The objective of this course is to give an introduction to the probabilistic techniques required to understand the most widely used models of mathematical finance. The course is intended for undergraduate and graduate students in mathematics, but it might also be useful for students in economics and operations research. The focus is on stochastic models in discrete time. This has at least two immediate benefits. First, the underlying probabilistic machinery is much simpler; second the standard paradigm of complete financial markets akin to many continuous time models where all derivative securities admit a perfect hedge does typically not hold and we are confronted with incomplete models at a very early stage. Our discussion of (dynamic) arbitrage theory and risk measures follows the textbook by Föllmer & Schied (2004). The chapter on optimal stopping and American options follows Section 2 of Lamberton & Lapeyre (1996). The final section on optimal derivative design and risk transfer is based on a array of research papers.

Contents

1 Introduction
   1.1 The put-call parity .................................................. 4
   1.2 Naive approaches to option pricing ................................. 5

2 Mathematical finance in one period ................................. 6
   2.1 Assets portfolios and arbitrage opportunities .................. 7
   2.2 The Fundamental Theorem of Asset Pricing ..................... 8
   2.3 Derivative securities ................................................ 14
      2.3.1 Arbitrage bounds and the super-hedging price ............ 16
      2.3.2 Attainable claims .............................................. 19
   2.4 Complete market models and perfect replication ............... 21
   2.5 Good deal bounds and super-deals ............................... 23

3 Dynamic hedging in discrete time .................................. 29
   3.1 The multi-period market model .................................. 29
   3.2 Arbitrage opportunities and martingale measures ............. 32
   3.3 European options and attainable claims ......................... 36
   3.4 Complete markets .................................................. 40
4 Binomial trees and the Cox-Ross-Rubinstein Model 42
  4.1 Delta-Hedgeing in discrete time 44
  4.2 Exotic derivatives 45
    4.2.1 The reflection principle 46
    4.2.2 Valuation formulae for up-and-in call options 47
    4.2.3 Valuation formulae for lookback options 49
  4.3 Convergence to Black-Scholes Prices 49

5 Introduction to Optimal Stopping and American Options 50
  5.1 Motivation and introduction 50
  5.2 Stopping times 52
  5.3 The Snell envelope 53
  5.4 Decomposition of Supermartingales and pricing of American options 55
  5.5 American options in the CRR model 57

6 Introduction to risk measures 60
  6.1 Risk measures and their acceptance sets 60
  6.2 Robust representations of risk measures 64
    6.2.1 Robust representations of convex risk measures 66
    6.2.2 Robust representations in terms of probability measures 68
  6.3 V@R, AV@R and Shortfall Risk 71
    6.3.1 V@R 72
    6.3.2 AV@R 73
    6.3.3 Shortfall Risk 74
  6.4 Law Invariance 75

7 Indifference valuation and optimal derivative design 75
  7.1 A simple model of optimal derivative design 76
    7.1.1 Model characteristics and optimal claims 76
    7.1.2 Bayesian Heterogeneity 77
  7.2 Introducing a financial market 77

8 Optimal risk transfer in principal agent games 77
  8.1 The Microeconomic Setup 78
  8.2 Proof of the Main Theorem 80
    8.2.1 Redistributing risk exposures among agents 85
    8.2.2 Minimizing the overall risk 87
<table>
<thead>
<tr>
<th>CONTENTS</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A Equivalent measures and dominated convergence</td>
<td>88</td>
</tr>
</tbody>
</table>
1 Introduction

Our presentation concentrates on options and other derivative securities. Options are among
the most relevant and widely spread financial instruments. The need to price and hedge
options has been the key factor driving the development of mathematical finance. An option
gives its holder the right, but not the obligation, to buy or sell

- a financial asset (underlying)
- at or before a certain date (maturity)
- at a predetermined price (strike).

Underlyings include, but are not limited to stocks, currencies, commodities (gold, copper,
oil, ...) and options. One usually distinguishes

- European Options that can only be exercised at maturity, and
- American Options that can be exercised any time before maturity.

An option to buy the underlying is referred to as a Call Option while an option to sell
is called at Put Option. Typically $T$ denotes the time to maturity, $K$ the strike and $S_t$ the
price of the underlying at time $t \in [0, T]$. The writer (seller) of European Call option needs
to pay the buyer an amount of

$$\max\{S_T - K, 0\} := (S_T - K)^+$$

at time $T$: if the underlying trades at some price $S_T < K$ at maturity the buyer will not
exercise the option while she will exercise her right to buy the option at the predetermined
price $K$ when the market price of the underlying at time $T$ exceeds $K$. As a result the writer
of the option bears an unlimited risk. The question, then, is how much the writer should charge
the buyer in return for taking that risk (Pricing the Option) and how that money should be
invested in the bond and stock market to meet his payment obligations at maturity (Hedging
the Option). To answer these questions we will make some basic assumptions. A commonly
accepted assumption in every financial market model is the absence of arbitrage opportunities
(“No free Lunch”). It states that there is no riskless profit available in the market.

1.1 The put-call parity

Based solely on the assumption of no arbitrage we can derive a formula linking the price of
a European call and put option with identical maturities $T$ and strikes $K$ written on the
same underlying. To this end, let $C_t$ and $P_t$ be the price of the call and put option at time $t \in [0, T]$, respectively. The no free lunch condition implies that

$$C_t - P_t = S_t - K e^{-r(T-t)}$$

where $r \geq 0$ denotes the risk free interest rate paid by a government bond. If fact, if we had

$$C_t - P_t > S_t - K e^{-r(T-t)}$$

we would buy one stock and the put and sell the call. The net value of this transaction is

$$C_t - P_t - S_t.$$

If this amount is positive, we would deposit it in a bank account where it earns interest at the rate $r$; if it were negative we would borrow the amount paying interest at rate $r$. If

$$S_T > K$$

the option will be exercised and we deliver the stock. In return we receive $K$ and close the bank account. The net value of the transaction is positive:

$$K + e^{r(T-t)}(C_t - P_t - S_t) > 0.$$

If, on the other hand,

$$S_T \leq K$$

then we exercise our right to sell our stock at $K$ and close the bank account. Again the net value of the transaction is positive:

$$K + e^{r(T-t)}(C_t - P_t - S_t) > 0.$$

In both cases we locked in a positive amount without making any initial investment which were an arbitrage opportunity. Similar considerations apply when

$$C_t - P_t < S_t - K e^{-r(T-t)}.$$

### 1.2 Naive approaches to option pricing

The “no free lunch condition” implies that discounted conditional expected payoffs are typically no appropriate pricing schemes as illustrated by the following example.

**Example 1.1** Consider a European option with strike $K$ and maturity $T = 1$ written on a single stock. The asset price $\pi_0$ at time $t = 0$ is known while the price at maturity is given by the realization of a random variable $S$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. More
specifically, consider a simple “coin flip” where \( \Omega = \{ w_-, w_+ \} \) and \( \mathbb{P} = (0.5, 0.5) \) and assume that

\[
\pi = 100, \quad K = 100, \quad \text{and} \quad S(\omega) = \begin{cases} 
120 & \text{if } \omega = w_+ \\
90 & \text{if } \omega = w_- .
\end{cases}
\]

If the price \( \pi(C) \) of the call option with payoff \( C = (S - 100)^+ \) were \( \frac{10}{1+r} \), then the following is an arbitrage opportunity as long as the risk-less interest rate is less than 5%:

- buy the option at \( \frac{10}{1+r} \);
- borrow the dollar amount \( \frac{60}{1+r} \) at the risk free rate;
- buy \( \frac{2}{3} \) shares of the stock.

A direct calculation shows that the value of the portfolio at time \( t = 1 \) is zero in any state of the world while it yields the positive cash-flow \( \frac{70}{1+r} - \frac{2}{3} 100 \) at time \( t = 0 \).

A second approach that dates back to Bernoulli is based on the idea of indifference valuation. The idea is to consider an investor with an initial wealth \( W \) whose preferences over income streams are described by a strictly concave, increasing utility function \( u : \mathbb{R} \to \mathbb{R}^+ \) and to price the option by its certainty equivalent. The certainty equivalent \( \pi(C) \) is defined as the unique price that renders an expected-utility maximizing investor indifferent between holding the deterministic cash amount \( W + \pi(C) \) and the random payoff \( W + C \); that is:

\[
u(W + \pi(C)) = \mathbb{E}[u(W + C)].
\]

Among the many drawbacks of this approach is the fact that \( \pi(C) \) is not a market price. Investors with heterogeneous preferences are charged different prices. It will turn out that options and other derivative securities can in fact be priced without any preference to preferences of market participants.

2 Mathematical finance in one period

Following Chapter 1 of Föllmer & Schied (2004), this section studies the mathematical structure of a simple one-period model of a financial market. We consider a finite number of assets whose initial prices at time 0 are known while their future prices at time 1 are described by random variables on some probability space. Trading takes place at time \( t = 0 \).
2.1 Assets portfolios and arbitrage opportunities

Consider a financial market model with \( d + 1 \) assets. In a one-period model the assets are priced at the initial time \( t = 0 \) and the final time \( t = 1 \). We assume that the \( i \)-th asset is available at time 0 for a price \( \pi^i \geq 0 \) and introduce the price system
\[
\pi = (\pi^0, \ldots, \pi^d) \in \mathbb{R}^{d+1}_+.
\]

In order to model possible uncertainty about prices in the following period \( t = 1 \) we fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and describe the asset prices at time 1 as non-negative measurable functions
\[
S^0, S^1, \ldots, S^d
\]
on \((\Omega, \mathcal{F})\). Notice that not all asset prices are necessarily uncertain. In fact, there is usually a riskless bond that pays a sure amount at time 1. We include such a bond assuming that
\[
\pi^0 = 1 \quad \text{and} \quad S^0 \equiv 1 + r
\]
for a constant deterministic rate of return \( r \geq 0 \). To distinguish the bond from the risky assets we conveniently write
\[
\bar{S} = (S^0, S^1, \ldots, S^d) = (S^0, S) \quad \text{and} \quad \bar{\pi} = (1, \pi).
\]

Every \( \omega \in \Omega \) corresponds to a particular scenario of market evolution, and \( S^i(\omega) \) is the price of the \( i \)-th asset at time 1 if the scenario \( \omega \) occurs.

A portfolio is a vector \( \tilde{\xi} = (\tilde{\xi}^0, \xi) \in \mathbb{R}^{d+1} \) where \( \xi^i \) represents the number of shares of the \( i \)-th asset. Notice that \( \xi^i \) may be negative indicating that an investor sells the asset short. The price for buying the portfolio \( \tilde{\xi} \) equals
\[
\bar{\pi} \cdot \tilde{\xi} = \sum_{i=0}^{d} \pi^i \xi^i.
\]

At time \( t = 1 \) the portfolio has the value
\[
\tilde{\xi} \cdot \bar{S}(\omega) = \xi^0(1 + r) + \xi \cdot S(\omega)
\]
depending on the scenario \( \omega \). With this we are now ready to formally introduce the notion of an arbitrage opportunity, an investment strategy that yields a positive profit in some states of the world without being exposed to any downside risk.

**Definition 2.1** A portfolio \( \tilde{\xi} \) is called an arbitrage opportunity if \( \bar{\pi} \cdot \tilde{\xi} \leq 0 \) but
\[
\mathbb{P}[\tilde{\xi} \cdot \bar{S} \geq 0] = 1 \quad \text{and} \quad \mathbb{P}[\tilde{\xi} \cdot \bar{S} > 0] > 0.
\]
We notice that the “real-world-probability-measure” \( \mathbb{P} \) enters the definition of arbitrage only through the null sets of \( \mathbb{P} \). In particular, the assumption can be stated without reference to probabilities if \( \Omega \) is countable (or finite). In this case we can with no loss of generality assume that
\[
\mathbb{P}[\{\omega\}] > 0 \quad \text{for all} \quad \omega \in \Omega
\]
and an arbitrage strategy is simply a portfolio \( \bar{\xi} \) such that \( \bar{\pi} \cdot \bar{\xi} \leq 0 \) but
\[
\bar{\xi} \cdot \bar{S}(\omega) \geq 0 \quad \text{for all} \quad \omega \in \Omega \quad \text{and} \quad \bar{\xi} \cdot \bar{S}(\omega_0) > 0 \quad \text{for at least one} \quad \omega_0 \in \Omega.
\]

**Definition 2.2** A portfolio \( \xi \) is call self-financing if \( \bar{\pi} \cdot \bar{\xi} = 0 \), i.e., if the investor merely re-balances her wealth.

It should be intuitively clear that in the absence of arbitrage opportunities any investment in risky assets which yields with positive probability a better result than investing the same amount in the risk-free asset must carry some downside risk. The following lemma confirms this intuition. Its proof is left as an exercise.

**Lemma 2.3** The following statements are equivalent.

(i) The market model admits an arbitrage opportunity.

(ii) There is a vector \( \xi \in \mathbb{R}^d \) such that
\[
\mathbb{P}[\xi \cdot S \geq (1 + r)\xi \cdot \pi] = 1 \quad \text{and} \quad \mathbb{P}[\xi \cdot S > (1 + r)\xi \cdot \pi] > 0.
\]

(iii) There exists a self-financing arbitrage opportunity.

The assumption of no arbitrage is a condition imposed for economic reasons. In the following section we characterize arbitrage free models in a mathematically rigorous manner.

### 2.2 The Fundamental Theorem of Asset Pricing

In this section we link the “no free lunch” condition on a financial market model to the existence of equivalent martingale measures. To this end, let a financial market model along with a price system be given:
\[
\bar{\pi} = (1, \pi), \quad \bar{S} = (1 + r, S) \quad \text{where} \quad (S^i)_{i=1}^d \text{ are random variables on } (\Omega, \mathcal{F}, \mathbb{P}).
\]

**Definition 2.4** The financial market model is called arbitrage-free if no free lunch exists, i.e., for all portfolios \( \xi \in \mathbb{R}^{d+1} \) that satisfy \( \bar{\xi} \cdot \bar{\pi} \leq 0 \) and \( \bar{\xi} \cdot \bar{S} \leq 0 \geq 0 \) almost surely, we have
\[
\bar{\xi} \cdot \bar{S} \leq 0 = 0 \quad \mathbb{P}-a.s.
\]
The notion is risk-neutral or martingale measures is key in mathematical finance.

**Definition 2.5** A probability measure $P^*$ on $(\Omega, \mathcal{F})$ is called a risk-neutral or martingale measure if asset prices at time $t = 0$ can be viewed as expected discounted future payoffs under $P^*$, i.e., if

$$\pi^i = E^*[\frac{S^i}{1+r}].$$

The fundamental theorem of asset pricing states that the set of arbitrage free prices can be linked to the set of equivalent martingale measures

$$\mathcal{P} = \{P^* : P^* \text{ is a martingale measure and } P^* \approx P\}.$$ 

We notice that the characterization of arbitrage free prices does not take into account the preferences of the market participants. It is entirely based on the assumption of no arbitrage. Furthermore, recall that the real-world-measure $P$ enters the definition of no-arbitrage only through its null sets. The assumption that any $P^* \in \mathcal{P}$ is equivalent to $P$, i.e., that

$$P[A] = 0 \iff P^*[A] = 0 \quad (A \in \mathcal{F})$$

guarantees that the null sets are the same; we recall some of the basic properties of equivalent measures as well Lebesgue’s theorem on dominated convergence in the appendix.

In order to state and prove the fundamental theorem of asset pricing it will be convenient to use the following notation: we denote by

$$X^i = \frac{S^i}{1+r} \quad \text{and} \quad Y^i = X^i - \pi^i \quad (i = 0, 1, \ldots, d)$$

the discounted payoff of the $i$-th asset and the net gain from trading asset $i$, respectively. In terms of these quantities the no arbitrage condition reads

$$P[\xi \cdot Y \geq 0] = 1 \Rightarrow P[\xi \cdot Y = 0] = 1$$

and $P^*$ is an EMM if and only if $E^*[Y^i] = 0$ for every asset.

**Theorem 2.6** A market model is arbitrage free if and only if $\mathcal{P} \neq \emptyset$. In this case, there exists a $P^* \in \mathcal{P}$ which has a bounded density with respect to $P$.

**Proof:**

(i) In a first step we assume that $P^*$ is an equivalent martingale measure and fix a portfolio $\xi \in \mathbb{R}^d$ such that

$$P[\xi \cdot Y \geq 0] = 1.$$
This property remains valid if we replace \( P \) by \( P^* \). Hence,
\[
E^*[\xi \cdot Y] = \sum_i \xi^i E^*[Y^i] = 0
\]
because \( E^*[Y^i] = 0 \). Thus, any portfolio that has no downside risk and some upside potential has a strictly positive price so there are no arbitrage opportunities.

(ii) For the converse implication we first consider the case where \( Y \in L^1(P) \). Let \( Q \) be the convex set of all probability measures that are equivalent to \( P \) with a bounded density. For \( Q \in Q \) we have
\[
E_Q[|Y^i|] = E_P \left[ \frac{dQ}{dP} |Y^i| \right] < \infty
\]
because the densities are bounded so
\[
\mathcal{K} := \{ E_Q[Y] : Q \in Q \} \subset \mathbb{R}^d
\]
is convex. By definition \( Q \) is a martingale measure if and only if \( E_Q[Y] = 0 \). Hence an EMM with bounded density exists if and only if
\[
0 \in \mathcal{K}.
\]
Suppose to the contrary that \( \mathcal{K} \) does not contain the origin. We show that in this case they were arbitrage opportunities. In fact the separating hyperplane theorem implies the existence of some vector \( \xi \in \mathbb{R}^d \) such that \( \xi \cdot x \geq 0 \) for all \( x \in \mathcal{K} \) and \( \xi \cdot x_0 > 0 \) for some \( x_0 \in \mathcal{K} \). In other words
\[
E_Q[\xi \cdot Y] \geq 0 \quad \forall \ Q \in Q \quad \text{and} \quad E_{Q_0}[\xi \cdot Y] > 0 \quad \text{for some} \quad Q_0 \in Q.
\]
As a result
\[
Q_0[\xi \cdot Y > 0] > 0 \quad \text{and hence} \quad P[\xi \cdot Y > 0] > 0
\]
because \( P \) and \( Q_0 \) are equivalent. Our goal is then to show that the preceding inequality along with the fact that \( E_Q[\xi \cdot Y] \geq 0 \) for all \( Q \in Q \) implies that
\[
\xi \cdot Y \geq 0 \quad P\text{-a.s.}
\]
i.e., an arbitrage opportunity. To this end, let us put
\[
A := \{ \xi \cdot Y < 0 \} \quad \text{and put} \quad \varphi_n := \left( 1 - \frac{1}{n} \right) 1_A + \frac{1}{n} 1_{A^c}.
\]
If \( P[A] = 0 \), the proof is finished. Let us therefore assume to the contrary that \( P[A] > 0 \) and take \( \varphi_n \) as a density for a new probability measure \( Q_n \):
\[
\frac{dQ_n}{dP_n} := \frac{1}{E[\varphi_n]} \cdot \varphi_n
\]
We have that $Q_n \in Q$ and hence that

$$0 \leq \xi \cdot E_{Q_n}[Y] = \frac{1}{E[\varphi_n]} \cdot E[\xi \cdot Y \varphi_n]$$

Since $Y \in L^1(\mathbb{P})$ we can take the limit as $n \to \infty$ due to Lebesgue’s dominated convergence theorem to obtain $0 \leq \frac{E[\xi Y 1_A]}{P[A]}$. In particular,

$$P[\xi \cdot Y \geq 0] = 1$$

contradicting our assumption $P[A] > 0$. Hence $P[A] = 0$ so $\mathcal{K}$ contains the origin and there exists an EMM with bounded density.

If $Y \notin L^1(\mathbb{P})$ we change the reference measure. Specifically, we choose a probability measure $\hat{\mathbb{P}} \approx \mathbb{P}$ with bounded density $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}$ and for which $\hat{\mathbb{E}}[|Y|] < \infty$. Such a measure can for instance be obtained by taking

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \frac{c}{1 + |Y|} \quad \text{for} \quad c = \left(\mathbb{E} \left[ \frac{1}{1 + |Y|} \right] \right)^{-1}.$$ 

Replacing $\mathbb{P}$ by $\hat{\mathbb{P}}$ does not affect the absence of arbitrage opportunities. Thus, the first part of the proof yields a martingale measure $\mathbb{P}^*$ that has a bounded density with respect to $\hat{\mathbb{P}}$. But then

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \frac{d\mathbb{P}^*}{d\hat{\mathbb{P}}} \cdot \frac{d\hat{\mathbb{P}}}{d\mathbb{P}}$$

is bounded so $\mathbb{P}^*$ is the desired EMM.

The following example that shows that the implication “$\Rightarrow$” of the fundamental theorem of asset pricing that the absence of arbitrage implies the existence of an EMM may not hold in markets with infinitely many assets.

