pull back of \( d^2 \) under \( \phi \) is equivalent to the Euclidean metric on \( N_p \times \mathfrak{m}_p \) near \((0, 0)\) which is the inequality (10).

It is now easy to see that \( \mathcal{L} \) is a submanifold of \( M \times G \times G \) and only if the \( G \) action has no singular orbits. In fact, in the local coordinates above we have \( \phi^{-1}((x, y), g) := (x, y) \in W(|d_{G_p}(x)| = x) \) i.e. \( \mathcal{L} \) is a submanifold of \( M \times G \) if and only if \( \mathcal{L} := (x, y) \in N_p \times G_p \mid d_{G_p}(x) = x \) is a submanifold of \( N_p \times G_p \).

But then the components of \( \mathcal{L} \) and \( N_p \times G_p \) containing \( (0, 0) \) must be equal meaning that the identity component of \( G_p \) acts trivially on \( N_p \) which in turn is equivalent to the nonexistence of singular orbits in view of the slice theorem.

We need one more definition. For a \( G \)-invariant open set \( U \subset M \) we put

\[
K(U, G) := \# (\dim G_p) \cdot g \in U
\]

and for \( p \in M \)

\[
K(p, G) := \inf \{ (K(U, G)) : U | U \ G \text{-invariant}, p \in U \}.
\]

Finally, let

\[
K(G, M) := \sup_{p \in M} K(p, G).
\]

We can now construct the desired weak resolution of \( \psi_0 \).

Theorem 3. There is a neighborhood \( U \) of \( \mathcal{L} \) such that \( \psi_0 | U \subset C^0 \) and has a weak resolution of order at most \( K(G, M) \) with degrees contained in \((0, 2)\).

Proof. It is sufficient to prove existence of such a weak resolution in a neighborhood \( V \) of any given point \((p, g_0) \in \mathcal{L} \).

Choosing \( W \) as in Lemma 8 and using Lemma 6 we need only construct a weak resolution for \( |x - d_{G_p}(x)|^2 + |y|^2 \) in \( W \). Picking a product neighborhood \( V \subset W \) and applying Lemma 7 repeatedly we can further reduce to \( |x - d_{G_p}(x)|^2 \) in a neighborhood of \((0, g_0)\) in \( N_p \times G_p \). Now denote by \( G^0 \) the identity component of \( G \) and put

\[
\bar{N} := \{ x \in N_p \mid d_{G_p}(x) = x \ | g \in G^0 \},
\]

\[
N_1 := \{ x \in N_p \mid d_{G_p}(x) = x \}.
\]

We decompose orthogonally

\[
N := N_1 \oplus N_2,
\]

where \( N_1 \oplus N_2 \).
and note that $\bar{N}$ is $G_J$-invariant. Writing $x = x_1 + x_2 + x_3$ accordingly, we find in a connected neighborhood of $(0, G_J)$

$$
|x - d_g(x)| = |(x_1 + x_2) - d_{g_0}(x_1 + x_2)| + |x_3 - d_{g_0}(x_3)|^2
$$

$$
= |(id - d_{g_0})(x_3)|^2 + |x_3 - d_{g_0}(x_3)|^2.
$$

Since $id - d_{g_0}$ is nonsingular on $N_J$, we can use Lemma 7 again and are left with $f(x, g) := |x_3 - d_{g_0}(x_3)|^2$ in a neighborhood of $(0, g_0)$ in $N_J \times G_J$.

We now use induction on $K(M, G)$. If $K(M, G) = 1$, we have no singular orbits hence $N_J = 0$ and the proof is complete in this case. Assume then that the theorem is true for all compact manifolds $M'$ with isometric $G'$ actions such that $K(M', G') < K(M, G)$. Denoting by $S$ the unit sphere in $N_J$, we have a $G_J$ action on $S$ with $K(S, G_J) < K(M, G)$ since $G_J$ has no orbits of dimension 0 on $S$. The induction hypothesis gives a weak resolution of $f := f_S \times S$ of order at most $K(M, G) - 1$ and with degrees contained in $(0, 2)$; namely, in a neighborhood of $f' = \{(0, g) \in S \times S \mid d_g(\omega) = \omega \}$ is equivalent to $d(u, g) = 0$ with $d$, the standard distance on the sphere, hence the assertion follows from Lemma 6.

