Integral Representation of Analytical Solutions of the Equation $yf'_x - xf'_y = g(x,y)$

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Equations of the form
\[
\frac{\partial f}{\partial \varphi} = y \frac{\partial f}{\partial x} - x \frac{\partial f}{\partial y} = g(x,y)
\]  

often arise in different averaging problems for systems of differential equations [1–4]. Suppose that $g(x,y)$ is an analytic function of two variables on the plane $\mathbb{R}^2$, $(\rho, \varphi)$ are polar coordinates: $x = \rho \cos \varphi$, $y = \rho \sin \varphi$, and $g$ has zero mean over the angle variable $\varphi$.

The integration of Eq. (1) is a trivial problem, and the solution can be written out up to an arbitrary function $F(\rho)$. Averaging procedures involve work with analytic functions of the two variables $x, y$, and the solution must be estimated by the $C$-norm of the function $g$. Such a problem does not arise if the averaging domain is disjoint with a neighborhood of the origin $x = 0, y = 0$. However, in physical applications, it is precisely this neighborhood that corresponds to solutions with small amplitudes and it may play the central role\(^1\). A straightforward application of the Newton–Leibniz formula gives a solution which is analytic in the variables $\varphi$ and $\rho$, but it is not analytic in $x$ and $y$ in a neighborhood of the origin. On the other hand, the theory justifying estimates of a solution represented by the Fourier series in the $C$-norm is rather laborious (1).

In this short communication, we propose a simple integral formula representing the solution of equation (1), which is analytic in the variables $x$ and $y$ and implies simple estimates in the $C$-norm. In fact, this formula was used earlier in [4], but since it has a general mathematical value and has many potential applications, we believe its separate publication is advisable.

**Theorem.** Equation (1) has a solution $f(x,y)$ which is analytic in all variables if and only if the mean value of the function $g$ of the variable $\varphi$ is trivial:

\[
\int_0^{2\pi} g(\rho \cos \varphi, \rho \sin \varphi) d\varphi = 0,
\]

and each solution has the form

\[
f(x,y) = \frac{1}{2} \left( \int_0^\varphi g(\rho \cos \psi, \rho \sin \psi) d\psi + \int_{\pi}^{\varphi} g(\rho \cos \psi, \rho \sin \psi) d\psi \right) + F(\rho^2),
\]

where $F(Z)$ is an arbitrary analytic function.

\(^1\)An example of such a setup is given by the problem of averaging the motion of a charged particle in the sum of a constant magnetic field and a periodic electric potential $v(x,y)$ (see [4]). The Hamiltonian of this system has the form $H = (p^2 + q^2)/2 + \varepsilon v(p + x, q + y) \equiv I + \varepsilon v(\sqrt{T} \cos \varphi + x, \sqrt{T} \sin \varphi + y)$, $0 < \varepsilon \ll 1$. This function is analytic in the variables $(q,p)$ and is not analytic in the angle-action variables $\varphi$, $I = \rho^2/2$ in a neighborhood of the origin $x = 0, y = 0$.
**Proof.** First, let us make the following change of variables:

\[ x = u + iv, \quad y = v + iu. \]

Consider the Taylor expansion of the function \( g \) in the variables \((u, v)\):

\[
g(u + iv, v + iu) = \sum_{k, l \in \mathbb{N}} g_{kl} u^k v^l \frac{(k + l)!}{k! l!}.
\]

The polar coordinates for the variables \((u, v)\) have the form

\[
u = \frac{pe^{-i\varphi}}{\sqrt{2}}, \quad v = \frac{pe^{i(\varphi - \pi/2)}}{\sqrt{2}}.
\]

Then the function \( g \) can be represented as follows:

\[
g(\rho \cos \varphi, \rho \sin \varphi) = \sum_{k, l \in \mathbb{N}} g_{kl} \frac{(k + l)!}{k! l!} \left( \frac{\rho}{\sqrt{2}} \right)^{k+l} e^{i(l-k)\varphi - il\pi/2}.
\]

This expansion may be regarded as the Fourier series for the function \( g(\rho \cos \varphi, \rho \sin \varphi) \). Now Eq. (1) can readily be solved:

\[
f = \int_0^\varphi g(\rho \cos \psi, \rho \sin \psi) \, d\psi + F_0(\rho) = \sum_{k, l \in \mathbb{N}} g_{kl} \frac{(k + l)!}{k! l!} \left( \frac{\rho}{\sqrt{2}} \right)^{k+l} e^{i(l-k)\psi - il\pi/2} \bigg|_0^\varphi + F_0(\rho)
\]

\[
= \sum_{k, l \in \mathbb{N}} g_{kl} \frac{(k + l)!}{k! l!} \left( \frac{\rho}{\sqrt{2}} \right)^{k+l} e^{i(l-k)\psi - il\pi/2} \bigg|_0^\varphi - \sum_{k, l \in \mathbb{N}} g_{kl} \frac{(k + l)!}{k! l!} \left( \frac{\rho}{\sqrt{2}} \right)^{k+l} e^{-il\pi/2} + F_0(\rho),
\]

where \( F_0(\rho) \) is a “constant” of integration. The first term in Eq. (3) is an analytic function, because it can be rewritten as a polynomial in the variables \( x = \rho \cos \varphi, \ y = \rho \sin \varphi \). The second term is not an analytic function at the origin \( x = 0, \ y = 0 \) for \((k + l)\) odd; therefore, we must cancel the nonanalytic summand by a proper choice of \( F_0(\rho) \). Let us define \( F_0 \) as follows:

\[
F_0(\rho) = \left( \frac{1}{2} \int_0^\varphi g(\rho \cos \psi, \rho \sin \psi) \, d\psi - \frac{1}{2} \int_\pi \varphi g(\rho \cos \psi, \rho \sin \psi) \, d\psi \right) + F(\rho^2).
\]

Now Eq. (4) cancels the nonanalytic term in Eq. (3), and \( f \) is an analytic function of both variables \( x \) and \( y \) at the origin \( x = 0, \ y = 0 \). This proves the theorem. \( \square \)

The corresponding estimates can easily be derived from this formula. Indeed, in the \( C(\Omega) \)-norm, where \( \Omega \) is a domain in \( \mathbb{R}^2 \), we have

\[
\|f\| \leq \frac{1}{2} \left\| \int_0^\varphi g(\rho \cos \psi, \rho \sin \psi) \, d\psi + \int_\pi^\varphi g(\rho \cos \psi, \rho \sin \psi) \, d\psi \right\|_{C(\Omega)} + \|F(\rho^2)\|_{C(\Omega)}
\]

\[
\leq \frac{1}{2} \left( \int_0^{2\pi} \max_{C(\Omega)} |g| \, d\psi + \int_\pi^{2\pi} \max_{C(\Omega)} |g| \, d\psi \right) + \max_{C(\Omega)} |F(\rho^2)| = \frac{3}{2} \pi \max_{C(\Omega)} |g| + \max_{C(\Omega)} |F(\rho^2)|.
\]
REFERENCES


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