**Example 2.7** Let $\Omega = \{1, 2, \ldots\}$ and the risk free rate be zero and consider assets with initial prices and payoffs given by

$$\pi^i = 1 \quad \text{and} \quad S^i(\omega) = \begin{cases} 
0 & \omega = i \\
2 & \omega = i + 1 \\
1 & \text{else}
\end{cases}$$

We consider only portfolios $\xi \in l^1$ that is, $\sum_{i=0}^{\infty} |\xi^i| < \infty$ and assume that there exists an EMM $\mathbb{P}^*$. In this case

$$1 = \pi^i = \mathbb{E}^*[S^i] = 2\mathbb{P}^*\{i + 1\} + \sum_{k=1,k\neq i,i+1} \mathbb{P}\{k\} = 1 + \mathbb{P}^*\{i + 1\} - \mathbb{P}^*\{i\}$$
so

\[ P^*[\{i\}] = P^*[\{i + 1\}] = P^*[\{i + 2\}] = \ldots \]

which is not possible. The model is, however, free of arbitrage. In fact, let \( \bar{\xi} = (\xi^0) \in l^1 \) be such that

\[ P[\bar{\xi} \cdot \bar{S} \geq 0, \bar{\xi} \cdot \bar{\pi} \leq 0] = 1. \]

For \( \omega = 1 \) this yields

\[ 0 \leq \bar{\xi} \cdot \bar{S}(1) = \xi^0 + \sum_{k=2}^{\infty} \xi^k = \bar{\pi} \cdot \bar{\xi} - \xi^1 \leq -\xi^1 \]

and by analogy for \( \omega > 1 \):

\[ 0 \leq \bar{\xi} \cdot \bar{S}(\omega) \leq \xi^{i-1} - \xi^i. \]

Thus,

\[ 0 \geq \xi^1 \geq \xi^2 \geq \ldots \]

which is compatible with \( \xi \in l^1 \) only if \( \xi^i = 0 \) in which case

\[ \xi \cdot S = 0 \quad P\text{-a.s.} \]

**Remark 2.8** For the special case of a finite set \( \Omega = \{\omega_1, \ldots, \omega_n\} \) and a single risky asset equivalent martingale measures satisfy two simple linear equations. To see this, let \( p_i = P[\{\omega_i\}] \) and \( s_i = S^1(\omega_i) \) and assume without loss of generality that

\[ p_i > 0 \quad \text{and} \quad s_1 < s_2 < \ldots < s_n. \]

An equivalent martingale measure is then a vector \( p^* = (p_1^*, \ldots, p_n^*) \) with positive entries such that

\[ s_1 p_1^* + \cdots + s_n p_n^* = \pi^1(1 + r) \]

\[ p_1^* + \cdots + p_n^* = 1. \] (2)

If a solution exists it will be unique if and only if \( n = 2 \). If there are more than two states of the world and just one asset, there will be infinitely many solutions.

Let us denote by \( V \) the linear space of all attainable payoffs:

\[ V := \{\bar{\xi} \cdot \bar{S} : \bar{\xi} \in \mathbb{R}^{d+1}\}. \] (3)

The portfolio that generates \( V \in V \) is not necessarily unique but in an arbitrage free market the law of one price prevails:

If \( V \in V \) can be written as \( V = \bar{\xi} \cdot \bar{S} = \bar{\zeta} \cdot \bar{S} \) then \( \bar{\pi} \cdot \bar{\xi} = \bar{\pi} \cdot \bar{\zeta} \).
In particular, it makes sense to define the price of $V \in \mathcal{V}$ in terms of a linear form $\pi$ on the finite-dimensional vector space $\mathcal{V}$. For any $\mathbb{P}^* \in \mathcal{P}$ we have

$$\pi(V) = \mathbb{E}^* \left[ \frac{V}{1 + r} \right].$$

For an attainable payoff $V$ such that $\pi(V) \neq 0$ the return of $V$ is defined by

$$R(V) = \frac{V - \pi(V)}{\pi(V)}.$$

For the special case of the risk free asset $S^0$ we have that

$$r = \frac{S^0 - \pi_0}{\pi_0}.$$

It turns out that in an arbitrage free market the expected return under any EMM equals the risk free rate.

**Proposition 2.9** Suppose that the market model is free of arbitrage and let $V \in \mathcal{V}$ be an attainable payoff with price $\pi(V) \neq 0$.

(i) Under any $\mathbb{P}^* \in \mathcal{P}$ the expected return of $V$ is equal to the risk free rate:

$$\mathbb{E}^*[R(V)] = r.$$

(ii) Under any measure $\mathbb{Q} \approx \mathbb{P}$ such that $\mathbb{E}_\mathbb{Q}[|\bar{S}|] < \infty$ the expected return is given by

$$\mathbb{E}_\mathbb{Q}[R(V)] = r - \text{cov}_\mathbb{Q} \left( \frac{d\mathbb{P}^*}{d\mathbb{Q}}, R(V) \right)$$

where $\mathbb{P}^*$ is any martingale measure and $\text{cov}_\mathbb{Q}$ is the covariance with respect to $\mathbb{Q}$.

**Proof:**

(i) Let $\bar{\xi}$ be such that $V = \bar{\xi} \cdot \bar{S}$. Since $\mathbb{E}^*[\bar{\xi} \cdot \bar{S}] = (1 + r)\bar{\pi} \cdot \bar{\xi}$ we have

$$\mathbb{E}^*[R(V)] = \mathbb{E}^*[\bar{\xi} \cdot \bar{S}] - \bar{\pi} \cdot \bar{\xi} = r.$$

(ii) Let $\mathbb{P}^*$ be any martingale measure and $\varphi^* = \frac{d\mathbb{P}^*}{d\mathbb{Q}}$. Then

$$\text{cov}_\mathbb{Q}(\varphi^*, R(V)) = \mathbb{E}_\mathbb{Q}[\varphi^* R(V)] - \mathbb{E}_\mathbb{Q}[\varphi^*] \cdot \mathbb{E}_\mathbb{Q}[R(V)]$$

$$= \mathbb{E}^*[R(V)] - \mathbb{E}_\mathbb{Q}[R(V)].$$

Hence the assertion follows form part (i).
Remark 2.10 So far we considered only the (standard) case where the riskless bond is the numéraire, that is, where all prices were expressed in terms of shares of the bond. The numéraire can be changed and prices be quoted in terms of, for instance, the 1st asset (provided its price $S^1$ is almost surely strictly positive) without altering the main results of this section. In order to see this, let 

$$\tilde{\pi}^i = \frac{\pi^i}{\pi^1} \quad \text{and} \quad \tilde{S}^i = \frac{S^i}{S^1}.$$ 

The no arbitrage condition with respect to the new numéraire reads:

$$\tilde{\pi}^i \equiv \tilde{E}^* \left[ \tilde{S}^i \right] = \tilde{E}^* \left[ \frac{S^i}{S^1} \right].$$

It is satisfied, for instance, for the measure $\tilde{P}^*$ with density

$$\frac{d\tilde{P}^*}{dP^*} = \frac{S^1}{E^*[S^1]} \quad (P^* \in \mathcal{P}^*).$$

2.3 Derivative securities

In real financial markets not only primary but also a large variety of derivative securities such as options and futures are traded. A derivative’s payoff depends in a possibly non-linear way on the primary assets $S^0, S^1, \ldots, S^n$.

Definition 2.11 A contingent claim is a random variable $C$ on the probability space $(\Omega, \mathcal{F}, P)$ such that

$$0 \leq C < \infty \quad \mathbb{P}\text{-a.s.}$$

A contingent claim is called a derivative if

$$C = f(S^0, S^1, \ldots, S^d)$$

for some non-negative measurable function $f$ on $\mathbb{R}^{d+1}$.

Example 2.12 (i) For a European Call Option on the first risky asset with strike $K$ we have that

$$f(S^1) = (S^1 - K)^+. \quad \text{(i)}$$

(ii) For a European Put Option on the first risky asset with strike $K$ we have that

$$f(S^1) = (K - S^1)^+. \quad \text{(ii)}$$

By the call-put-parity discussed in the introduction the price of a put option is determined by the price of the corresponding call option and vice versa.
(iii) For a forward contract on the first risky asset with strike $K$ we have that
\[ f(S^1) = S^1 - K. \]

(iv) A straddle is a bet that the price $\pi(V)$ of a portfolio with payoff $V$ moves, no matter in what direction:
\[ C = |V - \pi(V)|. \]
A butterfly spread, by contrast is a bet that the price does not move much: for $0 \leq a \leq b$
\[ C = (V - a)^+ + (V - b)^+ - 2(V - (a + b)/2)^+. \]

(v) A reverse convertible bond pays interest that is higher than that paid by a riskless bond. However, at maturity the issuer has the right to convert the bond into a predetermined number of shares of a given asset $S^i$ rather than paying the nominal value in cash. Suppose that the reverse convertible bond trades at $\$1$ at $t = 0$, that its nominal value at maturity $t = 1$ is $1 + \tilde{r}$ and that it can be converted into $x$ shares of the $i$-th asset. The conversion will happen if
\[ S^i < K := \frac{1 + \tilde{r}}{x}. \]
As a result, the purchase of the bond is equivalent to a risk-free investment of 1 with interest $r$ and the sale of $x$ put options with payoff $(K - S^i)$ for a unit price $(\tilde{r} - r)/(1 + r)$.

Our goal is to identify those possible prices for $C$ which do not generate arbitrage opportunities. To this end we observe that trading $C$ at time 0 for a price $\pi^C$ corresponds to introducing a new asset with prices
\[ \pi^{d+1} := \pi^C \quad \text{and} \quad S^{d+1} = C. \]
We call $\pi^C$ an arbitrage free price of $C$ if the market model extended in this manner is free of arbitrage. The set of all arbitrage free prices is denoted $\Pi(C)$. It can be characterized in terms of the equivalent martingale measures.

\textbf{Theorem 2.13} Suppose that $\mathcal{P} \neq \emptyset$. Then the following holds:
\[ \Pi(C) = \left\{ \mathbb{E}^* \left[ \frac{C}{1 + r} \right] : \mathbb{P}^* \in \mathcal{P} \text{ such that } \mathbb{E}^*[C] < \infty \right\} \neq \emptyset. \]

\textbf{Proof:}

“$\subset$”: Let $\pi^C$ be an arbitrage free price. By definition there exists a measure $\hat{\mathbb{P}} \approx \mathbb{P}$ such that
\[ \pi^i = \hat{\mathbb{E}} \left[ \frac{S^i}{1 + r} \right], \quad \pi^C = \hat{\mathbb{E}} \left[ \frac{C}{1 + r} \right] < \infty. \]
In particular, $\hat{\mathbb{P}}$ is an EMM of the original model.
“⊃”: Let $P^*$ be an EMM of the original model such that $E^*[C] < \infty$. If we put

$$\pi^C = E^* \left[ \frac{C}{1 + r} \right]$$

then $P^*$ is an EMM of the extended model which is therefore free of arbitrage.

“≠ ∅”: It remains to show that $\Pi(C) \neq \emptyset$, i.e., that there is an EMM with respect to which $C$ is integrable. To this end, we fix some measure $\tilde{P}$ such that $\tilde{E}[C] < \infty$. We can, for instance, take

$$\frac{d\tilde{P}}{dP} = \frac{c}{1 + C}$$

where $c$ is the normalizing constant. Under $\tilde{P}$ the market is arbitrage free and by Theorem 2.6 there exists $P^* \in \mathcal{P}$ that has a bounded density with respect to $\tilde{P}$. In particular, $E^*[C] < \infty$ and

$$\pi(C) = E^*[C/(1 + r)] \in \Pi(C).$$

\[\Box\]

2.3.1 Arbitrage bounds and the super-hedging price

The preceding theorem yields a unique arbitrage free price of a contingent claim $C$ if and only if there exists a unique EMM. For the special case of a finite set $\Omega = \{\omega_1, \ldots, \omega_n\}$ and a single risky asset this the case only if $\Omega$ has at most two elements; see (2). In general $\Pi(C)$ is a convex interval:

$$\Pi(C) = [\pi_{\text{min}}(C), \pi_{\text{max}}(C)].$$

Our next goal is thus to identify the arbitrage bounds $\pi_{\text{min}}(C)$ and $\pi_{\text{max}}(C)$. We first identify the upper bound $\pi_{\text{max}}(C)$.

**Lemma 2.14** For an arbitrage free market model the upper arbitrage bound is given by

$$\pi_{\text{max}}(C) = \sup_{P^* \in \mathcal{P}} E^* \left[ \frac{C}{1 + r} \right]$$

$$= \min \{m \in [0, \infty] : \exists \xi \in \mathbb{R}^d \text{ s.t. } (1 + r)m + \xi \cdot Y \geq C \text{ } \mathbb{P}\text{-a.s.} \}$$

**Proof:** Let

$$\mathcal{M} := \{m \in [0, \infty] : \exists \xi \in \mathbb{R}^d \text{ s.t. } (1 + r)m + \xi \cdot Y \geq C \text{ } \mathbb{P}\text{-a.s.} \}$$

Taking the expectation with respect to $P^*$ yields $m \geq E^*[C/(1 + r)]$ and hence

$$\inf \mathcal{M} \geq \sup_{P^* \in \mathcal{P}} E^* \left[ \frac{C}{1 + r} \right] \geq \sup \left\{ E^* \left[ \frac{C}{1 + r} \right] : P^* \in \mathcal{P}, \ E^*[C] < \infty \right\} = \sup \Pi(C).$$
We need to prove that all the inequalities are in fact equalities. This is trivial if $\sup \Pi(C) = \infty$. Hence we may assume that $\sup \Pi(C) < \infty$ and for this case we show that $m > \sup \Pi(C)$ implies $m \geq \inf M$.

By definition $\sup \Pi(C) < m < \infty$ requires the existence of an arbitrage opportunity $(\xi, \xi^{d+1})$ in the market model extended by $\pi^{d+1} := m$ and $S^{d+1} := C$. We will short the asset so $\xi^{d+1} < 0$ and we can define

$$\zeta := -\frac{1}{\xi^{d+1}} \xi \in \mathbb{R}^d$$

to obtain

$$(1 + r)m + \zeta \cdot Y \geq C \quad \mathbb{P}\text{-a.s.} \quad \text{hence} \quad m \geq \inf M.$$  

Let us then show that the infimum is in fact attained. For this we may w.l.o.g. assume that $\inf M < \infty$. In view of the previous step we may, for a sequence $\{m_n\}$ that converges to $\inf M$ consider portfolios $\xi_n \in \mathbb{R}^d$ such that

$$(1 + r)m_n + \xi_n \cdot Y \geq C \quad \mathbb{P}\text{-a.s.}$$

If $\lim \inf_n |\xi_n| < \infty$ we can extract a subsequence that converges to $\xi$ and pass to the limit in the equation above:

$$(1 + r)\inf M + \xi \cdot Y \geq C \quad \mathbb{P}\text{-a.s.}$$

Suppose now to the contrary that $\lim \inf_n |\xi_n| = \infty$ and consider a convergent subsequent of the normalized sequence $\eta_n := \xi_n / |\xi_n|$. We obtain

$$\frac{1 + r}{|\xi_n|} m_n + \eta_n \cdot Y \geq \frac{C}{|\xi_n|} \quad \mathbb{P}\text{-a.s.}$$

Passing to the limit we see that

$$\eta \cdot Y \geq 0 \quad \mathbb{P}\text{-a.s.}$$

The absence of arbitrage implies $\eta \cdot S = 0$ almost surely whence $\eta = 0$. This, however, contradicts $|\eta| = 1$ so $\lim \inf_n |\xi_n| < \infty$.

\begin{remark}
When calculating the arbitrage bounds we may as well take the inf and sup over the set of risk neutral measures that are merely absolutely continuous with respect to $\mathbb{P}$.
\end{remark}

\textbf{Proof:} Let

$$\tilde{\mathcal{P}} = \{\tilde{\mathbb{P}} \ll \mathbb{P} : \mathbb{P} \text{ risk neutral MM} \} \supset \mathcal{P}.$$  

The set $\tilde{\mathcal{P}}$ is convex. Moreover, we have that

$$\pi_{\max}(C) \leq \sup_{\tilde{\mathbb{P}} \in \tilde{\mathcal{P}}} \tilde{\mathbb{E}} \left[ \frac{C}{1 + r} \right] \quad \text{and} \quad \pi_{\min}(C) \geq \inf_{\tilde{\mathbb{P}} \in \tilde{\mathcal{P}}} \tilde{\mathbb{E}} \left[ \frac{C}{1 + r} \right].$$
In order to see the converse inequalities we fix \( \varepsilon > 0 \). For any \( \tilde{\mathbb{P}} \in \tilde{\mathcal{P}} \) and \( \mathbb{P}^* \in \mathcal{P} \) let us then define
\[
\mathbb{P}_\varepsilon = \varepsilon \mathbb{P}^* + (1 - \varepsilon) \tilde{\mathbb{P}},
\]
Now the assertion follows from
\[
E_\varepsilon[C] = \varepsilon E^*[C](1 - \varepsilon) + \tilde{E}[C]
\]
if we let \( \varepsilon \to 0 \).

The upper arbitrage bound \( \pi_{\text{max}}(C) \) is the so-called super-hedging price. This is the minimal amount of money the writer of the option has to ask for in order to be able to buy a portfolio \( \xi \) that allows her to meet her obligations from selling the option in any state of the world. In many situations this price is trivial. The writer of a Call option, for instance, can always hedge her risk from selling the option via a “buy-and-hold-strategy”, that is, by buying the underlying asset \( S_i \) at \( \pi_i \) in \( t = 0 \) and holding it until maturity. The following example shows that \( \pi_i \) may in fact be the super-hedging price.

**Example 2.16** Consider a market model with a single risky asset \( S^1 \) and assume that under the real world measure \( \mathbb{P} \) the random variable \( S^1 \) has a Poisson distribution:
\[
\mathbb{P}[S^1 = k] = \frac{e^{-1}}{k!} \quad (k = 0, 1, \ldots)
\]

For \( r = 0 \) and \( \pi = 1 \) the measure \( \mathbb{P} \) is risk neutral\(^1\) so the model is arbitrage free. By Remark 2.15 we may take the inf and sup over the set of risk neutral measures that are merely absolutely continuous with respect to \( \mathbb{P} \). Let us then consider probability measures \( \tilde{\mathbb{P}}_n \) with densities
\[
g_n(k) := (e - \frac{e}{n}) * 1_{\{0\}}(k) + (n - 1)! * e * 1_{\{n\}}(k) \quad (k = 0, 1, 2, \ldots)
\]

It is straightforward to check that
\[
\tilde{\mathbb{E}}_n[(S^1 - K)^+] = \left(1 - \frac{K}{n}\right)^+.
\]

Letting \( n \to \infty \) we that the upper arbitrage bound is in fact attained:
\[
\pi_{\text{max}}((S^1 - K)^+) = \lim_{n \to \infty} \left(1 - \frac{K}{n}\right)^+ = 1 = \pi.
\]

The next example establishes upper and lower arbitrage bounds for derivatives whose payoff is a convex function of some underlying.

\(^1\)Recall that the expected value of a standard Poisson random variable is 1.
Example 2.17 Let $V$ be the payoff of some portfolio and $f : [0, \infty) \to \mathbb{R}_+$ convex such that
\[
\beta := \lim_{x \to \infty} \frac{f(x)}{x}
\]
exists and is finite. Convexity of $f$ implies that
\[
f(x) \leq \beta x + f(0)
\]
so the derivative with payoff $C(V) = f(V)$ satisfies
\[
C \leq \beta V + f(0).
\]
Thus, any arbitrage free price $\pi_C$ satisfies
\[
\pi_C = \mathbb{E}^* \left[ \frac{C}{1 + r} \right] \leq \beta \pi(V) + f(0).
\]
In order to obtain a lower bound, we apply Jensen’s inequality to obtain
\[
\pi_C \geq \frac{1}{1 + r} f(\mathbb{E}^*[V]) = \frac{1}{1 + r} f(\pi(V)(1 + r)).
\]
We close this section with a result on the lower arbitrage bound. The proof proceeds by analogy to that of Lemma 2.14.

Lemma 2.18 For an arbitrage free market model the lower arbitrage bound is given by
\[
\inf \Pi(C) = \inf_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^* \left[ \frac{C}{1 + r} \right] = \max \{ m \in [0, \infty] : \exists \xi \in \mathbb{R}^d \text{ s.t. } (1 + r)m + \xi \cdot Y \leq C \text{ } \mathbb{P} \text{-a.s.} \}
\]

2.3.2 Attainable claims

For a portfolio $\bar{\xi}$ the resulting payoff $V = \bar{\xi} \cdot \bar{S}$, if positive, may be viewed as a contingent claim. Those claims that can be replicated by a suitable portfolio will play a special role in the sequel.

Definition 2.19 A contingent claim $C$ is called attainable (replicable, redundant) if $C = \bar{\xi} \cdot \bar{S}$ for some $\bar{\xi} \in \mathbb{R}^{d+1}$. The portfolio $\bar{\xi}$ is called a replicating portfolio for $C$.