Now introducing polar coordinates by $\varphi: \mathbb{R} \times S \times G_J \ni (r, \omega, g) \mapsto (r, \omega, g) \in N_J \times G_J$, we have $f = f(\varphi, \omega, g) = r^2(\omega, g)$. Applying Lemma 5 the proof is completed.

4. The asymptotic expansion. We now derive the asymptotic expansion of

$$
L_0(t) = \sum_{\chi} e^{-\lambda t} \dim \text{Hom}_G(V, E_\chi).
$$

With an orthonormal basis $(\tilde{e}_1)$ for $E_\chi$ we have for the character $\chi_0$ of the representation of $G$ in $E_\chi$

$$
\chi_0(g) = \sum_{\chi} \int_{M} \tilde{e}_1(g \cdot \tilde{e}_1^*) \tilde{e}_1^*(g \cdot \tilde{e}_1) d\phi.
$$

Denoting by $\chi_0$ the character of $\rho$ we obtain from the orthogonality relations [12], p. 189,

$$
\dim \text{Hom}_G(V, E_\chi) = \frac{1}{\text{vol} G} \int_{M} \tilde{e}_1(g) \tilde{e}_1^*(g) d\phi.
$$

On the other hand, the kernel $\Gamma_\chi$ of $e^{-\chi}$ is given by the convergent series [41], p. 205

$$
\Gamma_\chi(p, q) = \sum_{\chi} e^{-\lambda t} \tilde{e}_1^*(p \cdot \tilde{e}_1^*) \tilde{e}_1^*(q).
$$

The Cauchy–Schwarz inequality gives for every $N \in \mathbb{N}$

$$
\left| \sum_{\chi} e^{-\lambda t} \tilde{e}_1(p \cdot \tilde{e}_1^*) \tilde{e}_1^*(q) \right| \leq \Gamma_\chi(p, p) \Gamma_\chi(q, q).
$$

hence the Lebesgue–Fatou Lemma implies the identity

$$
L_0(t) = \frac{1}{\text{vol} G} \int_{M} \tilde{e}_1(g) \tilde{e}_1^*(p \cdot \tilde{e}_1^*) \tilde{e}_1^*(q) d\phi.
$$

Now we have the estimate [41], p. 50

$$
|\Gamma_\chi(p, q)| \leq C(t, g) \varphi(p, q), \quad p, q \in M, \quad t > 0,
$$

with some positive constants $C(t, g)$. Thus for any $f \in C^\infty(M \times G)$ with $f = 1$ in a neighborhood of $\mathcal{S}$ we obtain as $t \to 0$

$$
L_0(t) = \frac{1}{\text{vol} G} \int_{M \times G} \tilde{e}_1(g) \tilde{e}_1^*(p \cdot \tilde{e}_1^*) \tilde{e}_1^*(q) d\phi.
$$

Now we choose a neighborhood $U$ of $\mathcal{S}$ in $M \times G$ such that Theorem 3 holds in $U$ and the map $(p, g) \mapsto (p, g(p))$ maps $U$ into a neighborhood of the diagonal in $M \times M$ where the Minakshisundaram–Pleijel expansion (2) is valid. Combining these facts with Theorem 2 and Theorem 3.1 in [9] we have proved the main result of this paper.