If a contingent claims is attainable the law of one price implies that the price of $C$ equals the cost of its replicating portfolio. The following corollary shows that the attainable contingent claims are the only one that admit a unique arbitrage free price.

Corollary 2.20 Suppose that the market model is free of arbitrage and that $C$ is a contingent claim.
(i) $C$ is attainable if and only if it admits a unique arbitrage free price.

(ii) If $C$ is not attainable, then $\pi_{\min}(C) < \pi_{\max}(C)$ and hence there is a non-trivial interval of possible arbitrage free prices.

Proof:

(i) Clearly $|\Pi(C)| = 1$ if $C$ is attainable so (ii) implies (i).

(ii) The set $\Pi(C)$ is non-empty and convex due to the convexity of $P$. Hence $\Pi(C)$ is an interval. To show that the interval is open it suffices to prove that $\inf \Pi(C) \notin \Pi(C)$. By Theorem 2.13 there exists $\xi \in \mathbb{R}^d$ such that

$$(1 + r)\inf \Pi(C) + \xi \cdot Y \leq C \quad \mathbb{P}\text{-a.s.}$$

Since $C$ is not attainable this inequality cannot be an almost sure identity. Hence with $\xi^0 := -(1 + r)\inf \Pi(C)$ the strategy $(\xi^0, \xi)$ is an arbitrage opportunity in the extended market so $\inf \Pi(C)$ is not an arbitrage free price.

Example 2.21 (Portfolio Insurance) The idea of portfolio insurance is to enhance exposure to rising prices while reducing exposure to falling prices. For a portfolio with payoff $V \geq 0$ it is natural to consider a payoff $h(V)$ for a convex increasing function $h$. Since a convex function is almost surely differentiable and $h$ can be expressed

$$h(x) = h(0) + \int_0^x h'(y) \, dy$$

where $h'$ is the right-hand derivative.

Furthermore $h'' > 0$ so $h'$ is increasing. Hence $h'$ can be represented as the distribution functions of a positive measure $\gamma$:

$$h'(x) = \gamma([0, x]).$$

Thus, Fubini’s theorem yields

$$h(x) = h(0) + \int_0^x \int_0^y \gamma(dz) \, dy$$

$$= h(0) + \gamma(\{0\})x + \int_{(0, \infty)} \int_{\{y : z \leq y \leq x\}} dy \gamma(dz)$$

$$= h(0) + \gamma(\{0\})x + \int_{(0, \infty)} (x - z)^+ \gamma(dz).$$
Thus,
\[ h(V) = h(0) + \gamma(\{0\})V + \int_{(0,\infty)} (V - z)^+ \gamma(dz) \]
\[ = \text{investment in the bond + direct investment in } V + \text{investment in call options}. \]

If call options on \( V \) with arbitrary are available in the market the payoff \( h(V) \) is replicable.

In general there is no reason to assume that a contingent claim is attainable. Typical examples are CAT- (catastrophic) bonds which are often written on non-tradable underlyings such as weather or climate phenomena.

**Example 2.22** A couple of years ago, the Swiss insurance company Winterthur issued a bond that paid a certain interest \( \tilde{r} > r \) but only if - within a certain period of time - the number of cars insured by Winterthur and damaged due to hail-storms within a 24 hour period did not exceed some threshold level. This bond allowed Winterthur to transfer insurance related risks to the capital markets. Apparently its payoff cannot be replicated by investments in the financial markets alone.

It turns out that for stochastic models in discrete time the paradigm of a complete market where all contingent claims admit a perfect hedge is the exception rather than the rule. The exception of a complete market is discussed in the following section.

### 2.4 Complete market models and perfect replication

In this section we characterize the more transparent situation in which all contingent claims are attainable and hence allow for a unique price.

**Definition 2.23** An arbitrage free market is called complete if every contingent claim is attainable. That is, for every claim \( C \) there exists a portfolio \( \xi \in \mathbb{R}^{d+1} \) such that

\[ C = \xi \cdot \bar{S} \quad \mathbb{P}\text{-a.s.} \]

In a complete arbitrage free market any claim is integrable with respect to any EMM. In particular, the set \( \mathcal{V} \) of attainable claims defined in (3) satisfies

\[ \mathcal{V} \subseteq L^1(\mathbb{P}^*) \subseteq L^0(\mathbb{P}^*) = L^0(\mathbb{P}). \]

In a complete market any claim \( C \subset L^0(\mathbb{P}) \) is attainable so the above inequalities are in fact equalities. Since \( \mathcal{V} \subset \mathbb{R}^{d+1} \) is finite dimensional the same must be true for \( L^0(\mathbb{P}) \). As a result, \( L^0(\mathbb{P}) \) has at most \( d + 1 \) atoms; an atom of the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is an element \( A \in \mathcal{F} \) that contains no measurable subset of positive measure.
Lemma 2.24 For all \( p \in [0, \infty] \) the dimension of \( L^p \) is given by

\[
\dim L^p = \sup \{ n : \exists \text{ partition } A^1, \ldots, A^n \text{ of } \Omega \text{ with } \mathbb{P}[A^i] > 0 \}. 
\]

Proof: If there exists a partition \( A^1, \ldots, A^n \) then the dimension is at least \( n \) because the associated indicator functions are linear independent in \( L^p \). Conversely, if the right hand side equals \( n_0 < \infty \), then there exists a partition \( A^1, \ldots A^{n_0} \) into atoms and any \( Z \in L^p \) is constant on each \( A^i \). Thus, \( \dim L^p = n_0 \). □

We are now ready to show that a market model is complete if and only if there exists a unique EMM.

Theorem 2.25 An arbitrage free market model is complete if and only if \( |\mathcal{P}| = 1 \). In this case \( \dim L^0 \leq d + 1 \).

Proof: If the model is complete, then the indicator \( 1_A \) of each set \( A \in \mathcal{F} \) is an attainable claim. Hence Corollary 2.20 implies that the quantity \( \mathbb{P}^*[A] = \mathbb{E}^*[1_A] \) is the same for all \( \mathbb{P}^* \in \mathcal{P} \) so there exists only one risk neutral probability measure. Conversely, suppose that \( \mathcal{P} = \{ \mathbb{P}^* \} \), and let \( C \) be a bounded claim so that \( \mathbb{E}^*[C] < \infty \). Then \( C \) has a unique arbitrage free price and by Corollary 2.20 it is attainable. Hence

\[
L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subset \mathcal{V}.
\]

In view of the preceding lemma this implies that \( (\Omega, \mathcal{F}, \mathbb{P}) \) has at most \( d + 1 \) atoms. Since every claim is constant on atoms it is bounded and hence attainable. □

For a model with finitely many states of the world and one risky asset the previous theorem implies that a market model is complete only when \( \Omega \) consists of only two elements \( \omega^+ \) and \( \omega^- \). With \( p := \mathbb{P}[(\omega^+)] \) and \( S(\omega^+) = b \) and \( S(\omega^-) = a \) any risk neutral measure \((p^*, 1 - p^*)\) must satisfy

\[
\pi(1 + r) = \mathbb{E}^*[S] = a(1 - p^*) + bp^*.
\]

Hence the risk neutral measure is given in terms of \( p^* \) by

\[
p^* = \frac{\pi(1 + r) - a}{b - a}.
\]

The arbitrage free price for a call option \( C = (S - K)^+ \) with strike \( K \in [a, b] \) is given by the expected discounted payoff under the risk neutral measure:

\[
\pi(C) = \frac{b - K}{b - a} \cdot \pi - \frac{(b - K)a}{b - a} \cdot \frac{1}{1 + r}.
\]

Notice that the price does not depend on the real world probability \( p \). In fact, we observed earlier that the real world measure enters the set of EMMs only through its null sets, a trivial
condition if \( \Omega \) is finite. Furthermore \( \pi(C) \) is increasing in the risk-free rate while the classical discounted expectation with respect to the objective measure \( p \) is decreasing in \( r \) because

\[
E \left[ \frac{C}{1 + r} \right] = \frac{p(b - K)}{1 + r}.
\]

The central result of this section is that arbitrage free pricing requires the price of a contingent claim to be calculated as the discounted expected payoff with respect to an equivalent martingale measure rather than the objective, real-world measure.

### 2.5 Good deal bounds and super-deals

So far, the only condition we imposed on our financial market model was the assumption of no arbitrage. Ruling out arbitrage opportunities we ruled out “infinitely good deals”. In this section we discuss a refinement of the no free lunch condition, due to Carr et al. (2001). More precisely, we consider a financial market model \((\bar{\pi}, \bar{S})\) on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). In addition to the real world measure \( \mathbb{P} \) we are also given valuation measures \( \mathbb{Q}_i \ll \mathbb{P} \) \((i = 1, \ldots, n)\) that satisfy \( E_{\mathbb{Q}_i}[|Y^1|] < \infty \). The valuation measures can be viewed as capturing uncertainty about the - perhaps unknown and/or misspecified “real world” measure \( \mathbb{P} \). We put

\[
\mathbb{Q} := \{\mathbb{Q}_1, \ldots, \mathbb{Q}_n\}
\]

and assume that

\[
\mathbb{Q} \approx \mathbb{P} \quad \text{in the sense that for all } A \in \mathcal{F} \text{ with } \mathbb{P}[A] > 0 \text{ there exists } i \text{ s.t. } \mathbb{Q}_i[A] > 0.
\]

We are now ready to define the notion of a good deals and super deals.

**Definition 2.26** A portfolio \( \xi \) (or the associated net payoff \( \xi \cdot Y \)) is called a good deal of

\[
E_{\mathbb{Q}_i}[\xi \cdot Y] \geq 0 \quad \text{for all } i \in \{1, \ldots, n\}.
\]

It is called a super deal if, in addition, \( E_{\mathbb{Q}_{i_0}}[\xi \cdot Y] > 0 \) for at least one \( i_0 \in \{1, \ldots, n\} \). The model if called \( \mathbb{Q} \)-normal if no super deals exist.

Thus, a portfolio is a good deal if it yields an non-negative expected payoff under all possible models. It is a super deal if it yields a strictly positive payoff under at least one possible model. The following remark shows that notion of \( \mathbb{Q} \)-normality is in fact a refinement of the no free lunch condition.
Remark 2.27 Every free lunch is a super-deal.

Proof: Let \( \xi \) be a free lunch. Then \( \mathbb{P}[\xi \cdot \geq 0] = 1 \) so \( \mathbb{Q}_i[\xi \cdot \geq 0] = 1 \) because all the valuation measures are absolutely equivalent w.r.t. \( \mathbb{P} \). In particular \( \mathbb{E}_{\mathbb{Q}_i}[\xi \cdot Y] \geq 0 \) so \( \xi \) is a good deal. Moreover,

\[
\mathbb{P}[\xi \cdot Y > 0] > 0 \Rightarrow \mathbb{Q}_{i_0}[\xi \cdot Y > 0] > 0 \quad \text{for at least one } i_0.
\]

This yields \( \mathbb{E}_{\mathbb{Q}_{i_0}}[\xi \cdot Y] > 0 \) so \( \xi \) is a super deal. \( \square \)

The notion of \( \mathbb{Q} \)-normal prices follows the definition of arbitrage-free prices.

Definition 2.28 Let \( C \) be a contingent claim. A price \( \pi^C \) is called a \( \mathbb{Q} \)-normal price is the extended financial market model is \( \mathbb{Q} \)-normal. The set of all \( \mathbb{Q} \)-normal prices is denoted \( \Pi^\mathbb{Q}(C) \).

Our goal is now to characterize the set \( \Pi^\mathbb{Q}(C) \) in terms of a subset of equivalent martingale measures. More specifically, our aim is a characterization of the form

\[
\Pi^\mathbb{Q}(C) = \left\{ \mathbb{E}^* \left[ \frac{C}{1 + r} \right] : \mathbb{P}^* \in \mathcal{P} \cap \mathcal{R} \right\}
\]

for some set \( \mathcal{R} \). It will turn out that \( \mathcal{R} \) is given by the representative mixtures of the valuation measures so we define:

\[
\mathcal{R} := \left\{ \sum_{i=1}^{n} \lambda_i \mathbb{Q}_i : \lambda_i > 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}.
\]

We notice that the valuation measures do not belong to \( \mathcal{R} \) because all the \( \lambda_i \) are strictly positive. Furthermore, \( \mathcal{R} \) is convex and each element from the class \( \mathcal{R} \) is equivalent to \( \mathbb{P} \).

Proposition 2.29 The financial market model is \( \mathbb{Q} \)-normal if and only if \( \mathcal{P} \cap \mathcal{R} \neq \emptyset \).

Proof:

\[ \Rightarrow \]: Let \( \xi \) be a good deal. Then

\[
\mathbb{E}_R[\xi \cdot Y] \geq 0 \quad \text{for all } R \in \mathcal{R}.
\]

If \( R^* \in \mathcal{P} \cap \mathcal{R} \), then

\[
0 = \mathbb{E}_{R^*}[\xi \cdot Y] = \sum_{i=1}^{n} \lambda_i \mathbb{E}_{\mathbb{Q}_i}[\xi \cdot Y].
\]

Since \( \xi \) is a good deal this yields \( \mathbb{E}_{\mathbb{Q}_i}[\xi \cdot Y] = 0 \) for all \( i \in \{1, \ldots, n\} \) so \( \xi \) is no super deal.
The converse implication uses a separating hyperplane argument. Let
\[ C := \{ E_R[Y] = (E_R[Y^1], \ldots, E_R[Y^d]) : R \in \mathcal{R} \} \subset \mathbb{R}^d. \]

The set \( C \) is convex because \( \mathcal{R} \) and there exists an equivalent martingale measure that belongs to \( \mathcal{R} \) if and only if 0 \( \in \) \( C \).

Let us thus assume to the contrary that 0 \( \notin \) \( C \). Then there exists a portfolio \( \xi \in \mathbb{R}^d \) that separates \( C \) from the origin:
\[ \xi \cdot x \geq 0 \quad \text{for all} \quad x \in C, \quad \xi \cdot x_0 > 0 \quad \text{for some} \quad x_0 \in C. \]

Thus \( \xi \) satisfies
\[ E_R[\xi \cdot Y] = \xi \cdot E_R[Y] \geq 0 \quad \text{for all} \quad R \in \mathcal{R} \]

and
\[ E_{R_0}[\xi \cdot Y] = \xi \cdot E_{R}[Y] > 0 \quad \text{for some} \quad R_0 \in \mathcal{R} \quad \text{so} \quad E_{\mathbb{Q}i_0} [\xi \cdot Y] > 0 \quad \text{for some} \quad i_0. \]

The former inequality shows that \( \xi \) is a good deal; the latter shows that \( \xi \) is super deal. This contradicts the assumption of \( \mathbb{Q} \)-normality so 0 \( \in \) \( C \).

\[ \square \]

We are now ready to state and prove the main result of this section.

**Theorem 2.30** If the financial market model is \( \mathbb{Q} \)-normal, then the characterization (7) of \( \mathbb{Q} \)-normal prices holds.

**Proof:** Let the financial market model be \( \mathbb{Q} \)-normal.

"\( \neq \emptyset \): In view of the preceding proposition the set \( \{ E^*[C] : \mathbb{P}^* \in \mathcal{P} \cap \mathcal{R} \} \) is non-empty because for any \( \mathbb{Q} \in \mathcal{Q} \) we have \( E_{\mathbb{RQ}}[C] < \infty \); this is part of the definition of \( \mathbb{Q} \)-normality.

"\( \subset \):" Let \( \pi^C \) be a \( \mathbb{Q} \)-normal price and \( \hat{\mathcal{P}} \) the set of EMM associated with the extended model. By the preceding proposition there exists \( \hat{\mathcal{P}} \in \mathcal{P} \cap \mathcal{R} \) such that \( \pi^i = \hat{E}[S^i/(1 + r)] \) for \( i = 1, \ldots, d \) so \( \hat{\mathcal{P}} \in \mathcal{P} \). Moreover, \( \pi^C = \hat{E}[C/(1 + r)] \).

"\( \supset \):" Let \( \pi^C = E^*[C/(1 + r)] \) for some \( \mathbb{P}^* \in \mathcal{P} \cap \mathcal{R} \). Then \( \mathbb{P} \) belongs to the set \( \hat{\mathcal{P}} \cap \mathcal{R} \) so the extended model is \( \mathbb{Q} \)-normal.
In the sequel the following notation turns out to be useful. We denote by $L^1(Q)$ the class of all random variables that are integrable with respect to every valuation measure. For $X, Y \in L^1(Q)$ we write

$$X \geq_Q Y \text{ iff } \mathbb{E}^i_{Q}[X] \geq \mathbb{E}^i_{Q}[Y] \quad (i = 1, ..., n).$$

In terms of this notation $X$ is a good deal if and only if

$$X \geq_Q 0.$$

Let us now denote by $\pi^Q_{\max}(C)$ the largest possible $Q$-normal price of the claim $C$:

$$\pi^Q_{\max}(C) = \sup_{Q \in \mathcal{P} \cap R} \mathbb{E}^i_{Q} \left[ \frac{C}{1+r} \right].$$

This quantity is finite because $\max_i \mathbb{E}^i_{Q}[C] < \infty$. By analogy to the superhedgeing price $\pi^Q_{\max}(C)$ can be viewed as the minimal costs such that the seller of $C$ can generate a good deal.

**Theorem 2.31** Let $C \in L^1(Q)$. Then

$$\pi^Q_{\max}(C) = \min \left\{ m : \exists \xi \in \mathbb{R}^d : m + \xi \cdot Y \geq_Q \frac{C}{1+r} \right\}.$$

The proof of this theorem is based on the following two auxiliary results.

**Lemma 2.32** Let the financial market model be $Q$-normal. For $X \in L^1(Q)$ the set

$$C := \left\{ (\mathbb{E}^i_{Q}[\xi \cdot Y])_{i=1}^n + y : y \in \mathbb{R}^n_+, \xi \in \mathbb{R}^d \right\} \subset \mathbb{R}^n$$

is a closed convex cone that contains $\mathbb{R}^n_+$.

**Proof:** Convexity and the cone property are obvious. The fact that $\mathbb{R}^n_+ \subset C$ follows for the special case $\xi = 0$. In order to see that $C$ is indeed closed let $y(\xi) := (\mathbb{E}^i_{Q}[\xi \cdot Y])_{i=1}^n$. Any $x^* \in C$ is of the form

$$x^* = y(\xi^*) + y^*.$$

Here we may assume that

$$\xi^* \in N^\perp \quad \text{where } \quad N := \{ \eta \in \mathbb{R}^d : \mathbb{E}^i_{Q}[\eta \cdot Y] = 0, \ i = 1, ..., n \}.$$

Let us then consider a sequence $\{x_n\} \subset C$ that converges to $x$:

$$x_n = y(\xi_n) + z_n \rightarrow x \quad \text{as } n \rightarrow \infty.$$
In order to see that \( x \in C \) let us first assume that \( \liminf_n \| \xi_n \| < \infty \). In this case we may with no loss of generality assume that

\[
\xi_n \to \xi \quad \text{as } n \to \infty \quad \text{so} \quad y(\xi_n) \to y(\xi) \quad \text{as } n \to \infty.
\]

Thus, convergence of the sequence \( \{ x_n \} \) yields convergence of \( \{ z_n \} \) to some \( z \) and

\[
x = y(\xi) + z \in C.
\]

The case \( \liminf_n \| \xi_n \| < \infty \) is a little more involved. First, we consider the bounded sequence

\[
\zeta_n := \frac{\xi_n}{\| \xi_n \|}
\]

and assume with no loss of generality that \( \zeta_n \to \zeta \). Hence

\[
y(\zeta_n) \to y(\zeta) \quad \text{as } n \to \infty.
\]

Since \( C \) is a cone,

\[
\frac{x_n}{\| \xi_n \|} \in C \quad \text{and} \quad \frac{x_n}{\| \xi_n \|} = y(\zeta_n) + \frac{z_n}{\| \xi_n \|}.
\]

Convergence of \( \{ x_n \} \) therefore implies

\[
y(\zeta) = \lim_{n \to \infty} y(\zeta_n) = - \lim_{n \to \infty} \frac{z_n}{\| \xi_n \|} \in \mathbb{R}_-^d.
\]

Furthermore,

\[
\| \zeta \| = 1 \quad \text{so} \quad \zeta \neq 0.
\]

I: \( y(\zeta) \neq 0 \). In this case \( y(-\zeta) \geq 0 \) which is equivalent to

\[
\mathbb{E}_{Q_i} [-\zeta \cdot Y] \geq 0 \quad \text{and} \quad \mathbb{E}_{Q_{i_0}} [-\zeta \cdot Y] > 0 \quad \text{for some valuation measure } Q_{i_0}.
\]

Thus, \( -\zeta \) is a super deal which contradicts the assumption of \( Q \)-normality.

II: \( y(\zeta) = 0 \). This is equivalent to saying that the vectors

\[
\mathbb{E}_{Q_i}[Y], \ldots, \mathbb{E}_{Q_n}[Y]
\]

are co-linear, i.e., have a common normal vector \( \eta^* \). In this case we may choose a vector \( \xi^* \) in the representation (8) that is orthogonal to \( \eta^* \). This carries over to the limit portfolio \( \eta \) which renders \( y(\zeta) = 0 \) impossible.

Overall, \( \liminf_n \| \xi_n \| < \infty \) is not possible which proves the assertion. \( \square \)
Lemma 2.33 Let $\mathcal{A}$ be the class of all contingent claims which, when combined with a suitable portfolio, yield a good deal:

$$\mathcal{A} := \{X \in L^1(\mathcal{Q}) : \exists \xi : X + \xi \cdot Y \geq_\mathcal{Q} 0\}.$$

Furthermore, let $\bar{\mathcal{R}}$ be the convex hull of $\mathcal{Q}$. Then $\mathcal{A} = \mathcal{A}^*$ where

$$\mathcal{A}^* := \{X \in L^1(\mathcal{Q}) : \mathbb{E}^*[X] \geq 0 \forall \mathbb{P}^* \in \mathcal{P} \cap \bar{\mathcal{R}}\}.$$

Proof:

\[\subset\]: Let $X \in \mathcal{A}$ and $X + \xi \cdot Y \geq_\mathcal{Q} 0$. For $\mathbb{P}^* \in \mathcal{P} \cap \bar{\mathcal{R}}$ this yields

$$\mathbb{E}^*[X] = \mathbb{E}^*[X + \xi \cdot Y]$$

because $\mathbb{P}^*$ is an EMM

$$\geq 0$$

because $\mathbb{P}^* \in \bar{\mathcal{R}}$.

Hence $X \in \mathcal{A}^*$.

\[\supset\]: We have that $X \in \mathcal{A}$ iff there exists $\xi \in \mathbb{R}^d$ such that

$$\mathbb{E}_{\mathcal{Q}_i}[X] + \mathbb{E}_{\mathcal{Q}_i}[\xi \cdot Y] \geq 0$$

for all $i = 1, \ldots, d$

$$\Leftrightarrow (\mathbb{E}_{\mathcal{Q}_1}[X], \ldots, \mathbb{E}_{\mathcal{Q}_n}[X]) \in \mathcal{C}$$

Let $X \in \mathcal{A}^*$ and assume to the contrary that $X \notin \mathcal{A}$. Then

$$x_0 := (\mathbb{E}_{\mathcal{Q}_i}[X])_{i=1}^n \notin \mathcal{C}.$$

Since $\mathcal{C}$ is closed and convex the separating hyperplane theorem yields a vector $\lambda \in \mathbb{R}^n$ such that

$$\lambda \cdot x_0 < \inf_{x \in \mathcal{C}} \lambda \cdot x \quad (9)$$

Since $\mathbb{R}_+^n \subset \mathcal{C}$ and $\mathcal{C}$ is a cone all the entries $\lambda_i$ of $\lambda$ are non-negative. Indeed, if $e_i$ denotes the $i$-th normal vector, then $x_n = ne_i$ belongs to $\mathcal{C}$ and $\lambda_i < 0$would imply that $\lambda \cdot x_n \to -\infty$. We may with no loss of generality assume that $\sum_{i=1}^n \lambda_i = 1$ so that

$$\mathbb{R} := \sum_{i=1}^n \lambda_i \mathcal{Q}_i \in \bar{\mathcal{R}}.$$

In order to see that $\mathbb{R}$ is an EMM notice that (9) implies

$$\lambda \cdot \bar{x} = 0$$

for all $\bar{x} \in \{(\mathbb{E}_{\mathcal{Q}_i}[\xi \cdot Y])_{i=1}^n : \xi \in \mathbb{R}^d\}$.

In particular,

$$\mathbb{E}_R[\xi \cdot Y] = \sum_{i=1}^n \lambda_i \mathbb{E}_{\mathcal{Q}_i}[\xi \cdot Y] = 0$$

for all $\xi \in \mathbb{R}^d$. 

This implies $R \in \mathcal{P}$ and so $R \in \mathcal{P} \cap \mathcal{\bar{R}}$.

From this we deduce that $\inf_{x \in \mathcal{C}} \lambda \cdot x = 0$ else the infimum would be $-\infty$ because $\mathcal{C}$ is a cone. As a result,

$$\mathbb{E}_\mathcal{Q}[X] = \lambda \cdot x_0 < 0$$

contradicting the fact that $X \in \mathcal{A}^\ast$.

\[\square\]

We are now ready to prove the upper good deal bound.

**Proof of Theorem 2.31:** For a portfolio $\xi \in \mathbb{R}^d$ the following is equivalent, due to the receding lemma:

$$m + \xi \cdot Y \geq \mathbb{Q} \left( \frac{C}{1 + r} \right) \iff m - \frac{C}{1 + r} \in \mathcal{A}$$
$$\iff m - \frac{C}{1 + r} \in \mathcal{A}^\ast$$
$$\iff m - \mathbb{E}^\ast \left( \frac{C}{1 + r} \right) \geq 0 \quad \text{for all } \mathbb{P}^\ast \in \mathcal{P} \cap \mathcal{\bar{R}}$$
$$\iff m \geq \sup_{\mathbb{P}^\ast \in \mathcal{P} \cap \mathcal{\bar{R}}} \mathbb{E}^\ast \left( \frac{C}{1 + r} \right).$$

\[\square\]

3 Dynamic hedging in discrete time

We are now going to develop a dynamic version of the arbitrage theory of the previous Chapter. Here we are in a multi-period setting, where the financial price fluctuations are described by a stochastic process. This section follows Chapter 5 of Föllmer & Schied (2004).

3.1 The multi-period market model

Throughout we consider a financial market in which $d + 1$ assets are priced at time $t = 0, 1, \ldots, T$. The price of the $i$th asset at time $t$ is modelled as a non-negative random variable $S^i_t$ on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The random vector

$$\bar{S}_t = (S^0_t, S_t) = (S^0_t, \ldots, S^d_t)$$

is measurable with respect to a $\sigma$-field $\mathcal{F}_t \subset \mathcal{F}$. We think of $\mathcal{F}_t$ as the set of all events that are observable up to and including time $t$. It is hence natural to assume that

$$\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_T.$$
Such a family of $\sigma$-fields is called a filtration and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ is called a filtered probability space. To simplify the presentation we assume that

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \quad \text{and} \quad \mathcal{F}_T = \mathcal{F}.$$ 

**Definition 3.1** Let $Y = (Y_t)_{t=0}^T$ be a stochastic process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$.

(i) The process $Y$ is called adapted with respect to $(\mathcal{F}_t)_{t=0}^T$ if each $Y_t$ is measurable with respect to $\mathcal{F}_t$.

(ii) The process $Y$ is called predictable with respect to $(\mathcal{F}_t)_{t=0}^T$ if for $t \geq 1$ each $Y_t$ is measurable with respect to $\mathcal{F}_{t-1}$.

The asset price process $(\tilde{S}_t)$ forms an adapted process with values in $\mathbb{R}^{d+1}$. That is to say that

$$S_t : (\Omega, \mathcal{F}_t) \rightarrow (\mathbb{R}^{d+1}, \mathcal{B}^{d+1})$$

where $\mathcal{B}$ denotes the Borel $\sigma$-field on the real line. By convention the 0-th asset is a riskless asset such as a government bond or a bank account and

$$S_0^0 = 1.$$ 

Its returns is denoted by $r \geq 0$ so

$$S_t^0 = (1 + r)^t.$$ 

The discount factor is $(S_t^0)^{-1}$ and the discounted price processes will be denoted

$$X_t^i = \frac{S_t^i}{S_0^0}.$$ 

Then $X_t^0 \equiv 1$ and $X_t = (X_t^1, \ldots, X_t^d)$ expresses the value of the remaining assets in units of the numeraire $S_0^0$.

**Definition 3.2** A trading strategy is a predictable $\mathbb{R}^{d+1}$-valued process

$$\xi = (\xi^0, \xi) = (\xi_t^0, \xi_t^1, \ldots, \xi_t^d)_{t=1}^T$$

where $\xi_t^i$ corresponds to the number of shares of the $i$-th asset held during the $t$-th trading period between $t - 1$ and $t$.

**Remark 3.3** Economically, the fact that $\xi_t^i$ is predictable, i.e., $\mathcal{F}_{t-1}$-measurable means that portfolio decisions at time $t$ are based on the information available at time $(t - 1)$ and that portfolios are kept until time $t$ when new quotations become available.
The total value of the portfolio $\bar{\xi}_t$ at time $t-1$ is

$$(\bar{\xi}_t \cdot \bar{S}_{t-1})(\omega) = \sum_{i=0}^{d} \xi^i_t(\omega) S^i_{t-1}(\omega).$$

By time $t$ the value has changed to

$$(\bar{\xi}_t \cdot \bar{S}_t)(\omega) = \sum_{i=0}^{d} \xi^i_t(\omega) S^i_{t}(\omega).$$

If no funds are added or removed for consumption purposes the trading strategy is self-financing, that is,

$$\bar{\xi}_t \cdot \bar{S}_t = \bar{\xi}_{t+1} \cdot \bar{S}_t \quad \text{for} \quad t = 1, \ldots, T-1.$$

In particular the accumulated gains and losses resulting from the asset price fluctuations are the only source of variations of the portfolio value:

$$\bar{\xi}_t \cdot \bar{S}_t - \bar{\xi}_{t-1} \cdot \bar{S}_{t-1} = \bar{\xi}_t \cdot (\bar{S}_t - \bar{S}_{t-1}). \quad (10)$$

In fact, it is easy to see that a portfolio is self financing if and only if (10) holds.

**Remark 3.4** An important special case are the price dynamics of a money market account. Let $r_k$ be the "short rate", i.e., the short term interest rate for the period $[k,k+1)$. Then

$$S^0_t = \prod_{k=1}^{t} (1 + r_k).$$

Typically, the short rate is predictable so $(S^0_t)$ is predictable as well. Notice that in contrast to a savings account the short rate may change stochastically over time, though. For a savings account, one would usually assume that $r_k \equiv r$.

The discounted value process $V = (V_t)_{t=0}^{T}$ associated with a trading strategy $\bar{\xi}$ is given by

$$V_0 = \bar{\xi}_1 \cdot \bar{X}_0 \quad \text{and} \quad V_t = \bar{\xi}_t \cdot \bar{X}_{t-1} \quad \text{for} \quad t = 1, \ldots, T$$

while the associated gains process is defined by

$$G_0 = 0 \quad \text{and} \quad G_t = \sum_{k=1}^{t} \xi_k \cdot (X_k - X_{k-1}) \quad \text{for} \quad t = 1, \ldots, T.$$

The notion of a self-financing trading strategy can be expressed in terms of the value and gains processes as shown by the following proposition.

**Proposition 3.5** For a trading strategy $\bar{\xi}$ the following conditions are equivalent:
(i) $\bar{\xi}$ is self-financing.

(ii) $\bar{\xi}_t \cdot \bar{X}_t = \xi_{t+1} \cdot \bar{X}_t$ for $t = 1, \ldots, T - 1$.

(iii) $V_t = V_0 + G_t$ for all $t$.

Remark 3.6 The numéraire component of a self-financing portfolio $\bar{\xi}$ satisfies

$$\xi_{t+1}^0 - \xi_t^0 = -(\xi_{t+1} - \xi_t) \cdot X_t.$$ 

Since $\xi_1^0 = V_0 - \xi_1 \cdot X_0$ we see that the entire process $\xi^0$ is determined by the initial investment $V_0$ along with the process $\xi$. As a result, when $V_0$ and a $d$-dimensional predictable process $\xi$ are given, we can define a predictable process $\xi^0$ such that $\bar{\xi} = (\xi^0, \xi)$ is a self-financing strategy. In dealing with self-financing strategies it is thus sufficient to focus on initial investments and holdings in the risky assets.

3.2 Arbitrage opportunities and martingale measures

An arbitrage opportunity is an investment strategy that yields a positive profit with positive probability but without any downside risk.

Definition 3.7 A self-financing trading strategy is called an arbitrage opportunity if the associated value process satisfies

$$V_0 \leq 0, \quad \mathbb{P}[V_T \geq 0] = 1 \quad \text{and} \quad \mathbb{P}[V_T > 0] > 0.$$ 

In this section we characterize arbitrage-free market models, i.e., those models that do not allow arbitrage opportunities. It will turn out that a model is free of arbitrage if and only if there exists a probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ such that the discounted asset prices are martingales with respect to $\mathbb{P}^*$.

Definition 3.8 A stochastic process $M = (M_t)$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$ is called a martingale if $M$ is adapted, satisfies $\mathbb{E}_\mathbb{Q}[|M_t|] < \infty$ for all $t$ and

$$M_t = \mathbb{E}_\mathbb{Q}[M_{t+1} | \mathcal{F}_t].$$ 

The process is called a sub- and super-martingale, respectively, if, respectively,

$$M_t \leq \mathbb{E}_\mathbb{Q}[M_{t+1} | \mathcal{F}_t] \quad \text{and} \quad M_t \geq \mathbb{E}_\mathbb{Q}[M_{t+1} | \mathcal{F}_t].$$ 

A martingale can be regarded as the mathematical formalization of a fair game: for each time $s$ and any time horizon $t - s > 0$, the conditional expectation of the future gain $M_t - M_s$
is zero, given the information available at time \( s \). In our context of a finite time horizon a martingale \( M \) arises as a sequence of successive conditional expectations \( \mathbb{E}_Q[F|\mathcal{F}_t] \) for some \( F \in L^1(\Omega, \mathcal{F}_T, Q) \). Whether or not \( M \) is a martingale depends on the underlying probability measure and filtration. While the filtration will be fixed throughout this section, we shall deal with an array of different probability measures. To emphasize the dependence of the martingale on \( Q \) we say that \( M \) is a \( Q \)-martingale or a martingale with respect to \( Q \).

**Definition 3.9** A probability measure \( Q \) on \((\Omega, \mathcal{F}_T)\) is called a martingale measure if the discounted price process \( X \) is a \( d \)-dimensional martingale with respect to \( Q \). A martingale measure \( \mathbb{P}^* \) is called an equivalent martingale measure (EMM) if it is equivalent to \( \mathbb{P} \). The set of equivalent martingale measures will be denoted \( \mathbb{P} \).

The following result states that a fair game admits no realistic gambling system which produces a positive expected gain. Here \( Y^- \) denotes the negative part of a random variable \( Y \).

**Theorem 3.10** For a probability measure \( Q \) the following conditions are equivalent:

(i) \( Q \) is a martingale measure.

(ii) If \( \bar{\xi} \) is self-financing and \( \xi \) is bounded, then the value process associated with \( \bar{\xi} \) is a \( Q \)-martingale.

(iii) If \( \bar{\xi} \) is self-financing and the associated value process \( V \) satisfies \( \mathbb{E}_Q[V_T^-] < \infty \), then \( V \) is a \( Q \)-martingale.

(iv) If \( \bar{\xi} \) is self-financing and the associated value process \( V \) satisfies \( V_T \geq 0 \) \( Q \)-a.s. then \( \mathbb{E}_Q[V_T] = V_0 \).

**Proof:** (i) \( \Rightarrow \) (ii): Let \( V \) be the value process of a self-financing trading strategy \( \bar{\xi} = (\xi^0, \xi) \) such that \( |\xi| \) is bounded by a constant \( c \). Then

\[
|V_t| \leq |V_0| + \sum_{k=1}^t c(|X_k| + |X_{k-1}|).
\]

Since each \( |X_k| \) belongs to \( L^1(\mathbb{Q}) \) we see that \( \mathbb{E}_Q[|V_t|] < \infty \). Since \( \xi \) is predictable and bounded the martingale property of \( X \) yields

\[
\mathbb{E}_Q[V_{t+1}|\mathcal{F}_t] = V_t + \mathbb{E}_Q[\xi_t \cdot (X_{t+1} - X_t)|\mathcal{F}_t] = V_t + \xi_t \cdot \mathbb{E}_Q[X_{t+1} - X_t|\mathcal{F}_t] = V_t.
\]

(ii) \( \Rightarrow \) (iii): It is enough to show that \( \mathbb{E}_Q[V_t^-] < \infty \) then \( \mathbb{E}_Q[V_t|\mathcal{F}_{t-1}] = V_{t-1} \). To this end, notice first that \( \mathbb{E}_Q[V_t|\mathcal{F}_{t-1}] \) is well defined when \( \mathbb{E}_Q[V_t^-] < \infty \). Next, let

\[
\xi_t^a := \xi_t 1_{\{\xi_t \leq a\}}
\]
for $a > 0$. Then $\xi^a \cdot (X_t - X_{t-1})$ is a martingale increment by condition (ii); that is

$$\xi^a_t \cdot (X_t - X_{t-1}) \in L^1(\mathbb{Q}) \quad \text{and} \quad \mathbb{E}_Q [\xi^a \cdot (X_t - X_{t-1}) | \mathcal{F}_t] = 0.$$ 

Hence using predictability of the trading strategy a standard calculation shows that

$$\mathbb{E}_Q [V_t | \mathcal{F}_{t-1}] 1_{\{|\xi_t| \leq a\}} = V_{t-1} 1_{\{|\xi_t| \leq a\}}.$$ 

By sending $a \uparrow \infty$ we obtain the assertion.

(iii) $\Rightarrow$ (iv): Since the initial filtration is trivial, any martingale $M$ satisfies

$$M_0 = \mathbb{E}_Q[M_T | \mathcal{F}_0] = \mathbb{E}_Q[M_T].$$ 

(iv) $\Rightarrow$ (i): To prove that $X^i_t \in L^1(\mathbb{Q})$, consider the deterministic process $\xi$ defined by $\xi^i_s := 1_{\{s \leq t\}}$ and $\xi^i_s = 0$ otherwise. In view of Remark 3.6 we can complement $\xi$ with a predictable process $\xi^0$ such that $\xi = (\xi^0, \xi)$ is a self-financing trading strategy with initial investment $V_0 = X^i_0$. The corresponding value process satisfies

$$V_T = V_0 + \sum_{s=1}^T \xi^i_s \cdot (X_t - X_{t-1}) = X^i_t \geq 0.$$ 

Hence (iv) yields

$$\mathbb{E}_Q[X^i_t] = \mathbb{E}_Q[V_T] = V_0 = X^i_0 \quad \text{so} \quad X^i_t \in L^1(\mathbb{Q}).$$ 

(11)

To prove (i) it remains to show that $\mathbb{E}_Q[X^i_t | A] = \mathbb{E}_Q[X^i_{t-1} | A]$ for any $A \in \mathcal{A}_{t-1}$. For this we define

$$\eta^i_s := 1_{\{s < t\}} + 1_A 1_{\{s = t\}} \quad \text{and} \quad \eta^j_s = 0 \quad \text{for} \quad j \neq i.$$ 

As above we find a predictable process $\eta^0$ so that $(\eta^0, \eta)$ is self-financing and use (iv) to obtain

$$X^i_0 = V_0 = \mathbb{E}_Q[V_T] = \mathbb{E}_Q[X^i_T | A^c] + \mathbb{E}_Q[X^i_{t-1} | A].$$

Comparing this identity with (11) we conclude that $\mathbb{E}_Q[X^i_t | A] = \mathbb{E}_Q[X^i_{t-1} | A]$. \qed

**Remark 3.11** If the “real-world-measure” $\mathbb{P}$ is itself a martingale measure, the preceding theorem shows that there are no realistic self-financing strategies that would generate a positive expected gain. The assumption $\mathbb{P} \in \mathcal{P}$ is a version of the efficient market hypothesis. This hypothesis implies that risk-averse investors would not be attracted towards investing into risky assets if their expectations are consistent with $\mathbb{P}$.

We are now ready to state the following dynamic version of the fundamental theorem of asset pricing which links the absence of arbitrage to the existence of equivalent martingale measures.
Theorem 3.12 The market model is free of arbitrage if and only if \( \mathcal{P} \neq \emptyset \). In this case there exists \( \mathcal{P}^* \in \mathcal{P} \) with bounded density \( \frac{d\mathcal{P}^*}{d\mathcal{P}} \).

To prove the preceding result we first state a proposition that shows that a market model is arbitrage free if and only if there are no arbitrage opportunities for each single trading period. Its proof is left as an exercise.

Proposition 3.13 The market model admits an arbitrage opportunity if and only if there exists \( t \in \{1, 2, \ldots, T\} \) and a “trading strategy” \( \eta \in L^0(\Omega, \mathcal{F}_{t-1}, \mathcal{P}) \) such that

\[
\mathcal{P}[\eta \cdot (X_t - X_{t-1}) \geq 0] = 1 \quad \text{and} \quad \mathcal{P}[\eta \cdot (X_t - X_{t-1}) > 0] > 0.
\]

We are now ready to prove the main result of this section.