Theorem 4. We have the following asymptotic expansion as $t \to 0$

$$
L_0(t) = \frac{1}{(4\pi)^{n/2}} \sum_{a \in \mathbb{Q}^n, \gamma} a_k \alpha_k \log^a
$$

$$
\frac{1}{(2\pi)^{n/2}} \sum_{\gamma \in \mathbb{Q}^n, K(M, G)} e^{-\chi(g)}
$$

where $m := \dim M$ and $K(M, G)$ was defined in (11). Moreover

$$
a_{00} = \text{vol} M / \dim V^n
$$

$H$ a principal isotropy group, and

$$
a_k = 0, \quad k > 0.
$$

Remark. If $G$ is connected we may reduce the integral over $G$ in the formula for $L_0$ to an integral over $T$ a maximal torus, by Weyl's integration formula. In particular, we may replace $K(M, G)$ by $K(M, T)$ in Theorem 4 showing that the exponent of the logarithmic factor is at most $\dim T = \text{rank} G$. But $K(M, T) > K(M, G)$ is possible, e.g., if $G = \text{SO}(3)$ acts on $S^2$ in the standard way.

5. The case of the sphere. As mentioned in the introduction, the logarithmic terms do occur, but so far their dependence on the geometry is not clear. In the remainder of this paper we present some nonexistence results.

It is clear from the proof of Theorem 4 that the log terms are somehow related to the singularities of $\mathcal{S}$. By Lemma 8 $\mathcal{S}$ is locally (up to Euclidean factors) diffeomorphic to

$$
\{ (x, g) \in \mathbb{R}^n \mid g(x) = x \}.
$$
where $\mathcal{G} \subset O(n)$ is an isometry group of $G$. Thus the local singularities of $\mathcal{L}$ can be realized already by isometric actions on the standard sphere $S^n$ (by embedding $\mathcal{G}$ in $O(n+1)$ in the usual way). Somewhat surprisingly we have the following result which seems to indicate a strong dependence of the log terms on the Riemannian metric.

**Theorem 5.** For $M = S^n$ the standard sphere, $G$ a closed subgroup of $O(n+1)$, and any finite dimensional representation $p$ of $G$ on a complex vector space $V$ we have an asymptotic expansion of the form

$$L_A(t) = \sum_{j=0}^\infty \hat{a}_j t^{j/2}, \quad t \to 0.$$  

The proof is independent of Theorem 4 and will follow from the next two propositions recalling that $\lambda_j(S^n) = k(k + n - 1)$.

**Proposition 1.** There exist an integer $m \geq 1$ and polynomials $p_0, \ldots, p_{m-1} \in \mathbb{Q}[x]$ of degree at most $n - 1$ such that

$$\dim \text{Hom}_\mathcal{G}(V, E_{k}(S^n)) = p_k(k)$$

if $k = r \mod m$ and $k$ is sufficiently large.

**Proposition 2.** For $m, r, s \in \mathbb{R}$ and $m > 0$ and $0 < r < s$ we have with certain $a_i \in \mathbb{R}$

$$\sum_{k=0}^\infty \sum_{j=0}^\infty \hat{a}_j t^{j/2} = \sum_{k=0}^\infty \sum_{j=0}^\infty \frac{f(z)}{(1-z^k)...(1-z^{k-s})}$$

as $t \to 0$.

We begin with the proof of Proposition 1. Denote by $\mathcal{P}$ the polynomial ring $\mathbb{Q}[x_1, \ldots, x_{n+1}]$ and by $\mathcal{P}_k$ the homogeneous polynomials of degree $k > 0$ in $\mathcal{P}$. Put $P_{-1} := \mathcal{P}_{-1} := 0$. The Laplace operator $\Delta$ in $\mathbb{R}^{n+1}$ maps $\mathcal{P}$ to itself. We denote by $\Delta : \mathcal{P}_k \to \mathcal{P}_{k-2}$ the restriction to $\mathcal{P}_k$. It is well known that $E_k(S^n) = \mathcal{P}_k = \ker \Delta$ the isomorphism being given by restriction to the sphere. A direct computation shows the following:

**Lemma 9.** $\Delta : \mathcal{P}_k \to \mathcal{P}_{k-2}$ is surjective, $k > 0$.

Now let $G \subset O(n+1)$ be a closed subgroup and $p : G \to GL(V)$ a finite dimensional representation of $G$. Then $G$ acts naturally on $\mathcal{P}$ leaving $\mathcal{P}_k$ in $\mathcal{P}$ invariant and the isomorphism $\mathcal{P}_k \subset E_k(S^n)$ is $G$-equivariant. Since $G$ is compact the exact sequence

$$0 \to \text{Hom}(V, \mathcal{P}_k) \to \text{Hom}(V, \mathcal{P}_k) \to \text{Hom}(V, \mathcal{P}_{k-2}) \to 0$$

implies the exact sequences

$$0 \to \text{Hom}_\mathcal{G}(V, \mathcal{P}_k) \to \text{Hom}_\mathcal{G}(V, \mathcal{P}_k) \to \text{Hom}_\mathcal{G}(V, \mathcal{P}_{k-2}) \to 0,$$

Thus we have

**Lemma 10.**

$$\dim \text{Hom}_\mathcal{G}(V, E_k(S^n)) = \dim \text{Hom}_\mathcal{G}(V, \mathcal{P}_k) - \dim \text{Hom}_\mathcal{G}(V, \mathcal{P}_{k-2}), \quad k > 0.$$

Now we come to the crucial point.

**Lemma 11.** The Poincaré series of $\mathcal{P}_\mathcal{G} := \mathcal{P}_\mathcal{G} := \mathcal{P}_\mathcal{G}$ has the form

$$\sum_{k=0}^\infty \dim \text{Hom}_\mathcal{G}(V, E_k(S^n)) t^k = \frac{f(z)}{(1-z^k)...(1-z^{k-s})}$$

where $f \in \mathbb{Q}[z]$ and $d_1, \ldots, d_s$ are certain positive integers (namely the degrees of a set of generators for $\mathcal{P}_\mathcal{G}$).

**Proof.** Since by Lemma 5

$$\sum_{k=0}^\infty \dim \text{Hom}_\mathcal{G}(V, E_k(S^n)) t^k = (1-z^k) \sum_{k=0}^\infty \dim \text{Hom}_\mathcal{G}(V, \mathcal{P}_k) t^k$$

it is enough to consider the Poincaré series of $\mathcal{P}_\mathcal{G}$ for which the corresponding statement is essentially known. In fact by a classical result of Weyl [15] (cf. also [14], Ch. 10, Theorem 5) $\mathcal{P}_\mathcal{G}$, the subspace of $G$-invariant polynomials, is a finitely generated algebra over $\mathbb{C} = \mathcal{P}_\mathcal{G}$. Since $A := \mathcal{P}_\mathcal{G}$ is a finitely generated graded $\mathcal{P} = \oplus_{k=0}^\infty \mathcal{P}_k$-module it follows from [23], 2.4.14 that $A^G = \mathcal{P}_\mathcal{G}$ is a finitely generated graded $\mathcal{P}_\mathcal{G}$-module.

But then by a result of Hilbert–Serre [23], Proposition 2.5.4 or [3], Theorem 11.1 the Poincaré series of $\text{Hom}_\mathcal{G}(V, \mathcal{P}_k)$ is a rational function of the desired form. □

**Lemma 12.** Let $(a_k)_{k \geq 0}$ be a sequence of integers and assume

$$\sum_{k=0}^\infty a_k z^k = \frac{f(z)}{(1-z^k)...(1-z^{k-s})}$$

for some $f \in \mathbb{Q}[z]$ and some integers $d_1, \ldots, d_s > 0$. Put $m := \text{lcm}(d_1, \ldots, d_s)$. Then there exist $P_0, \ldots, P_{m-1} \in \mathbb{Q}[z]$ such that $a_k = P_k(k)$ if $k \equiv r \mod m$ and $k$ is sufficiently large.