Proof of Theorem 3.12: It is clear that the existence of an EMM implies the absence of arbitrage opportunities so we only need to show the converse assertion. For this we define, for any \( t \in \{1, 2, \ldots, T\} \) the set

\[
\mathcal{X}_t := \{ \eta \cdot (X_t - X_{t-1}) : \eta \in L^0(\Omega, \mathcal{F}_{t-1}, \mathcal{P}) \}.
\]

In view of the preceding proposition the market is arbitrage free if and only if

\[
\mathcal{X}_t \cap L^0_+(\Omega, \mathcal{F}_t, \mathcal{P}) = \{0\}
\]

for all \( t \). Note that this condition depends on \( \mathcal{P} \) only through its null sets. Applying the fundamental theorem of asset pricing for one-period models yields a probability measure \( \mathcal{P}_T \approx \mathcal{P} \) with bounded density such that

\[
\mathbb{E}_T[X_T - X_{T-1} | \mathcal{F}_{T-1}] = 0.
\]

We now proceed by induction. For this suppose now that we already have a probability measure \( \mathcal{P}_k \approx \mathcal{P} \) with bounded density such that

\[
\mathbb{E}_k[X_k - X_{k-1} | \mathcal{F}_{T-1}] = 0 \quad \text{for} \quad t + 1 \leq k \leq T.
\]  \hspace{1cm} (12)

Applying Theorem 2.6 to the \( t \)-th trading period yields a probability measure \( \mathcal{P}_t \approx \mathcal{P}_{t+1} \) with bounded density \( Z_t \) such that

\[
\mathbb{E}_t[X_t - X_{t-1} | \mathcal{F}_{t-1}] = 0.
\]

Clearly, \( \mathcal{P}_t \) is equivalent to \( \mathcal{P} \) with bounded density\[
\frac{d\mathcal{P}_t}{d\mathcal{P}} = Z_t \frac{d\mathcal{P}_{t+1}}{d\mathcal{P}}.
\]
Moreover, if \( t + 1 \leq k \leq T \) the \( \mathcal{F}_t \) measurability of \( Z_t \) yields
\[
\mathbb{E}_t[X_k - X_{k-1}|\mathcal{F}_{k-1}] = \frac{\mathbb{E}_{t+1}[(X_k - X_{k-1})Z_t|\mathcal{F}_{k-1}]}{\mathbb{E}_{t+1}[Z_t|\mathcal{F}_{k-1}]} = \mathbb{E}_{t+1}[X_k - X_{k-1}|\mathcal{F}_{k-1}] = 0.
\]
Hence (12) carries over from \( \mathbb{P}_{t+1} \) to \( \mathbb{P}_t \). We can repeat this recursion until finally \( \mathbb{P}^* := \mathbb{P}_1 \) yields the desired EMM. \( \square \)

### 3.3 European options and attainable claims

A European contingent claim can be viewed as an asset which yields at \textit{maturity} \( T \) a random amount \( C \). Its payoff typically depends on the behavior of the primary assets \( S_0, \ldots, S^d \).

**Definition 3.14** A non-negative random variable \( C \) on \((\Omega, \mathcal{F}_T, \mathbb{P})\) is called a European contingent claim. It is called a derivative of the underlyings \( S_0, \ldots, S^d \) if it is measurable with respect to the \( \sigma \)-algebra generated by the price process \((\bar{S}_t)_{t=0}^T \).

For a European call and put option, respectively, on the \( i \)-th asset with maturity \( T \) and strike \( K \) we have that
\[
C^{\text{call}} = (S^i_T - K)^+ \quad \text{and} \quad C^{\text{put}} = (K - S^i_T)^+.
\]

**Example 3.15** (i) The payoff of an Asian option depends on the average price
\[
S^i_{\text{av}} := \frac{1}{T} \sum_{t=1}^T S^i_t
\]
of the underlying. For instance, an average price call with strike \( K \) corresponds to a contingent claim
\[
C^{\text{call}}_{\text{av}} = (S^i_{\text{av}} - K)^+
\]

(ii) The payoff of a barrier option depends on whether the price of the underlying reaches a certain level before maturity. Most barrier options are either knock-ins or knock-outs. For instance, a down-and-in put with strike \( K \) and barrier \( B \) pays
\[
C^{\text{put}}_{\text{di}} = \begin{cases} (K - S^i_T)^+ & \text{if } \min_{0 \leq t \leq T} S^i_t \leq B \\ 0 & \text{else} \end{cases}
\]
An up-and-out call with strike \( K \) and barrier \( B \) corresponds to
\[
C^{\text{call}}_{\text{uo}} = \begin{cases} (S^i_T - K)^+ & \text{if } \max_{0 \leq t \leq T} S^i_t < B \\ 0 & \text{else} \end{cases}
\]

(iii) A digital or binary option is essentially a bet on whether or not the price of the underlying reaches a certain threshold $B$ before maturity:

$$C^{digital} = \begin{cases} 1 & \text{if } \max_{0 \leq t \leq T} S_t^i \geq B \\ 0 & \text{else} \end{cases}.$$ 

From now on we will assume that our market model is free of arbitrage, i.e., that $\mathcal{P} \neq \emptyset$ and denote by $H$ the discounted value of a European contingent claim $C$.

**Definition 3.16** A contingent claim $C$ is attainable (replicable, redundant) if there exists a self-financing trading strategy $\bar{\xi}$ whose terminal portfolio value coincides almost surely with $C$, i.e.,

$$C = \bar{\xi}_T \cdot \bar{S}_T \text{ a.s.}$$

Such a trading strategy will be called a replicating strategy for $C$.

A contingent claim $C$ is attainable if and only if the corresponding discounted claim $H$ is of the form

$$H = V_0 + \sum_{t=1}^{T} \xi_t \cdot (X_t - X_{t-1})$$

for a self-financing strategy $\bar{\xi}$ with associated value process $V$. The following theorem shows that an attainable contingent claim is automatically integrable with respect to any equivalent martingale measure.

**Theorem 3.17** Any attainable discounted claim $H$ is integrable with respect to each equivalent martingale measure. Moreover, for any $\mathbb{P}^* \in \mathcal{P}$ the value process of any replicating strategy satisfies $\mathbb{P}$-a.s.

$$V_t = \mathbb{E}^{*}[H | \mathcal{F}_t] \text{ for } t = 0, \ldots, T.$$ 

**Proof:** The assertion follows from $V_T = H \geq 0$ and Theorem 3.10.

It follows from the preceding theorem that $V_t$ is a version of the conditional expected payoff $\mathbb{E}^{*}[H | \mathcal{F}_t]$ under any equivalent martingale measure and that all replicating strategies have the same value process. Hence for an attainable claim $H$ the (discounted) initial investment $V_0 = \mathbb{E}^{*}[H]$ needed for the replication of $H$ can be interpreted as the unique “fair value” of $H$. More generally, we formalize the idea of an arbitrage-free price of a general (not necessarily attainable) claim $H$ as follows.
Definition 3.18 A real number $\pi^H \geq 0$ is called an arbitrage-free price of a discounted claim $H$ if there exists an adapted stochastic process $X^{d+1}$ such that

$$X_0^{d+1} = \pi^H, \quad X_t^{d+1} \geq 0 \quad \text{for all } t = 1, \ldots, T, \quad X_T^{d+1} = H,$$

and such that the market model with the price process $(X^0, \ldots, X^d, X^{d+1})$ is free of arbitrage. The set of all arbitrage free prices is denoted $\Pi(H)$. It upper and lower bounds are

$$\pi_{\inf}(H) := \inf \Pi(H) \quad \text{and} \quad \pi_{\sup}(H) := \sup \Pi(H).$$

Notice that when $\mathbb{P}^*$ is an EMM for the original model with respect to which the claim $H$ has finite expectation and if we define the process $(X_t^{d+1})$ by

$$X_0^{d+1} = \mathbb{E}^*[H] \quad \text{and} \quad X_t^{d+1} = \mathbb{E}^*[H|\mathcal{F}_t]$$

then $\mathbb{P}^*$ is an EMM for the extended model. In this way every EMM of the original model can be viewed as an EMM for the extended model. At the same time, any EMM of the extended model is an EMM for the original model. With this we are now ready to state the following dynamic version of Theorem 2.13 above.

Theorem 3.19 Suppose that $\mathcal{P} \neq \emptyset$. Then the set of arbitrage free prices of a discounted claim $H$ is non-empty and given by

$$\Pi(H) = \{ \mathbb{E}^*[H] : \mathbb{P}^* \in \mathcal{P} \text{ and } \mathbb{E}^*[H] < \infty \}. \quad (13)$$

Furthermore, the lower and upper arbitrage bounds are given by

$$\pi_{\inf}(H) = \inf_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*[H] \quad \text{and} \quad \pi_{\sup}(H) = \sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*[H].$$

Proof: By Theorem 3.12 $\pi^H$ is an arbitrage free price if and only if there exists an EMM $\hat{\mathbb{P}}$ such that the extended market model satisfies

$$X_i^j = \hat{\mathbb{E}}[X_j^i|\mathcal{F}_t].$$

In particular, $\pi^H = \hat{\mathbb{E}}[H]$ which yields the inclusion “$\subseteq$” in (13). Conversely, if $\pi^H = \mathbb{E}^*[H] < \infty$ for some $\mathbb{P}^* \in \mathcal{P}$ we can define

$$X_t^{d+1} = \mathbb{E}^*[H|\mathcal{F}_t]$$

which makes $\mathbb{P}^*$ an EMM of the extended model. To show that $\Pi(H) \neq \emptyset$ let us fix some $\hat{\mathbb{P}} \approx \mathbb{P}$ such that $\hat{\mathbb{E}}[H] < \infty$. The market is arbitrage free under $\hat{\mathbb{P}}$ so Theorem 2.6 yields $\mathbb{P}^* \in \mathcal{P}$ such that $d\mathbb{P}^*/d\hat{\mathbb{P}}$ is bounded. In particular, $\mathbb{E}^*[H] \in \Pi(H)$. 
The formula for the lower arbitrage bound follows from (13). The upper arbitrage bound is obvious if $H$ is integrable with respect to any EMM. If $\mathbb{E}^\infty[H] = \infty$ for some $\mathbb{P}^\infty \in \mathcal{P}$ we need to show that for any $c > 0$ there exists some $\pi \in \Pi(H)$ such that $\pi > c$. To this end, let $n$ be such that $\tilde{\pi} := \mathbb{E}^\infty[H \wedge n] > c$ and define

$$X_t^{d+1} := \mathbb{E}^\infty[H \wedge n|\mathcal{F}_t] \quad (t = 0, \ldots, T).$$

Then $\mathbb{P}^\infty$ is an EMM for the extended market model which is hence free of arbitrage. For this model we can find an equivalent measure $\hat{\mathbb{P}}$ such that

$$\hat{\mathbb{E}}[H] < \infty$$

because $\Pi(H) \neq \emptyset$. We take $\hat{\mathbb{P}}$ as our reference measure. By the fundamental theorem of asset pricing there exists an EMM $\mathbb{P}^* \approx \hat{\mathbb{P}}$ with a bounded density. In particular $H$ has finite expectation under $\mathbb{P}^*$ and the process $(X^0, \ldots, X^{d+1})$ is a $\mathbb{P}^*$-martingale:

$$\mathbb{E}^*[H] < \infty \quad \text{and} \quad X_t^{d+1} = \mathbb{E}^*[H \wedge n|\mathcal{F}_t] \quad (t = 0, 1, \ldots, T).$$

Since $\mathbb{P}^*$ is also an EMM for the original model we see that $\pi := \mathbb{E}^*[H]$ is an arbitrage free price. Finally, note that

$$\pi \geq \mathbb{E}^*[H \wedge n] = \mathbb{E}^*[X_T^{d+1}] = X_0^{d+1} = \tilde{\pi} > c.$$

This proves the formula for the upper arbitrage bound.

Notice that $\pi_{in}(H)$ and $\pi_{sup}(H)$ if $H$ is attainable. In this case $H$ can be priced unambiguously. The converse is also true, i.e., $H$ is attainable if and only if $\pi_{in}(H) = \pi_{sup}(H)$.

**Theorem 3.20** Let $H$ be a discounted claim. Then the following holds:

(i) If $H$ is attainable, then $\Pi(H)$ contains a single element, namely $\mathbb{E}^*[H]$ where the expectation can be taken with any EMM $\mathbb{P}^*$.

(ii) If $H$ is not attainable, then $\pi_{in}(H) < \pi_{sup}(H)$ and $\Pi(H)$ is an interval:

$$\Pi(H) = (\pi_{in}(H), \pi_{sup}(H)).$$

For a European call option with strike $K$ and maturity $T$ the arbitrage free price is given by

$$\pi^{call} = \mathbb{E}^* \left[ \frac{C^{call}}{S_0^T} \right] = \mathbb{E}^* \left[ \left( X_T^1 - \frac{K}{S_T^0} \right)^+ \right].$$
Due to the convexity of the function \( x \mapsto x^+ \) the quantity \( \pi^\text{call} \) can be bounded from below as follows:

\[
\pi^\text{call} \geq \left( E^* \left[ X_T^1 - \frac{K}{S_T^1} \right] \right)^+ = \left( S_0^1 - E^* \left[ \frac{K}{S_T^1} \right] \right)^+ \geq (S_0^1 - K)^+.
\]
Thus, the option value \( \pi^\text{call} \) is higher than its intrinsic value \( (S_0^1 - K)^+ \), i.e., the payoff if the option were exercised immediately.

### 3.4 Complete markets

We have seen that any attainable contingent claim in an arbitrage-free market has a unique arbitrage-free price. This situation becomes particularly transparent when all claims are attainable.

**Definition 3.21** An arbitrage-free market model is called complete if every contingent claim is attainable.

Complete markets are appealing insofar as that every claim has a unique and unambiguous price. However, in discrete time, only a limited class of models enjoys this property as shown by the following theorem.

**Theorem 3.22** An arbitrage-free market model is complete if and only if there exists a unique EMM. In this case the number of atoms in \((\Omega, \mathcal{F}, \mathbb{P})\) is bounded above by \((d + 1)^T\).

**Proof:** If the model is complete then \( H = 1_A \) for \( A \in \mathcal{F}_T \) is an attainable discounted claim. In particular, the map

\[
\mathbb{P}^* \mapsto E^*[H] = P^*[A]
\]

is constant over the set \( \mathcal{P} \). Hence there can only be one EMM. Conversely, if \( |\mathcal{P}| = 1 \) then the set \( \Pi(H) \) of arbitrage-free prices of every discounted claim has a unique element. Hence \( H \) is attainable, due to Theorem 3.20. The second assertion follows essentially by an induction argument over \( T \). We leave the details as an exercise.

It follows from the preceding theorem that in a one-period model the binary case where only two states of the world are possible is the only example of a complete market model. In discrete time the usual paradigm of complete markets is therefore the exception rather than the rule.

**Theorem 3.23** Let \( \mathcal{Q} \) be the set of all martingale measures. For \( \mathbb{P}^* \in \mathcal{P} \) the following conditions are equivalent:
(i) $\mathcal{P} = \{\mathbb{P}^*\}$.

(ii) $\mathbb{P}^*$ is an extreme point of $\mathcal{P}$.

(iii) $\mathbb{P}^*$ is an extreme point of $\mathcal{Q}$.

(iv) Every $\mathbb{P}^*$-martingale can be represented as a “stochastic integral” of a $d$-dimensional predictable process $\xi$:

$$M_t = M_0 + \sum_{k=1}^{t} \xi_k \cdot (X_k - X_{k-1}).$$

**Proof:** The implications $(i) \Rightarrow (iii)$, $(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow$ are more or less obvious. The implication $(i) \Rightarrow (iv)$ follows from Theorem 3.22 applied separately to the positive and negative parts of the terminal value of the martingale $M$. As for the implication $(iv) \Rightarrow (i)$, we apply our assumptions to the martingale $M_t = \mathbb{P}^*[A|\mathcal{F}_t]$.

This shows that $H = 1_A$ can be replicated and hence the map $\mathbb{P} \mapsto \mathbb{P}[A]$ on $\mathcal{P}$ is constant. This shows $(i)$. To show that $(ii)$ implies $(i)$ suppose first that there exists a measure $\hat{\mathbb{P}} \in \mathcal{P}$ which is different from $\mathbb{P}^*$ and whose density is bounded by some $c > 0$. Then, if $0 < \epsilon < c^{-1}$ the density

$$\frac{d\mathbb{P}'}{d\mathbb{P}^*} := 1 + \epsilon - \epsilon \frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*}$$

defines another EMM $\mathbb{P}' \neq \mathbb{P}^*$ and $\mathbb{P}^*$ can be written as

$$\mathbb{P}^* = \frac{\epsilon}{1 + \epsilon} \hat{\mathbb{P}} + \frac{1}{1 + \epsilon} \mathbb{P}'.$$  

This contradicts $(ii)$ so it remains to show that $\hat{\mathbb{P}}$ can be chosen to have a bounded density. If $\hat{\mathbb{P}} \neq \mathbb{P}^*$ there exists a set $A \in \mathcal{F}_T$ such that $\hat{\mathbb{P}}[A] \neq \mathbb{P}^*[A]$. We can enlarge the market by introducing an additional asset $X_t^{d+1} := \hat{\mathbb{P}}[A|\mathcal{F}_t]$ and take $\mathbb{P}^*$ as our reference measure. By definition $\hat{\mathbb{P}}$ is an EMM for the extended market model. Hence the model is free of arbitrage and by Theorem 3.12 there exists an EMM $\mathbb{P}$ that has a bounded density with respect to $\mathbb{P}^*$. Moreover, $\hat{\mathbb{P}}$ must be different from $\mathbb{P}^*$ since $\mathbb{P}^*$ is not a martingale measure because

$$X_0^{d+1} = \hat{\mathbb{P}}[A] \neq \mathbb{P}^*[A] = \mathbb{E}^*[X_T^{d+1}].$$
In a complete market model the unique arbitrage-free price \( H_t \) of the a discounted claim \( H \) at time \( t \) is given by the conditional expected payoff under the EMM:

\[
H_t = \mathbb{E}^*[H|\mathcal{F}_t].
\]

In particular \((H_t)\) is a \( \mathbb{P}^*\)-martingale. In view of part (iv) of the preceding theorem this martingale can be represented as a stochastic integral:

\[
H_t = H_0 + \sum_{k=1}^{t} \xi_k \cdot (X_k - X_{k-1})
\]

for some predictable process \( \xi \). In economic terms \( H_0 = \mathbb{E}^*[H] \) is the initial investment needed to implement the replication strategy \( \tilde{\xi} = (\xi^0, \xi) \). Here \( \xi^0 \) is the unique trading strategy in the bond market associated with \( H_0 \) and \( \xi \). Hence the martingale representation theorem yields both the fair price of \( H \) and a replicating strategy in the stock and bond market.

## 4 Binomial trees and the Cox-Ross-Rubinstein Model

In view of Theorem 3.22 a complete market model with a single risky asset must have binary tree structure. In this section we consider a binary tree model that has originally been introduced by Cox, Ross & Rubinstein (1971). The underlying idea is that in any given period stock prices can only move up or down. More precisely, there is a riskless bond whose price at time \( t \) is given by \( S^0_t = (1 + r)^t \) and single risky asset whose price fluctuations satisfy

\[
R_t = \frac{S^1_t - S^1_{t-1}}{S^1_{t-1}} \in \{a, b\} \quad \text{for all} \quad t = 1, 2, \ldots, T
\]

and some \(-1 < a < b\). We construct the model on the sample space

\[
\Omega = \{-1, +1\}^T = \{\omega = (y_1, \ldots, y_T) : y_i \in \{-1, +1\}\}
\]

and put

\[
Y_t(\omega) = y_t.
\]

Then

\[
R_t(\omega) = a \frac{1 - Y_t(\omega)}{2} + b \frac{1 + Y_t(\omega)}{2}
\]

and the price dynamics of the risk asset is modeled as

\[
S_t = S_0 \prod_{k=1}^{t} (1 + R_k).
\]
The discounted price process is again denoted \((X_t)\). As the filtration we take
\[
F_t = \sigma(S_0, \ldots, S_t) = \sigma(X_0, \ldots, X_t) = \sigma(Y_1, \ldots, Y_t) = \sigma(R_1, \ldots, R_t).
\]
Finally, we fix a probability measure \(\mathbb{P}\) such that \(\mathbb{P}([\omega]) > 0\) for all \(\omega \in \Omega\). We notice that we do not make any a-priori assumptions on the serial dependence of returns. It turns out though that in an arbitrage free model, the returns are independent of each other.