**Proof.** Since $(1-z^k)^{-1} = g(z)(1-z^{k-s})^{-1}$ for some $g \in \mathbb{Q}[z]$ we may assume that $d_i = \cdots = d_i = m$. If

$$f(z) = \sum_{k=0}^\infty b_k z^k$$
it follows from
\[
\frac{(x-1)!}{(1-x)!} = \left(\frac{1}{1-x}\right)^{x-1} = \sum_{n=0}^{\infty} \binom{x-1}{n} x^{-n}
\]
that
\[
\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} b_n \binom{k}{n} x^{k-n} = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} b_n \binom{k-n}{n} x^{k-n}.
\]
Hence if \( k > N \) and \( k = r \mod m \)
\[a_k = \sum_{n=0}^{m} b_n \binom{k-n}{n} = P_n(k)\]
and the lemma follows. \( \square \)

The proof of Proposition 1 now follows from Lemma 11 and Lemma 12 noting that \( \dim \text{Hom}_G(V, E_0(S^r)) < \dim V \cdot \dim E_0(S^r) < Ck^{-1} \) hence \( \deg P_r < \kappa - 1, 0 < r < m - 1 \).

For the proof of Proposition 2 we need the following result which follows easily from [10], Lemmas 8.1 and 8.5.

**Lemma 13.** We have the following asymptotic expansions as \( t \to 0 \).

\[
(1) \quad \sum_{k=0}^{\infty} e^{-t/k} \binom{k}{n} \left(\frac{x}{4t}\right)^{1/2} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^n}{n!} e^{1/2 - (2n+1)/(2t)},
\]
\[
(2) \quad \sum_{k=0}^{\infty} e^{-t/k} \binom{k}{n} \left(-\frac{1}{2}\right)^{n-1/2} e^{-2t/(2n+1)^2}, \quad n \in \mathbb{N}.
\]
\[
(3) \quad \sum_{k=0}^{\infty} e^{-t/k} \binom{k}{n} \sum_{k=0}^{n} \frac{(-1)^k B_{k+n+1}}{2k+1} \left(\frac{2k+1}{t}\right)^{n-1} e^{t(k+n+1)/2},
\]
\( \theta \in \mathbb{Z}_+ \), where \( B_\theta \) denotes the \( \theta \)th Bernoulli number.

**Proof of Proposition 2.** We have
\[
\sum_{k=0}^{\infty} e^{-t/k} \binom{k}{n} \binom{k}{n} x^k = \sum_{k=0}^{\infty} e^{-t/(k+x)} x^k
\]
where \( \alpha = (x+\alpha)/m \). Thus it suffices to prove such an expansion for

\[
F_r(x, t) = \frac{\sum_{k=0}^{\infty} e^{-t/(k+x)} x^k, \alpha > 0}{\alpha}.
\]

and hence for \( N \in \mathbb{N} \)
\[
F_r(x, t) = \sum_{k=0}^{N} \frac{(-t)^k}{k!} P_k(x, t) + \frac{(-t)^N}{N!} \int_0^t F_{r+N+1}(x, t)(1-s)^N ds,
\]
the result follows from \( P_{r+N+1}(x, t) < P_{r+N+1}(0, t) \) and Lemma 8. \( \square \)

Theorem 5 generalizes easily to spaces which are the image of a sphere under a Riemannian submersion with minimal fibres and with decktransformations acting transitively on each fibre. Thus we get the following generalization.

**Theorem 6.** Let \( M \) be a spherical space form, \( G \). Then for any closed subgroup \( G \) of \( I(M) \) and any finite dimensional complex representation \( \rho \) of \( G \) we have an asymptotic expansion of the form
\[
L_A(t) = \sum_{r=0}^{\infty} \alpha_r \rho^{(2/3)} (x, t) \to 0.
\]

We conjecture that Theorem 5 generalizes to all compact symmetric spaces.

6. \( S^1 \) actions. We are going to show that the asymptotic expansion of \( L_s \) does not contain logarithmic terms when \( G = S^1 \). The basic observation leading to the proof of this fact is the following.