**Proposition 4.1** The model is free of arbitrage if and only if \(a < r < b\). In this case there exists a unique MM \(\mathbb{P}^*\) and the random variables are independent under \(\mathbb{P}^*\) with
\[
\mathbb{P}^*[R_t = b] = \frac{r-a}{b-a}.
\]

**Proof:** Existence, uniqueness and characterization of MM is easy and left as an exercise. As for the independence of returns, notice that any MM \(\mathbb{Q}\) satisfies
\[
X_t = E_Q[X_{t+1}|F_t] = X_t E_Q\left[\frac{1 + R_{t+1}}{1 + r} | F_t\right]
\]
\[
\iff r = b \mathbb{Q}[R_{t+1} = b | F_t] + a (1 - \mathbb{Q}[R_{t+1} = b | F_t])
\]
\[
\iff \mathbb{Q}[R_{t+1} = b | F_t] = \frac{r-a}{b-a}.
\]

Let us now consider the problem of pricing a (possibly path dependent) derivative \(H = h(S_0, \ldots, S_T)\) in a complete market model. We know already that the value \((V_t)\) process associated with any replication strategy satisfies
\[
V_t = \mathbb{E}^*[H | F_t].
\]
The following theorem further specifies the value process.

**Theorem 4.2** The value process is of the form
\[
V_t = v_t(S_0, \ldots, S_t)
\]
where
\[
v_t = \mathbb{E}^*\left[h \left( x_0, \ldots, x_t, x_t \frac{S_1}{S_0}, \ldots, x_t \frac{S_T}{S_0} \right) \right].
\]

**Proof:** We have
\[
V_t = \mathbb{E}^*\left[h \left( S_0, \ldots, S_t, S_t \frac{S_{t+1}}{S_t}, \ldots, S_t \frac{S_T}{S_t} \right) | F_t \right].
\]
Furthermore, each term \(S_{t+s}/S_t\) is independent of \(F_t\) under \(\mathbb{P}^*\) and has the same distribution as \(S_t/S_0\). Hence the assertion follows from the stochastic version of Fubini’s theorem (see Lemma 4.3 below).
Lemma 4.3 (Stochastic Fubini Theorem) For $i = 0, 1$ let $(E_i, \mathcal{E}_i)$ be measurable spaces and $U_i : (\Omega, \mathcal{F}) \to (E_i, \mathcal{E}_i)$ measurable. Let $\mathcal{F}_0 = \sigma(U_0)$ and $U_1$ independent of $U_0$. Then

$$\mathbb{E}[f(U_0, U_1)|\mathcal{F}_0](\omega) = \mathbb{E}[f(U_0(\omega), U_1)] =: h(U_0(\omega))$$

for all non-negative measurable functions $f$ on $E_0 \times E_1$.

**Proof:** The right hand side of the preceding equation is $\mathcal{F}_0$-measurable so by definitin of the conditional expectation we only to need to show that

$$\mathbb{E}[Zf(U_0, U_1)] = \mathbb{E}[Zh(U_0)]$$

for all $\mathcal{F}_0$-measurable random variables $Z$.

To this end, notice first that $Z$ allows a representation of the form $Z = g(U_0)$. Thus, Fubini’s theorem yields

$$\mathbb{E}[Zf(U_0, U_1)] = \mathbb{E}[g(U_0)f(U_0, U_1)]$$

$$= \int_{E_0} \int_{E_0} g(u_0) f(u_0, u_1) \mu_0(du_0) \mu_1(du_1)$$

$$= \int_{E_0} \int_{E_1} g(u_0) f(u_0, u_1) \mu_1(du_1) \mu_0(du_0)$$

$$= \int_{E_0} g(u_0) \left( \int_{E_1} f(u_0, u_1) \mu_1(du_1) \right) \mu_0(du_0)$$

$$= \mathbb{E}[g(U_0)h(U_0)]$$

$$= \mathbb{E}[Zh(U_0)].$$

\[ \Box \]

4.1 **Delta-Hedgeing in discrete time**

Since the value process of a replicating strategy satisfies $V_T = H$ and $V_t = \mathbb{E}[V_{t+1}|\mathcal{F}_t]$ we obtain a recursive structure for the function $v_t$. In fact,

$$v_T(x_0, \ldots, x_T) = h(x_0, \ldots, x_T)$$

$$v_t(x_0, \ldots, x_t) = p^* v_{t+1}(x_0, \ldots, x_t, x_t(1+b)) + (1-p^*) v_{t+1}(x_0, \ldots, x_t, x_t(1+a)).$$

It turns out that not only the value process, but also the hedging (replicating) strategy $\xi = (\xi^0, \xi)$ can be expressed in terms of $v_t$. To this end, we introduce the ”option delta” $\Delta_t$, i.e., a discrete derivative of the value function $v_t$ with respect to possible changes in the stock price:

$$\Delta_t := (1+r) \frac{v_t(x_0, \ldots, x_{t-1}, x_{t-1}(1+b)) - v_{t-1}(x_0, \ldots, x_{t-1}, x_{t-1}(1+a))}{x_{t-1}(1+b) - x_{t-1}(1+a)}.$$
Proposition 4.4  The delta yields a hedging strategy for the attainable claim $H$.

PROOF: For each $\omega$ the hedging strategy must satisfy

$$\xi_t(\omega)(X_t(\omega) - X_{t-1}(\omega)) = V_t(\omega) - V_{t-1}(\omega).$$

The random variables $\xi_t$, $X_{t-1}$ and $V_{t-1}$ depend on the first $t-1$ components of $\omega$ only. Let

$$\omega^\pm = (y_1, \ldots, y_{t-1}, \pm 1, y_{t+1}, \ldots, y_T).$$

Then

$$\xi_t(\omega) \cdot \left( X_{t-1}(\omega) \frac{1 + b}{1 + r} - X_{t-1}(\omega) \right) = V_t(\omega^+) - V_{t-1}(\omega^+)$$

and

$$\xi_t(\omega) \cdot \left( X_{t-1}(\omega) \frac{1 + a}{1 + r} - X_{t-1}(\omega) \right) = V_t(\omega^-) - V_{t-1}(\omega^-).$$

Solving for $\xi_t(\omega)$ using the recursive structure for $v_t$ yields the assertion. \qed

For a discounted claim $H = h(S_T)$ whose payoff is an increasing function of the terminal price $S_T$ (such as a European Call Option) the value function

$$v_t(x) = \mathbb{E}^* \left[ h(xS_T) / S_t \right]$$

is increasing in $x$ and hence the hedging strategy is non-negative, i.e., does not involve short sells:

$$\xi_t(\omega) = (1 + r)^t \frac{v_t(S_{t-1}(\omega)(1 + b)) - v_t(S_{t-1}(\omega)(1 + a))}{S_{t-1}(\omega)(1 + b) - S_{t-1}(\omega)(1 + a)} \geq 0.$$

### 4.2 Exotic derivatives

The results of the preceding section can be used for numerical computation of the value process associated with a contingent claim $H$. In this section we focus on barrier and lookback options whose value depends on the maximum

$$\hat{M}_t := \max_{0 \leq s \leq t} S_s$$

of the underlying stock price. The valuation formulas are based on the reflection principle for a symmetric random walk so from now on we work under the additional assumption that

$$\hat{a} = \frac{1}{\hat{b}} \quad \text{where} \quad \hat{a} = (1 + a), \hat{b} = (1 + b).$$

In this case the price process can be written as

$$S_t = S_0 \hat{b}^{Z_t} \quad \text{where} \quad Z_t = Y_1 + \ldots + Y_t.$$
If we denote by $P$ the uniform distribution on $\Omega$ then $(Y_t)$ is a sequence of iid random variables with $P[Y_t = +1] = \frac{1}{2}$ and $(Z_t)$ is a symmetric random walk. As a result,

$$P[Z_t = k] = 2^{-t} \binom{t + k}{\frac{t}{2}} \text{ if } t + k \text{ is even.}$$

### 4.2.1 The reflection principle

In order to calculate the value of an up-and-in barrier option we need the joint distribution of the terminal price and the running maximum of the underlying. This distribution can be given in closed form. To this end, it will be convenient to assume that the random walk $(Z_t)$ is defined up to time $T + 1$ and to put

$$M_t := \max_{0 \leq s \leq t} Z_t.$$

We are now ready to state and prove the reflection principle for a symmetric random walk. It states that the joint distribution of the running maximum of a symmetric random walk and its terminal value can be expressed in terms of the distribution of the terminal value only.

**Lemma 4.5** For all $k \in \mathbb{N}$ and $l \in \mathbb{N}_0$ the following holds:

(i) 

$$P[M_T \geq k, Z_T = k - l] = P[Z_T = k + l].$$

(ii) 

$$P[M_T = k, Z_T = k - l] = 2 \frac{k + l + 1}{T + 1} P[Z_{T + 1} = 1 + k + l].$$

**Proof:** The proof is based on the fact that each path $Z_l, \ldots, Z_k$ of our random walk has the same probability that its “shadow path” defined by $\hat{Z}_{t+1} = Z_t - Y_{t+1} \ (t = l, \ldots, k - 1)$. Specifically, we denote by

$$\tau(\omega) = \int \{t : Z_t(\omega) = k\} \wedge T$$

the first time the random walk hits the level $k$. If the level is reached before the terminal time $T$ we put

$$\phi(\omega) = (y_1, \ldots, y_{\tau(\omega)}, -y_{\tau(\omega) + 1}, \ldots, -y_T);$$

otherwise $\phi(\omega) = \omega$ and $\tau(\omega) = T$. In order to obtain (i), let

$$A_{k,l}^\pm := \{\omega : Z_T(\omega) = k \pm l, M_T \geq k\}.$$
Then $\phi : A_{k,l}^- \rightarrow A_{k,l}^+$ is bijective. Furthermore,

$$A_{k,l}^+ = \{Z_T = k + l\} \text{ because } l \geq 0.$$  

In particular, $\mathbb{P}[A_{k,l}^-] = \mathbb{P}[Z_T = k + l]$. As for (ii), let $j = (T + k + l)/2$ if $T + k + l$ is even. In this case

$$\mathbb{P}[M_T = k, Z_T = k - l] = \mathbb{P}[M_T \geq k, Z_T = k - l] - \mathbb{P}[M_T \geq k + 1, Z_T = k - l]$$

$$= 2^{-T} \left( \begin{array}{c} T \\ j \end{array} \right) - 2^{-T} \left( \begin{array}{c} T \\ j + 1 \end{array} \right)$$

$$= 2^{-T} \left( \begin{array}{c} T + 1 \\ j + 1 \end{array} \right) \frac{2j + 1 - T}{T + 1}$$

$$= 2 \frac{k + l + 1}{T + 1} \mathbb{P}[Z_{T+1} = 1 + k + l].$$

If $T + k + l$ is not even, then $T + k - l$ is odd as well and both sides of the equation equal zero.

In order to establish a pricing formula for exotic derivatives we need a reflection under the equivalent martingale measure $\mathbb{P}^*$ rather than $\mathbb{P}$. To this end, we first notice that the distribution of the random walk under $\mathbb{P}^*$ is given by

$$\mathbb{P}^*[Z_t = k] = (p^*)^{t+k/2} (1 - p^*)^{t-k/2} \left( \begin{array}{c} t \\ t+k \end{array} \right)$$

if $t + k$ is even.

With this, one can show that the reflection principle under $\mathbb{P}^*$ reads:

$$\mathbb{P}^*[M_T \geq k, Z_T = k - l] = \left( \frac{1 - p^*}{p^*} \right)^l \mathbb{P}[Z_T = k + l] = \left( \frac{p^*}{1 - p^*} \right)^l \mathbb{P}[Z_T = -k - l]$$

and

$$\mathbb{P}[M_T = k, Z_T = k - l] = \frac{1}{p^*} \left( \frac{1 - p^*}{p^*} \right)^l \frac{k + l + 1}{T + 1} \mathbb{P}[Z_{T+1} = 1 + k + l]$$

$$= \frac{1}{1 - p^*} \left( \frac{p^*}{1 - p^*} \right)^l \frac{k + l + 1}{T + 1} \mathbb{P}[Z_{T+1} = -1 - k - l].$$

### 4.2.2 Valuation formulae for up-and-in call options

Based on the reflection principle we are now going to compute a closed form (though cumbersome) valuation formula for an up-and-in call option. To this end, we may with no loss of generality assume that the barrier lies within the range of asset prices, that is,

$$B = S_0 \hat{b}^k \text{ for some } k \in \mathbb{N}.$$
Now we write
\[
\pi = \mathbb{E}^*[(S_T - K)^+; \hat{M}_t \geq B] = \mathbb{E}^*[(S_T - K)^+; S_T \geq B] + \mathbb{E}^*[(S_T - K)^+; \hat{M}_t \geq B, S_T < B].
\]

The first term can be computed using the binomial distribution. In fact, if there are \( T - n \) up movements, then \( S_T = S_0 \hat{b}^{T-2n} \) and \( S_T \geq B \) if and only if \( T - 2n \geq k \). We denote by \( n_k \) the greatest such integer so that
\[
\mathbb{E}^*[(S_T - K)^+; S_T \geq B] = \frac{1}{(1 + r)^T} \sum_{n=0}^{n_k} (S_0 \hat{b}^{T-2n} - K)^+(p^*)^{T-n}(1 - p^*)^n \left( \begin{array}{c} T \\ T - n \end{array} \right).
\]

In order to calculate the second term notice first that
\( S_T < B \iff Z_T = k - l \) for some \( l \geq 1 \).

Thus the reflections principle shows that the second term equals:
\[
\sum_{l \geq 1} \mathbb{E}^*[(S_T - K)^+; M_T \geq k, Z_T = k - l] = \sum_{l \geq 1} (S_0 \hat{b}^{k-l} - K)^+ \mathbb{P}^*[M_T \geq k, Z_T = k - l] = \sum_{l \geq 1} (S_0 \hat{b}^{k-l} - K)^+ \left( \frac{p^*}{1 - p^*} \right)^k \mathbb{P}^*[Z_T = -k - l] = \left( \frac{p^*}{1 - p^*} \right)^k \hat{b}^{2k} \sum_{l \geq 1} (S_0 \hat{b}^{k-l} - K)^+ \mathbb{P}^*[Z_T = -k - l].
\]

With
\[
\hat{K} = K \hat{b}^{-2k} = K \left( \frac{S_0}{B} \right)^2
\]
we see that
\[
\sum_{l \geq 1} \mathbb{E}^*[(S_T - K)^+; M_T \geq k, Z_T = k - l] = \left( \frac{p^*}{1 - p^*} \right)^k \left( \frac{S_0}{B} \right)^2 \mathbb{E}^*[(S_T - \hat{K}); S_T < B].
\]

The last term can be calculated by analogy to our receding considerations as
\[
\left( \frac{p^*}{1 - p^*} \right)^k \left( \frac{S_0}{B} \right)^2 \mathbb{E}^*[(S_T - \hat{K})^+; S_T < B] = \left( \frac{p^*}{1 - p^*} \right)^k \left( \frac{S_0}{B} \right)^2 \frac{1}{(1 + r)^T} \sum_{n=n_k+1}^{T} (S_0 \hat{b}^{T-2n} - \hat{K})^+(p^*)^{T-n}(1 - p^*)^n \left( \begin{array}{c} T \\ T - n \end{array} \right).
\]
4.2.3 Valuation formulae for lookback options

A lookback put option corresponds to the contingent claim
\[ C = \hat{M}_T - S_T. \]

The discounted arbitrage-free price is given by
\[ \pi = \frac{1}{(1+r)^T} \mathbb{E}^*[\hat{M}_T] - S_0. \]

The expectation of the running maximum can be computed as
\[ \mathbb{E}^*[\hat{M}_T] = S_0 \hat{b}^k \mathbb{P}^*[M_T = k]. \]

By the reflection principle
\[ \mathbb{P}^*[M_T = k] = \sum_{l \geq 0} \mathbb{P}^*[M_T = k, Z_T = k - l] \]
\[ = \sum_{l \geq 0} \frac{1}{1 - p^*} \left( \frac{p^*}{1 - p^*} \right)^k \frac{k + l + 1}{T + 1} \mathbb{P}^*[Z_{T+1} = -1 - k - l] \]
\[ = \frac{1}{1 - p^*} \left( \frac{p^*}{1 - p^*} \right)^k \frac{1}{T + 1} \mathbb{E}^*[-Z_{T+1}; Z_{T+1} \leq -1 - k]. \]

The latter formula can again be given in closed form.

4.3 Convergence to Black-Scholes Prices

In the preceding section we obtained closed form, though sometimes cumbersome valuation formula for European call and certain exotic options. In this section we consider a sequence of CRR models where the time lag between two consecutive trading dates tends to zero and study convergence properties of the associated sequence of options prices. Thus, for the remainder of this chapter \( T \) denotes the terminal date rather than the number of trading times. Instead we divide the time interval \([0, T]\) into \(N\) equidistant time steps \(t_1, ..., t_N\) where we interpret \(t^k\) as the \(k\)-th trading period. We assume that there is a single risky asset; in the \(N\)-th approximation we denote its price process by \(S^N\). For the riskless bond we assume that \((1 + r_N)^N \to e^T\) as \(N \to \infty\) for some constant \(r > 0\). For every \(N\) the process \(S^N\) is defined on a probability space \((\Omega, \mathcal{F}^N, \mathbb{P}^N)\) where \(\mathcal{F}^N\) is the \(\sigma\)-field generated by the asset prices and \(\mathbb{P}^N\) denotes the unique EMM. Thus, the discounted price process \(X^N\) is an \(\mathcal{F}^N\)-martingale with respect to \(\mathbb{P}^N\). The returns \(R^N_t\) take values in a set \(\{a_N, b_N\}\) where \(-1 < a_N < r_N < b_N\) and
\[ \lim_{N \to \infty} a_N = \lim_{N \to \infty} b_N = 0. \]
Finally, we assume that the variances $V_{Nt}$ of the return $R_{Nt}$ under $\mathbb{P}^N$ satisfy

$$\sigma_N^2 := \frac{1}{T} \sum_{k=1}^{N} V_{Nt} \to \sigma^2 \in (0, \infty).$$

The preceding assumptions are satisfied if, for instance,

$$a_N = \frac{a}{\sqrt{N}}, \quad b_N = \frac{b}{\sqrt{N}}.$$ 

This means that we can start with some benchmark CRR model and then derive a continuous time limit by suitably rescaling its dynamics in space and time.

Before stating the main convergence result for asset prices, we recall that a random variable $Y$ is called log-normally distributed if $\log X$ follows a normal distribution. We also recall that a sequence of probability distributions $\{\mu_n\}$ converges weakly to a probability measure $\mu$ if the integrals of bounded continuous functions with respect to $\mu_n$ converge to the integral with respect to $\mu$.

**Theorem 4.6** Under the above assumptions the distributions of the terminal prices $S_N^t$ under $\mathbb{P}^N$ converge weakly to the distribution of

$$S_T = S_0 \exp \left( \sigma W_T + (r - \frac{1}{2} \sigma^2)T \right)$$

where $W_T \sim N(0, T)$, i.e., $S_T$ has a log-normal distribution.

# 5 Introduction to Optimal Stopping and American Options

This section contains a brief introduction into the theory of optimal stopping and American options. Our analysis follows Section 2 of the textbook by Lamberton & Lapeyre (1996). Throughout we assume that all stochastic processes are defined on a probabilistic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, that the sample space is finite and that $\mathcal{F}_0$ is the trivial $\sigma$-field.

## 5.1 Motivation and introduction

Let is consider an American put option on an underlying with price process $(S_t)$ and assume that there exists a unique equivalent martingale measure $\mathbb{P}^*$. Since the option can be exercised at any time prior to maturity we shall define its value in terms of a positive sequence adapted $(Z_t)$, where $Z_t$ is the profit from immediate exercise at time $t$, that is,

$$Z_t = (K - S_t)^+.$$
If the option has not been exercised early, then its value $U_T$ at time $T$ is

$$U_T = Z_T.$$ 

At what price should one sell the option at time $T-1$? Straight exercise yields $Z_{T-1}$. At the same time, the seller must be prepared to meet his obligations at time $T$ should the owner of the option choose not to exercise it early. Hence,

$$U_{T-1} = \max\{Z_{T-1}, \frac{1}{1+r}E^*[Z_T|F_{T-1}]\} = \max\{Z_{T-1}, \frac{1}{1+r}E^*[U_T|F_{T-1}]\}.$$ 

Similarly,

$$U_{t-1} = \max\{Z_{t-1}, \frac{1}{1+r}E^*[U_t|F_{t-1}]\}.$$ 

In terms of the discounted sequences $\tilde{Z}_t := Z_t/(1+r)$ and $\tilde{U}_t := U_t/(1+r)$ we obtain that

$$\tilde{U}_{t-1} = \max\{\tilde{Z}_{t-1}, E^*[\tilde{U}_t|F_{t-1}]\}.$$ 

The following proposition characterizes the value of an American put option in terms of a super-martingale property.

**Proposition 5.1** The sequence $(\tilde{U}_t)$ is the smallest $\mathbb{P}*$ super-martingale that dominates $\tilde{Z}_t$.

**Proof:** From

$$\tilde{U}_{t-1} = \max\{\tilde{Z}_{t-1}, \frac{1}{1+r}E^*[\tilde{U}_t|F_{t-1}]\}$$

it follows that $\tilde{U}_{t-1} \geq \tilde{Z}_{t-1}$ and that $(\tilde{U})$ is a supermartingale. Let us then consider another super-martingale $\tilde{V}$ that dominates $(\tilde{Z}_t)$. Then

$$\tilde{V}_T \geq \tilde{U}_T = \tilde{Z}_T.$$

Furthermore, if $\tilde{V}_t \geq \tilde{U}_t$, then

$$\tilde{T}_{t-1} \geq E^*[\tilde{T}_t|F_{t-1}] \geq E^*[\tilde{U}_t|F_{t-1}] \geq \max\{\tilde{Z}_{t-1}, E^*[\tilde{U}_t|F_{t-1}]\} = \tilde{U}_{t-1}.$$ 

As a result the assertion follows from by backward induction. □
5.2 Stopping times

The buyer of an American option has the right to exercise the option at any time before maturity. The decision whether or not to exercise at time $t$ will be made according to the information available at that time. The exercise strategy is therefore defined by a stopping time.

**Definition 5.2** A random variable $\tau$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is called a stopping time if for any time $t$ the event

$$\{\tau = t\} \text{ belongs to } \mathcal{F}_t.$$ 

**Remark 5.3** If $\tau$ is stopping time, then

$$\{\tau \leq t\} = \{\tau = 1\} \cup \cdots \cup \{\tau = t\} \in \mathcal{F}_t.$$ 

On the other hand $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \leq T$ implies that

$$\{\tau = t\} = \{\tau \leq t\} \setminus \{\tau \leq t - 1\} \in \mathcal{F}_t.$$ 

Thus, $\tau$ is a stopping time if and only if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t = 1, \ldots, T$.

In order to analyze American options we need to study stochastic processes stopped at a stopping time. For a adapted process $(X_t)$ and a stopping time $\tau$ we define the stopped process $(X_\tau^t)$ by

$$X_\tau^t(\omega) = X_{\tau(\omega) \land t}(\omega) = \begin{cases} 
  X_t(\omega) & \text{if } \tau(\omega) \geq t \\
  X_{\tau(\omega)}(\omega) & \text{else}
\end{cases}.$$ 

The following proposition shows that martingales are stable with respect to stopping. This result is important because it states that under the (any) risk neutral measure the martingale property of stock prices, and hence the assumption of no arbitrage, is preserved when early exercise of options is allowed.

**Proposition 5.4** Let $(X_t)$ be an adapted process and $\tau$ a stopping time. Then the following holds:

(i) The stopped sequence $(X_\tau^t)$ is adapted.

(ii) If $(X_t)$ is a (super-) martingale, then $(X_\tau^t)$ is a (super-) martingale.

**Proof:** For any $t \geq 1$ we have that

$$X_\tau^t = X_0 + \sum_{j=1}^{t} 1_{\{j \leq \tau\}}(X_j - X_{j-1}).$$
Since \( \{ j \leq \tau \} = \{ \tau < j \}^c \in \mathcal{F}_{j-1} \) we see that the sequence \( (1_{\{j \leq \tau\}}) \) is predictable so the stopped process is indeed adapted. If, in addition, \((X_t)\) is a martingale with respect to the filtration \((\mathcal{F}_t)\), then

\[
E[X^\tau_t | \mathcal{F}_{t-1}] = E\left[ X_0 + \sum_{j=1}^{t} 1_{\{j \leq \tau\}}(X_j - X_{j-1}) | \mathcal{F}_{t-1} \right]
\]

\[
= X_0 + \sum_{j=1}^{t} 1_{\{j \leq \tau\}}E[X_j - X_{j-1} | \mathcal{F}_{t-1}]
\]

\[
= X^\tau_{t-1}.
\]

The super-martingale property follows from similar arguments. 

\[\square\]

### 5.3 The Snell envelope

We have already seen that the value process of an American put option is given by the smallest \(\mathbb{P}^\star\) super-martingale associated with the gain process from immediate exercise. Such processes are called Snell envelopes. The goal of this section is study the properties of Snell envelopes in greater detail.

**Definition 5.5** Let \((Z_t)\) be an adapted process defined on \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\). The associated snell envelope \((U_n)\) is the smallest \(\mathbb{P}\) super-martingale that dominates \((Z_t)\), i.e.,

\[
\left\{ \begin{array}{l}
U_T = Z_T \\
U_t = \max\{Z_t, E[U_{t+1} | \mathcal{F}_t]\} \quad \text{for all } t \leq T_1
\end{array} \right.
\]

If \(U_t > Z_t\), i.e., if early exercise of the option is not beneficial, then \(U_t = E[U_{t+1} | \mathcal{F}_t]\). The following proposition shows that, by stopping adequately, it is indeed possible to obtain a martingale.

**Proposition 5.6** The random variable

\[
\tau := \inf\{t : U_t = Z_t\}
\]

is a stopping time and the stopped process \((U^\tau_t)\) is a martingale.

**Proof:** The random variable \(\tau\) is well defined because \(U_T = Z_T\). Furthermore,

\[
\{ \tau = 0 \} = \{ U_0 = Z_0 \} \in \mathcal{F}_0
\]

and for \(t \geq 1\) we have

\[
\{ \tau = t \} = \{ U_0 > Z_0 \} \cap \cdots \cap \{ U_{t-1} > Z_{t-1} \} \cap \{ U_t = Z_t \} \in \mathcal{F}_t.
\]
Thus, $\tau$ is a stopping time. In order to establish the martingale property, we write

$$U_t^\tau = U_0 + \sum_{j=1}^{t} \mathbf{1}_{\{j \leq \tau\}}(U_j - U_{j-1}).$$

Hence

$$U_{t+1}^\tau - U_t^\tau = \mathbf{1}_{\{t+1 \leq \tau\}}(U_{t+1} - U_t).$$

On the set $\{t + 1 \leq \tau\}$ we have that $U_t > Z_t$ so $U_t = \mathbb{E}[U_{t+1}|\mathcal{F}_t]$. Thus,

$$U_{t+1}^\tau - U_t^\tau = \mathbf{1}_{\{t+1 \leq \tau\}}(U_{t+1} - \mathbb{E}[U_{t+1}|\mathcal{F}_t]).$$

Taking conditional expectations yields

$$\mathbb{E}[U_{t+1}^\tau - U_t^\tau | \mathcal{F}_t] = \mathbf{1}_{\{t+1 \leq \tau\}}\mathbb{E}[U_{t+1} - \mathbb{E}[U_{t+1}|\mathcal{F}_t]] = 0$$

which proves that $(U_t^\tau)$ is a martingale.

Our goal is now to establish an optimality property of the stopping time $\tau$ defined in (14). More precisely, if we think of $Z_t$ as the gains of a gambler after $t$ games, then the strategy $\tau$ maximizes the expected gains. In order to see this, we denote by $\mathcal{T}_{t,T}$ the set of all stopping time taking values in the set $\{t, t+1, \ldots, T\}$.

Proposition 5.7 The stopping time $\tau$ satisfies

$$U_0 = \mathbb{E}[Z_\tau] = \sup_{\tau' \in \mathcal{T}_{0,T}} \mathbb{E}[Z_{\tau'}].$$

PROOF: Since $U^\tau$ is a martingale,

$$U_0 = U_0^\tau = \mathbb{E}[U_T^\tau] = \mathbb{E}[U_\tau] = \mathbb{E}[Z_\tau].$$

On the other hand, if $\nu \in \mathcal{T}_{0,T}$, then $U_\nu$ is a supermartingale so

$$U_0 \geq \mathbb{E}[U_T^\nu] = \mathbb{E}[U_\nu] \geq \mathbb{E}[Z_\nu].$$

This proves the assertion.

Let us call a stopping time $\tau^*$ optimal for the adapted process $(Z_t)$ if

$$Z_{\tau^*} = \sup_{\tau' \in \mathcal{T}_{0,T}} \mathbb{E}[Z_{\tau'}].$$

The preceding proposition stats that $\tau$ defined in (14) is optimal. The next theorem shows that among all the optimal stopping times, $\tau$ is minimal.
Theorem 5.8 A stopping time $\tau^*$ is optimal for $(Z_t)$ if and only if
\[ Z_{\tau^*} = U_{\tau^*} \quad \text{and} \quad (U_t^*) \text{ is a martingale.} \]

PROOF: If $\tau^*$ is optimal, then
\[ U_0 = \mathbb{E}[Z_{\tau^*}] \leq \mathbb{E}[U_{\tau^*}]. \]
Since the stopped process is a supermartingale, we also have the converse inequality. Hence
\[ \mathbb{E}[Z_{\tau^*}] = \mathbb{E}[U_{\tau^*}]. \]
Since $U_{\tau^*} \geq Z_{\tau^*}$ it follows that $Z_{\tau^*} = U_{\tau^*}$. In order to show that the stopped process is a martingale notice first that the supermartingale property yields
\[ \mathbb{E}[U_{\tau^*}] = U_0 \geq \mathbb{E}[U_t^*] \geq \mathbb{E}[U_{\tau^*}] = U_0. \]
Hence
\[ \mathbb{E}[U_t^*] = \mathbb{E}[\mathbb{E}[U_{\tau^*}|\mathcal{F}_t]]. \]
Since $U_t^* \geq \mathbb{E}[U_{\tau^*}|\mathcal{F}_t]$ it follows that
\[ U_t^* = \mathbb{E}[U_{\tau^*}|\mathcal{F}_t] \]
and hence the martingale property. The converse direction is obvious. $\Box$

The next step is to characterize the largest optimal stopping time. This will be based on the Doob-decomposition of supermartingales.

5.4 Decomposition of Supermartingales and pricing of American options
Supermartingales can written as the difference of a martingale and some non-decreasing predictable process as shown by the following proposition.

Proposition 5.9 Let $(U_t)$ be a supermartingale. There exists a unique decomposition
\[ U_t = M_t - A_t \]
where $(M_t)$ is a martingale and $(A_t)$ is a non-decreasing predictable process with $A_0 = 0$.

PROOF: It is obvious that $A_0 = 0$ implies $M_0 = U_0$. If a decomposition with the desired property exists than
\[ U_{t+1} - U_t = M_{t+1} - M_t - (A_{t+1} - A_t) \]
and the process $(A_t)$ is uniquely defined by taking conditional expectation in the equation above:
\[ -(A_{t+1} - A_t) = \mathbb{E}[U_{t+1}|\mathcal{F}_t] - U_t. \]
(15)
This process is predictable and non-decreasing because of the supermartingale property of \((U_t)\) With this, we see that the martingale is uniquely defined by

\[
M_{t+1} - M_t = U_{t+1} - \mathbb{E}[U_{t+1} | \mathcal{F}_t].
\]  

(16)

This shows that there is at most one such decomposition. On the other hand, the processes defined by (15) and (16) satisfy the desired properties. This proves the assertion.

We are now ready to characterize the largest optimal stopping time of a process \((Z_t)\) with associated Snell envelope \((U_t)\) in terms of the Doob decomposition \(U_t = M_t - A_t\).

**Theorem 5.10** The largest optimal stopping \(\nu\) time for \((Z_t)\) is given by \(\nu = A_T\) if \(A_T = 0\) and \(\nu = \inf\{t : A_{t+1} \neq 0\}\).

**Proof:** Since \((A_t)\) is predictable, \(\nu\) is a stopping time and apparently \(U_\nu = M_\nu\) so the stopped process is a martingale. In order to show optimality it is thus enough to show that

\[
U_\nu = Z_\nu.
\]

To this end, we notice that

\[
U_\nu = \sum_{j=0}^{T-1} 1_{\{\nu=j\}} U_j + 1_{\{\nu=T\}} U_T
\]

\[
= \sum_{j=0}^{T-1} 1_{\{\nu=j\}} \max\{Z_j, \mathbb{E}[U_{j+1} | \mathcal{F}_j]\} + 1_{\{\nu=T\}} Z_T.
\]

Since

\[
\mathbb{E}[U_{j+1} | \mathcal{F}_j] = M_j - A_{j+1}
\]

and \(A_j = 0\) and \(A_{j+1} > 0\) on \(\{\nu = j\}\) we see that \(U_j = M_j\) and \(\mathbb{E}[U_{j+1} | \mathcal{F}_j] = M_j - A_{j+1} < U_j\).

Hence

\[
U_j = \max\{Z_j, \mathbb{E}[U_{j+1} | \mathcal{F}_j]\} = Z_j
\]

so

\[
U_\nu = Z_\nu.
\]

It remains to show that \(\nu\) is the largest optimal stopping time. For this, we consider a stopping time \(\sigma \geq \nu\) with \(\mathbb{P}[\sigma > \nu] > 0\). Then

\[
\mathbb{E}[U_\sigma] = \mathbb{E}[M_\sigma] - \mathbb{E}[A_\sigma] = \mathbb{E}[U_0] - \mathbb{E}[A_\nu] < \mathbb{E}[U_0]
\]

and \((U_t)\) cannot be a martingale. \(\square\)
We are now going to apply the theory of optimal stopping to American options. We have already seen that the value process \((U_t)\) of an American option described by process \((Z_t)\) is given by the associated Snell envelope (after discounting). From our general theory we know that

\[
\tilde{U}_t = \sup_{\tau' \in \mathcal{I}_{t,T}} \mathbb{E}^* \left[ Z_{\tau'} | \mathcal{F}_t \right]
\]

so

\[
U_t = (1 + r)^t \sup_{\tau' \in \mathcal{I}_{t,T}} \mathbb{E}^* \left[ (1 + r)^{-t} Z_{\tau'} | \mathcal{F}_t \right].
\]

Furthermore,

\[
\tilde{U}_t = \tilde{M}_t - \tilde{A}_t
\]

for a \(\mathbb{P}^*\)-martingale \((\tilde{M}_t)\) and an increasing, predictable process \((\tilde{A}_t)\). Since the market is complete, there exists a self-financing strategy such that the associated discounted value process \((\tilde{V}_t)\) satisfies

\[
\tilde{V}_T = \tilde{M}_T.
\]

Since \((\tilde{V}_t)\) is a \(\mathbb{P}^*\)-martingale, \(\tilde{V}_t = \tilde{M}_t\) and

\[
\tilde{U}_t = \tilde{V}_t - \tilde{A}_t.
\]

As a result, the writer of an option can hedge himself perfectly: he can generate a portfolio worth \(V_t\) at time \(t\) and may even withdraw some funds \((A_t)\) for consumption. As for the buyer, there is no point executing the option after time

\[
\nu = \inf\{ j : A_{j+1} \geq 0 \}.
\]

Hence the optimal exercise time is given by a stopping time \(\tau \leq \nu\) that makes \(U^\tau\) a \(\mathbb{P}^*\)-martingale.

### 5.5 American options in the CRR model

We have seen above, a European call option has a non-negative time value. This suggests that an early exercise of a European call is not beneficial. In order to make this more precise, let is denote by \((U_t)\) the discounted value process associated with an American option that is characterized by an adapted process \((Z_t)\) and let \((u_t)\) be the discounted price process of a European option defined by the terminal payoff \(Z_T\). The supermartingale property of \((U_t)\) and the martingale property of \((u_t)\) under \(\mathbb{P}^*\) implies that

\[
U_t \geq \mathbb{E}^* [U_T | \mathcal{F}_t] = \mathbb{E}^* [Z_T | \mathcal{F}_t] = \mathbb{E}^* [u_T | \mathcal{F}_t] = u_t
\]
for all $t$. If, at the same time $u_t \geq \tilde{Z}_t$ then $(U_t)$ is a $\mathbb{P}^*$ martingale because $(U_t)$ is the smallest $\mathbb{P}^*$-supermartingale that dominates $(\tilde{Z}_t)$. For a call option

$$Z_t = (S_t - K)^+$$

so the martingale property of the discounted price process $(X_t)$ yields

$$u_t = \frac{1}{(1 + r)^T} \mathbb{E}^*[\left((S_T - K)^+\right)|\mathcal{F}_t]$$

$$\geq \mathbb{E}^*[\left(X_T - K(1 + r)^{-T}\right)|\mathcal{F}_t]$$

$$= X_t - K(1 + r)^{-T}$$

$$\geq \tilde{Z}_t$$

provides the risk free rate is non-negative. This shows that it is not beneficial to exercise an American call option early. Unlike the European call option, the time value

$$W_t := (1 + r)^t \mathbb{E}^*\left[\frac{(K - S_T)^+}{(1 + r)^T} | \mathcal{F}_t\right] - (K - S_T)^+$$

of a European put option usually becomes negative at a certain time. This corresponds to an early exercise premium, which, in turn, is the surplus which an owner of an American option would have over the value of a European put.

We illustrate this effect in the context of the CRR model. To this end, we recall that

$$S_t = S_0 \lambda_t \quad \text{where} \, \lambda_t := \prod_{j=1}^{t} R_j,$$

where the returns process $(R_t)$ is given by a sequence of independent and identically distributed binary random variables:

$$R \in \{a, b\} \quad \text{with} \quad -1 < a < r < b$$

and

$$\mathbb{P}^*[R_t = b] = p^* = \frac{r - a}{b - a}.$$ 

We assume that $r > 0$ and $a < 0$ and denote by

$$\pi(x) := \sup_{\tau \in \mathcal{T}, \tau} \mathbb{E}^*\left[\frac{(K - x\Lambda_{\tau})^+}{(1 + r)^\tau}\right]$$

the value of an American put as a function of the initial price $x = S_0$. If the option is far our of the money in the sense that

$$x \geq \frac{K}{(1 + a)^T}$$
then $S_t \geq K$ for all $t$ and the payoff of the put is always zero. In particular $\pi(x) = 0$. If

$$x \leq \frac{K}{(1 + b)^T},$$

then $S_t \leq K$ for all $t$ and the martingale property of asset prices yields

$$\pi(x) = \sup_{\tau \in \mathcal{T}, T} \mathbb{E}^* \left[ \frac{K - x \Lambda_{\tau}}{(1 + r)^T} \right] = \sup_{\tau \in \mathcal{T}, T} \mathbb{E}^* \left[ \frac{K}{(1 + r)^T} - x \right] = K - x.$$

In this case the price of the American put equals its intrinsic value at time $t = 0$ and immediate execution is optimal. Next, we consider the case

$$K \leq x < \frac{K}{(1 + a)^T}$$

where the option is “at the money” (that is, $K \approx x$) or not too far out of the money. Under the risk neutral measure $(R_t)$ essentially a symmetric random walk. Thus

$$\mathbb{P}^* \left[ \limsup_{t \to \infty} R_t = +\infty \right] = 1 \quad \text{and} \quad \mathbb{P}^* \left[ \liminf_{t \to \infty} R_t = -\infty \right] = 1.$$

This implies that for a large enough $t$ the probability of a non-zero payoff from immediate execution is strictly positive:

$$\mathbb{P}[Z_t > 0] > 0.$$

It follows that $\pi(x)$ is strictly positive while the intrinsic value $(K - x)^+$ vanishes. Hence immediate execution is not optimal. Thus, the map $x \mapsto \pi(x)$ has the following qualitative structure: there exists $x^*$ with

$$\frac{K}{(1 + b)^T} < x^* \leq K$$

such that

$$\begin{cases} 
\pi(x) = (K - x)^+ & \text{for } x \leq x^* \\
\pi(x) > (K - x)^+ & \text{for } x^* \leq x < K/(1 + a)^T \\
\pi(x) = 0 & \text{else}
\end{cases}$$

**Remark 5.11** In the context of the CRR model asset prices, and hence discounted asset prices follow a time-homogeneous Markov chain. Furthermore, the value from immediate execution is given in term of a time-dependent function of the stock price:

$$Z_t = h(t, S_t).$$

The general theory of Snell envelopes for Markov chain implies that the value process is also given in terms of a time-dependent transformation of asset prices:

$$U_t = u(t, S_t).$$
where the functions \( u(t, \cdot) \) can be determined recursively:

\[
u(T, x) = h(T, x)
\]

and

\[
u(t, x) = \max \{ h(t, x), u(t + 1, x(1 + b)p^* + u_{t+1}(x(1 + a)(1 - p^*)) \}.
\]

From this we see that the state space \([0, T] \times [0, \infty)\) can be decomposed into two regions, a stopping region

\[
\mathcal{R}_s = \{(t, x) : u(t, x) = h(t, x)\}
\]

and a continuation region

\[
\mathcal{R}_c = \{(t, x) : u(t, x) > h(t, x)\}
\]

and the minimal optimal stopping time can be viewed as the first exit time of the time-space process \((t, S_t)\) (a homogeneous Markov chain) from \(\mathcal{R}_c\).

6 Introduction to risk measures

In this section we follow Chapter 4 of Föllmer & Schied (2004) in discussing the problem of quantifying the risk of a financial position \(X\). In a probabilistic model specified by a set of scenarios and a probability measure on scenarios we could try to measure the risk in terms of moments and quantiles. A classical measure of risk is the variance. However, it does not capture a basic asymmetry in the financial interpretation of \(X\). Here, the downside risk matters. This asymmetry is taken into account by measures such as Value at Risk, V@R, however, fails to satisfy some natural consistency requirements. Such observations have motivated an axiomatic approach to risk measures.