**Lemma 14.** Let \( f \in C^\infty_0(\mathbb{R}^2) \) satisfy
\[
\frac{\partial^2 g}{\partial x^2} (0, 0) = 0, \quad j \geq 0.
\]
Then the asymptotic expansion of
\[
I(t) := \int_{\mathbb{R}^2} e^{-t/(x^2 + y^2)} f(x, y) dx dy
\]
for \( t \to 0 \) does not contain logarithmic terms.

**Proof.** It is known (cf. e.g., [8], Theorem 1) that there is an asymptotic expansion
\[
I(t) = \sum_{r=0}^{\infty} \alpha_r \rho^{(2/3)} (x, t) \to 0,
\]
where \( \alpha = (x+\alpha)/m \). Thus it suffices to prove such an expansion for
We put $\sigma := t / \varepsilon^2$: then upon substituting this in the above expansion we get an asymptotic expansion with respect to the functions $e^{(x^2+y^2) / \varepsilon^2} / \varepsilon^{2+1}$ and $e^{(t^2+y^2) / \varepsilon^2} / \varepsilon^{2+1}$ as $x^2 + \sigma^2 \to 0$, and in this expansion the coefficients of $e^{x^2/\varepsilon^2}$ and $e^{x^2/\varepsilon^2}$ must be equal. Now write

$$I(t) = \int_0^\infty ds \left( \int_0^s \int_0^{s/\varepsilon} dy e^{-s^2 / \varepsilon^2} f(x, y) \right) = I_1(s, \sigma) + I_2(s, \sigma).$$

An easy calculation gives as $x^2 + \sigma^2 \to 0$

$$I_1(s, \sigma) = \sum_{j=0}^{d+1} \frac{s^{d+1-j}}{j!} \int_0^s e^{-s^2 / \varepsilon^2} \frac{\partial^j f}{\partial y^j}(x, 0) dy \frac{dy}{\varepsilon^j},$$

which has no terms of the form $e^{x^2/\varepsilon^2}$. On the other hand, since $t / \varepsilon^2 \leq \sigma$ we get from Taylor's formula

$$I_2(s, \sigma) = \sum_{j=0}^{d+1} \frac{s^{d+1-j}}{j!} \int_0^s e^{-s^2 / \varepsilon^2} f(x,y) \frac{dy}{\varepsilon^j}.$$

By Taylor's formula again we find

$$\int_0^s \frac{dy}{\varepsilon^j} f(x,y) \frac{dy}{\varepsilon^j} = \int_0^s \frac{dy}{\varepsilon^j} f(x,y) \frac{dy}{\varepsilon^j} = C_j + \int_0^s \frac{dy}{\varepsilon^j} f(x,y) \frac{dy}{\varepsilon^j},$$

where

$$C_j := \int_0^s \frac{dy}{\varepsilon^j} f(x,y) \frac{dy}{\varepsilon^j}$$

and $R_j(s)$ has an asymptotic expansion in nonnegative powers of $\varepsilon$. The assumption (12) shows that $I_2(s, \sigma)$ contains no term of $e^{x^2/\varepsilon^2} \varepsilon^j$, too, and the Lemma is proved.

**Theorem 7.** Let $M$ be a compact Riemannian manifold of dimension $n$ with an effective isometric $G$-action, where $G$ is connected of rank one. Then we have the asymptotic expansion

$$I(t) = (4\pi)^{-n+1/2} \sum_{j=0}^{d+1} \phi_j e^{s/\varepsilon^2},$$

as $t \to 0$, where $\phi_j = \omega(\theta_j \delta / \varepsilon^j)$.

**Proof.** We only need to show that there are no logarithmic terms. The remark after Theorem 4 shows that we may assume $G = S^1$. Fix $(p, g_0) \in \mathcal{M}$ and

$$\Phi_\varepsilon = \Phi(q_\varepsilon, \varepsilon) \to 0$$

assumed that we can find a diffeomorphism $\Phi: W \to U$ where $U$ is a neighborhood of $(p, g_0)$ in $M \times S^1$ and $W$ an open subset of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ such that

$$\Phi_\varepsilon \cdot (\Phi(q_\varepsilon, \varepsilon) \circ (y_1, x_2), \theta) = \Phi(q_\varepsilon, \varepsilon)(y_1, x_2) \cdot (\theta, 0) \Phi_\varepsilon.$$  

(13)

Here $q_\varepsilon, \delta(\varepsilon), \varepsilon(\varepsilon)$ as before. As in the proof of Theorem 4 we have to determine the asymptotic expansion of the integrals

$$I(f) := \int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}} e^{-(1/\varepsilon^2)(y_1^2 + y_2^2 + \delta(\varepsilon))} \Phi(q_\varepsilon, \varepsilon)(y_1, x_2, \theta) d\theta dy$$

as $t \to 0$ where $f \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$. Introducing polar coordinates $(r, \sigma, \omega)$ in $\mathbb{R}^n$ and expanding the $z$-integral we do not encounter logarithmic terms. Hence we are left with the integrals

$$I(f) := \int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}} e^{-(1/\varepsilon^2)(y_1^2 + y_2^2 + \delta(\varepsilon))} \Phi(q_\varepsilon, \varepsilon)(y_1, x_2, \theta) d\theta dy.$$

Here $f \in C_0^\infty(\mathbb{R}^n \times \mathbb{R})$ and we can assume that $f$ is invariant under the transformations $z \mapsto -z$ and $\theta \mapsto -\theta$. Introducing polar coordinates $(r, \sigma, \omega)$ in $\mathbb{R}^n$ we find

$$I(f) := \int_{\mathbb{R}^n} e^{-r^2/r^2} f(r, \theta, \omega) d\theta$$

where $f \in C_0^\infty(\mathbb{R}^n)$. And its Taylor series around $0$ contains only even powers of $r$ and $\theta$ by the above invariance property. But then

$$\frac{\partial^2}{\partial \rho^2} f(x^2) = 0, j > 0,$$

and the proof is complete in view of Lemma 14.

For the proof of (13) we consider first the case that $S^1$ is finite. Using Lemma 8 we get a coordinate system $\Phi: W \to U$, where $U$ is a neighborhood of $(p, g_0)$ in $M \times S^1$ and $W$ open in $\mathbb{R}^n \times \mathbb{R}^n$, such that

$$\frac{1}{2} |y_2|^2 + q_\varepsilon \cdot \Phi(q_\varepsilon, \varepsilon)(y_1, x_2) \leq C |y_2|^2$$

which corresponds to the case $k = 0$ in (13). But then the critical set of $q_\varepsilon \cdot \Phi$ is given by $y_2 = 0$ and $q_\varepsilon \cdot \Phi$ is nondegenerate with Hess $q_\varepsilon \cdot \Phi(y_1, 0)$ being $n$-positive definite. The generalized Morse Lemma (c.f. [19]) proves (13) in this case. Now assume $S^1 = S^1$. We decompose $T_q M$ orthogonally as $T_q M = V_1 \oplus V_2$ where $V_1 := \langle x \in T_q M | \Phi_\varepsilon \cdot \Phi(q_\varepsilon, \varepsilon)(x) = x \rangle$ and $V_2$ orthogonally as $V_2 := V_2 \oplus V_3$, where $V_3$ is the subspace that carries the trivial part of the orthogonal representation of $S^1$ on $V$. In particular, dim $V_2 = 2k$ for some $k \in \mathbb{Z}$. Using Lemma 8 again and the explicit form of orthogonal $S^1$ actions we can find a
coordinate system \( \Phi : W \rightarrow \mathbb{U} \), \( W \) open in \( \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \), such that \( \Phi_0 = \Phi_1 \) satisfies

\[
\frac{1}{C} \left( |j_2|^2 + |z|^2 \right) < \Phi_0 \cdot \Phi_1 (j_1, j_2, z, \theta) < C \left( |j_2|^2 + |z|^2 \right).
\]  
(14)

Since \( S^1 \) can be covered by two coordinate systems centered at 1 and \(-1\) we may assume \( \Phi_0 = \Phi_1 \) for \( \Phi_0 \cdot \Phi_1 \) is even in \( \theta \). Also, by our previous considerations it is clear that we have to prove (13) for \( \Phi_0 \cdot \Phi_1 \) in a neighborhood of the points \((j_0, 0, 0, 0)\) only. We fix a point \((j_0, 0, 0, 0)\) and write

\[
\Phi_0 \cdot \Phi_1 (j_1, j_2, z, \theta) = \Phi_0 (j_1, j_2, z, \theta) + \delta \Phi_2 (j_1, j_2, z, \theta)
\]

since \( \Phi_0 = \Phi_1 \) is even in \( \theta \). From (14) we get

\[
\frac{1}{C} |j_2|^2 < |\Phi_0 (j_1, j_2, z, \theta)| < C |j_2|^2.
\]  
(15)

Using Taylor's formula we obtain

\[
\Phi_0 (j_1, j_2, z, \theta) = \Phi_0 (j_1, j_2, z, \theta) + 2 \Phi_2 (j_1, j_2, z, \theta) z, \\
\Phi_0 (j_1, j_2, z, \theta) = O(|j_2|), \quad \Phi_0 (j_1, j_2, z, \theta) = O(|j_2|).
\]

From (14) again we have

\[
\frac{1}{C} |j_2|^2 < |\Phi_0 (j_1, j_2, z, \theta)| < C |j_2|^2.
\]  
(16)

hence

\[
\Phi_0 (j_1, j_2, z, \theta) = \delta \Phi_2 (j_1, j_2, z, \theta) z.
\]

with \( \Phi_2 \) smooth and positive definite. Thus we find

\[
\Phi_2 (j_1, j_2, z, \theta) = (\Phi_2 (j_1, j_2, z, \theta) + \delta \Phi_2 (j_1, j_2, z, \theta)) z + \Phi_2 (j_1, j_2, z, \theta)
\]

with \( \Phi_2 \) again smooth and positive definite, \( \Phi_2 \) linear in \( j_2 \), and \( \Phi_2 = O(|j_2|^2) \). This gives

\[
\Phi_0 \cdot \Phi_1 (j_1, j_2, z, \theta) = (\Phi_0 (j_1, j_2, z, \theta) + \delta \Phi_2 (j_1, j_2, z, \theta)) z = \Phi_0 (j_1, j_2, z, \theta)
\]

where \( \Phi_0 \) also satisfies (15). Now changing coordinates \( j_1 = j_1, j_2 = j_2, z_1 = z + \delta z_2 \), \( \theta = \theta \) we find a local diffeomorphism \( \Phi \) such that

\[
\Phi_0 \cdot \Phi_1 (j_1, j_2, z, \theta) = \Phi_0 (j_1, j_2, z, \theta) + \delta \Phi_2 (j_1, j_2, z, \theta)
\]

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where \( \Phi_0 \) and \( \Phi_1 \) satisfy (15) and (16), respectively. Applying the generalized Morse Lemma we may assume \( \Phi_0 (j_1, j_2, z, \theta) = |j_2|^2 \) while \( \Phi_0 \) still satisfies (16). Applying this argument to \( \Phi_0 \) we obtain (13) since the required change of coordinates leaves \( j_2 \) and \( \theta \) unchanged.

Concluding remarks. It is clear from the proof of Theorem 7 that the asymptotic expansion of the integral

\[
\int_{\mathbb{R}^n} \Gamma(p, \xi(p))(p) \, dp
\]

does not contain logarithmic terms if \( G \subset S^1 \) for \( p \in \text{supp} \xi \). We can use the arguments also to prove a nonexistence in several other special situations, for example if \( \text{dim} \mathbb{V}/\mathbb{C} < 1 \).

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