6.1 Risk measures and their acceptance sets

Let \(\Omega\) be a set of scenarios. A financial position is described by a mapping \(X : \Omega \to \mathbb{R}\) where \(X(\omega)\) is the discounted net worth at the end of the trading period if \(\omega \in \Omega\) realizes. We assume throughout that \(X\) belongs to a given class \(\mathcal{X}\), where \(\mathcal{X}\) is a linear space of bounded functions. The risk associated with \(X\) is quantified by some number \(\varrho(X)\).

**Definition 6.1** A mapping \(\varrho : \mathcal{X} \to \mathbb{R}\) is called a monetary measure of risk if it satisfies the following conditions for all \(X, Y \in \mathcal{X}\):

- **Monotonicity:** \(\varrho(X) \geq \varrho(Y)\) if \(X \leq Y\).
- **Translation (Cash) Invariance:** If \(m \in \mathbb{R}\), then \(\varrho(X + m) = \varrho(X) - m\).
Translation invariance is motivated by the interpretation of \( \varrho(X) \) as a capital requirement. We view \( \varrho(X) \) as the amount of money that should be added to the position \( X \) in order to make it acceptable from the point of view of a supervising agency. Translation invariance implies that

\[
\varrho(X + \varrho(X)) = 0 \quad \text{and} \quad \varrho(m) = \varrho(0) - m \quad (m \in \mathbb{R}).
\]

**Remark 6.2** Monotonicity along with translation invariance implies that monetary risk measures are Lipschitz continuous with respect to the supremum norm:

\[
|\varrho(X) - \varrho(Y)| \leq \|X - Y\|.
\]

**Proof:** Clearly, \( X \leq Y + \|X - Y\| \) so by monotonicity and cash invariance

\[
\varrho(Y) - \|X - Y\| \leq \varrho(X).
\]

Reversing the roles of \( X \) and \( Y \) yields the assertion. \( \square \)

From a practical point of view monetary risk measures should encourage diversification. The risk associated with a diversified portfolio should be no greater than the risk associated with a non-diversified portfolio. This property is captured by the notion of “convexity”.

**Definition 6.3** A monetary risk measure \( \varrho : \mathcal{X} \to \mathbb{R} \) is called a convex measure of risk if it satisfies

- **Convexity:** \( \varrho(\lambda X + (1 - \lambda)Y) \leq \lambda \varrho(X) + (1 - \lambda) \varrho(Y) \) for \( 0 \leq \lambda \leq 1 \).

A convex measure of risk is called a coherent measure of risk if it is positively homogeneous, i.e., if it satisfies

- **Positive Homogeneity:** If \( \lambda \geq 0 \) then \( \varrho(\lambda X) = \lambda \varrho(X) \).

If a risk measure is positively homogeneous, it is normalized, i.e., \( \varrho(0) = 0 \). Under the assumption of positive homogeneity, convexity is equivalent to

- **Subadditivity:** \( \varrho(X + Y) \leq \varrho(X) + \varrho(Y) \).

This property allows to decentralize the task of managing the risk arising from a collection of different positions. A risk manager could, for instance, assign different risk limits to different trading “desks” and the overall risk would be bounded by the sum of the individual risks. However, in many situations risk grows in non-linear manner; a prominent example is liquidity risk. We shall hence focus on convex risk measures.
Example 6.4 (Entropic Risk Measure) Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$. The entropic risk measure is given by
\[ \varrho(X) = \frac{1}{\beta} \log \mathbb{E}_\mathbb{P}[e^{-\beta X}] . \]
The entropic risk measure is analytically convenient. It is primarily of academic interest, though, because it is closely related to exponential utility functions.

We denote the acceptance set associated with $\varrho$, i.e., the set of financial positions that are acceptable in the sense that they do not require any additional capital by
\[ \mathcal{A}_\varrho := \{ X \in \mathcal{X} : \varrho(X) \leq 0 \} . \]
The following proposition summarizes the link between risk measures and their acceptance sets.

Proposition 6.5 Suppose that $\varrho$ is a monetary measure of risk with acceptance set $\mathcal{A} = \mathcal{A}_\varrho$.

(i) $\mathcal{A} \neq \emptyset$ and satisfies the following two conditions:
\[ \inf \{ m \in \mathbb{R} : m \in \mathcal{A} \} > -\infty \quad \text{and} \quad X \in \mathcal{A}, Y \in \mathcal{X}, X \geq Y \Rightarrow Y \in \mathcal{A} . \quad (17) \]
Moreover, $\mathcal{A}$ is closed in the sense that for $X \in \mathcal{A}$ and $Y \in \mathcal{X}$ we have
\[ \{ \lambda \in [0, 1] : \lambda X + (1 - \lambda)Y \in \mathcal{A} \} \text{ is closed in } [0, 1] . \]

(ii) $\varrho$ can be recovered from $\mathcal{A}$:
\[ \varrho(X) = \inf \{ m \in \mathbb{R} : m + X \in \mathcal{A} \} . \]

(iii) $\varrho$ is a convex measure of risk if and only if $\mathcal{A}$ is convex.

(iv) $\varrho$ is positively homogeneous if and only if $\mathcal{A}$ is a cone.

Proof: (i) The first two properties are obvious. As for the closedness property, notice that the map $\lambda \mapsto \varrho(\lambda X + (1 - \lambda)Y)$ is continuous. Hence, the upper contour sets are closed, that is, the set of $\lambda \in [0, 1]$ such that
\[ \varrho(\lambda X + (1 - \lambda)Y) \leq 0 \]
is closed.
(ii) Cash invariance implies that
\[
\inf \{ m \in \mathbb{R} : m + X \in \mathcal{A}_\varrho \} = \inf \{ m \in \mathbb{R} : \varrho(m + X) \leq 0 \} = \inf \{ m \in \mathbb{R} : \varrho(X) \leq m \} = \varrho(X).
\]

(iii) Clearly, \( \mathcal{A} \) is convex if \( \varrho \) is convex. We shall prove the converse below.

(iv) Positive homogeneity clearly implies that \( \mathcal{A} \) is a cone; the converse follows as in (iii).

Conversely, we can take as given a set of acceptable positions \( \mathcal{A} \subset \mathcal{X} \) and define \( \varrho_{\mathcal{A}}(X) \) as the minimal capital requirement that makes \( X \) acceptable:
\[
\varrho_{\mathcal{A}}(X) = \inf \{ m \in \mathbb{R} : m + X \in \mathcal{A} \}.
\]

**Proposition 6.6** Suppose that \( \mathcal{A} \) is a non-empty subset of \( \mathcal{X} \) and satisfies (17). Then the functional \( \varrho_{\mathcal{A}} \) has the following properties:

(i) \( \varrho_{\mathcal{A}} \) is a monetary measure of risk.

(ii) If \( \mathcal{A} \) is convex, then \( \varrho_{\mathcal{A}} \) is a convex measure of risk.

(iii) If \( \mathcal{A} \) is a cone, then \( \varrho_{\mathcal{A}} \) is positively homogeneous.

Let us now consider some specific examples. We take \( \mathcal{X} \) as the set of all bounded measurable functions on a measurable space \((\Omega, \mathcal{F})\) and denote by \( \mathcal{M}_1 \) the class of all probability measures on \((\Omega, \mathcal{F})\).

**Example 6.7** (Worst Case Measure) The worst case measure is defined by
\[
\varrho_{\text{max}}(X) = - \inf_{\omega \in \Omega} X(\omega).
\]
The value \( \varrho_{\text{max}}(X) \) is the least upper bound for the potential loss than can occur in any scenario. It is the most conservative risk measure in the sense that any other other measure \( \varrho \) satisfies
\[
\varrho(X) \leq \varrho(\inf_{\omega} X(\Omega)) = \varrho_{\text{max}}(X).
\]
It can be represented as
\[
\varrho_{\text{max}}(X) = \sup_{Q \in \mathcal{M}_1} \mathbb{E}[X].
\]
Example 6.8 (Floors) Let $Q \subset M_1$ and consider a mapping $\gamma : Q \to \mathbb{R}$ with $\sup_Q \gamma < \infty$ that specifies for any $Q \in Q$ a “floor” $\gamma(Q)$. Suppose that a position is acceptable if

$$\mathbb{E}_Q[X] \geq \gamma(Q).$$

The acceptance set is convex so the associated risk measure $\varrho$ is convex. It takes the form

$$\varrho(X) = \sup_{Q \in Q} (\gamma(Q) - \mathbb{E}_Q[X]).$$

Alternatively, we can represent $\varrho$ in terms of the penalty function

$$\alpha(Q) = \begin{cases} -\gamma(Q) & \text{if } Q \in Q \\ +\infty & \text{else} \end{cases}$$

as

$$\varrho(X) = \sup_{Q \in M_1} (\mathbb{E}_Q[-X] - \alpha(Q)).$$

Example 6.9 (V@R) Suppose that we have specified a model, i.e., a probability measure $\mathbb{P}$ on $(\Omega, F)$. A position is often considered acceptable if the probability of a loss is bounded by a given level $\lambda \in (0, 1)$, i.e., if

$$\mathbb{P}[X < 0] \leq \lambda.$$

The corresponding monetary risk measure is called Value at Risk at level $\lambda$. It is defined by

$$V@R_\lambda(X) = \inf \{m \in \mathbb{R} : \mathbb{P}[m + X < 0] \leq \lambda \}.$$

$V@R$ is positively homogeneous but typically not convex, i.e., it may penalize diversification. The reason is that it only looks at the probability that something is happening but not at “what happens if something happens”.

In the preceding three Examples we encountered risk measures that allowed for a robust representation. In the next section we show how such representations arise in a systematic manner.

6.2 Robust representations of risk measures

Throughout this section we assume that $X$ denotes the set of bounded measurable functions $f : \Omega \to \mathbb{R}$ equipped with the sup-norm $\| \cdot \|$. We denote by $M_1$ the class of all probability measures on $(\Omega, F)$ and by

$$M_{1,f} = M_{1,f}(\Omega, F)$$

the class of all finitely additive set functions $Q : F \to [0, 1]$ which are normalized to $Q[\Omega] = 1$. 
Proposition 6.10 A functional \( \varrho \) on \( X \) is a coherent measure of risk if and only if there exists a set \( Q \subset M_{1,f} \) such that

\[
\varrho(X) = \sup_{Q \in Q} \mathbb{E}_Q[-X].
\]

Moreover, \( Q \) can be chosen as a convex set for which the supremum is attained:

\[
\varrho(X) = \max_{Q \in Q} \mathbb{E}_Q[-X].
\]

Proof: The necessity of being a convex risk measure for the representation to hold is obvious. The proof of the converse is based on the separating hyperplane theorem and the Riesz representation theorem. It states that any bounded continuous linear functional

\[
l : X \to \mathbb{R}
\]

can be represented as an expected value with respect to some finitely additive set function \( Q \) with finite total variation:

\[
l(Y) = \mathbb{E}_Q[Y] \quad (Y \in X).
\]

We put \( J(X) = \varrho(-X) \) construct for any position \( X \) a finitely additive set function \( Q_X \) such that

\[
J(X) = \mathbb{E}_{Q_X}[X] \quad \text{and} \quad J(Y) \leq \mathbb{E}_{Q_X}[Y] \quad (Y \in X).
\]

Then

\[
J(Y) = \min_{Q \in Q_0} \mathbb{E}_Q[Y] \quad \text{for all } Y \in X
\]

where \( Q_0 = \{Q_X : X \in X\} \). The above equation remains valid if we replace \( Q_0 \) by its convex hull. To construct \( Q_X \) we define the convex subsets of \( X \) by

\[
B = \{Y \in X : J(Y) > 1\}
\]

and

\[
C_1 = \{Y \in X : Y \leq 1\}, \quad C_2 = \{Y \in X : Y \leq \frac{X}{J(X)}\}.
\]

We denote by \( C \) the convex hull of the union of \( C_1 \) and \( C_2 \) so \( B \) and \( C \) are disjoint. The set \( C_1 \) and hence \( C \) contains the unit ball so \( C \) has a nonempty interior. Hence there exists a non-zero continuous linear functional \( l \) on \( X \) such that

\[
c := \sup_{Y \in C} l(Y) \leq \inf_{Z \in B} l(Z).
\]

Since \( C \) contains the unit ball, \( c \geq 1 \) so we may with no loss of generality assume that \( c = 1 \). In particular, \( l(Y) \leq 1 \) as \( 1 \in C \). On the other hand, any constant \( b > 1 \) is contained in \( B \) so \( l(1) = 1 \) because

\[
l(1) = \lim_{b \downarrow 1} l(b) \geq c = 1.
\]
If $A \in \mathcal{F}$ then $1_{A^c} \in \mathcal{C}_1 \subset \mathcal{C}$ which implies

$$l(1_A) = l(1) - l(1_{A^c}) \geq 1 - 1 = 0$$

so $l$ is bounded. As a result, it follows from the Riesz representation theorem that there exists a normalized set function $Q_X \in \mathcal{M}_{1,f}$ such that

$$l(Y) = E_{Q_X}[Y].$$

It remains to show that $E_{Q_X}[Y] \geq J(Y)$ for all $Y \in \mathcal{X}$ with equality for $Y = X$. By translation invariance of $J$ we need only consider the case in which $J(Y) > 0$. Then

$$Y_n := \frac{Y}{J(Y)} + \frac{1}{n} \in \mathcal{B},$$

and $Y_n \to Y/J(Y)$ uniformly, whence

$$\frac{E_{Q_X}[Y]}{J(Y)} = \lim_{n \to \infty} E_{Q_X}[Y_n] \geq 1.$$  

On the other hand, $X/J(X) \in \mathcal{C}_2 \subset \mathcal{C}$ yields the inequality

$$\frac{E_{Q_X}[X]}{J(X)} \leq c = 1.$$

Our goal is now twofold; (i) we would like to have a similar result for convex risk measures; (ii) it is an undesirable feature of the Riesz representation theorem that we only obtain a representation in terms of finitely additive set functions rather than probability measures. It turns out that (i) can be achieved by introducing a penalization; (ii) requires additional continuity or tightness conditions.

### 6.2.1 Robust representations of convex risk measures

Let $\alpha : \mathcal{M}_{1,f} \to \mathbb{R} \cup \{\infty\}$ be any function such that

$$\inf_{Q \in \mathcal{M}_{1,f}} \alpha(Q) \in \mathbb{R}.$$

For each $Q \in \mathcal{M}_{1,f}$ the function $X \mapsto E_Q[-X] - \alpha(Q)$ is convex, monotone and translation invariant. These properties are preserved when taking the supremum over the set of additive set functions. Hence

$$\varrho(X) := \sup_{Q \in \mathcal{M}_{1,f}} (E_Q[-X] - \alpha(Q))$$
defines a convex measure of risk such that
\[
\varrho(0) = -\inf_{Q \in \mathcal{M}_{1,f}} \alpha(Q).
\] (18)

The functional \(\alpha\) will be called a penalty function for \(\varrho\) on \(\mathcal{M}_{1,f}\) and we say that \(\varrho\) is represented by \(\alpha\) on \(\mathcal{M}_{1,f}\).

**Theorem 6.11** Any convex measure of risk \(\varrho\) on \(\mathcal{X}\) is of the form
\[
\varrho(X) = \max_{Q \in \mathcal{M}_{1,f}} (\mathbb{E}_Q[-X] - \alpha_\text{min}(Q))
\]
where the penalty function \(\alpha_\text{min}\) is given by
\[
\alpha_\text{min}(Q) := \sup_{X \in \mathcal{A}_\varrho} \mathbb{E}_Q[-X].
\]

Moreover, \(\alpha_\text{min}\) is the minimal penalty function that represents \(\varrho\).

**Proof:** We proceed in several steps:

i) We first notice that
\[
\varrho(X) \geq \max_{Q \in \mathcal{M}_{1,f}} (\mathbb{E}_Q[-X] - \alpha_\text{min}(Q)).
\]
This follows from the fact that \(X' = X + \varrho(X)\) is acceptable so that
\[
\alpha_\text{min}(X) \geq \mathbb{E}_Q[-X'] = \mathbb{E}_Q[-X] - \varrho(X) \quad (Q \in \mathcal{M}_{1,f}).
\]

ii) In order to prove the robust representation we now construct for a given position \(X\) a measure \(Q_X \in \mathcal{M}_{1,f}\) such that
\[
\varrho(X) \leq \mathbb{E}_{Q_X}[-X] - \alpha_\text{min}(Q_X).
\]
We may with no loss of generality assume that \(\varrho(X) = 0\) and \(\varrho(0) = 0\). In this case
\[
X \notin \mathcal{B} := \{Y \in \mathcal{X} : \varrho(Y) < 0\},
\]
the latter being an open convex set. By the separating hyperplane theorem there exists a continuous linear functional \(l : \mathcal{X} \to \mathbb{R}\) such that
\[
l(X) \leq \inf_{Y \in \mathcal{B}} l(Y) =: b.
\]
Monotonicity and translation invariance implies that \(l(Y) \geq 0\) if \(Y \geq 0\). Furthermore, \(l(1) > 0\).
By the Riesz representation theorem there exists some measure $Q_X$ such that
\[ \mathbb{E}_{Q_X}[Y] = \frac{l(Y)}{l(1)} \quad (Y \in \mathcal{X}). \]

Since all the elements of $\mathcal{B}$ are acceptable,
\[ \alpha_{\text{min}}(Q_X) = \sup_{Y \in \mathcal{A}_e} \mathbb{E}_{Q_X}[-Y] \geq \sup_{Y \in \mathcal{B}} \mathbb{E}_{Q_X}[-Y] = -\frac{b}{l(1)}. \]

Since $Y + \epsilon$ is acceptable for all $\epsilon > 0$ if $Y$ is acceptable, we obtain equality. This shows that
\[ \mathbb{E}_{Q_X}[-X] - \alpha_{\text{min}}(Q_X) = \frac{1}{l(1)} (b - l(X)) \geq 0 = \varrho(X). \]

iii) In order to show that $\alpha_{\text{min}}$ is the minimal penalty function, let $\alpha$ be any penalty function. Then
\[ \varrho(X) \geq \mathbb{E}_Q[-X] - \alpha(Q) \]
so
\[ \alpha(Q) \geq \sup_{X \in \mathcal{X}} (\mathbb{E}_Q[-X] - \varrho(X)) \]
\[ \geq \sup_{X \in \mathcal{A}_e} (\mathbb{E}_Q[-X] - \varrho(X)) \]
\[ \geq \alpha_{\text{min}}(Q). \]

The penalty function arising in (18) is not necessarily unique. For a coherent risk measure we already have a representation via some set $Q \subset \mathcal{M}_{1,f}$ that corresponds to the penalty function
\[ \alpha(Q) = \begin{cases} 0 & \text{if } Q \in Q \\ +\infty & \text{else} \end{cases}. \]

The following corollary shows that the minimal penalty function of a coherent risk measure is always of this form.

**Corollary 6.12** The minimal penalty function of a coherent risk measure takes only the values 0 and $+\infty$.

### 6.2.2 Robust representations in terms of probability measures

In the sequel we are interested in those convex measures of risk that admit a representation in terms of $\sigma$-additive probability measure. Such measures can be represented by penalty functions that are infinite outside the set $\mathcal{M}_1$ of probability measures on $(\Omega, \mathcal{F})$:
\[ \varrho(X) := \sup_{Q \in \mathcal{M}_1} (\mathbb{E}_Q[-X] - \alpha(Q)). \] (19)
This representation is closely related to continuity properties of $\varrho$. We say that a risk measure is continuous from above if

$$X_n \downarrow X \implies \varrho(X_n) \uparrow \varrho(X).$$

The following lemma shows that continuity from above is equivalent to lower-semicontinuity with respect to bounded pointwise convergence:

$$X_n \to X \in \mathcal{X} \quad \text{and} \quad \sup \|X_n\|_{\infty} < \infty$$
implies

$$\varrho(X) \leq \liminf_{n \to \infty} \varrho(X_n). \quad (20)$$

This property is usually referred to as the Fatou property.

**Lemma 6.13** Continuity from above is equivalent to the Fatou property.

**Proof:** If the Fatou property holds and $X_n \downarrow X$, then monotonicity yields

$$\varrho(X_n) \leq \varrho(X) \quad \text{for all} \quad n \in \mathbb{N}.$$ 

Taking the limit yields the assertion. Conversely, let $\varrho$ be continuous from above and $(X_n)$ a bounded sequence that converges pointwise to $X$. The sequence

$$Y_M := \sup_{n \geq m} X_n$$

decreases almost surely to $X$ so $\varrho(X_n) \geq \varrho(Y_n)$ yields

$$\liminf_{n \to \infty} \varrho(X_n) \geq \liminf_{n \to \infty} \varrho(Y_n) = \varrho(X).$$

We are now ready to show that a convex risk measure that admits a representation in term of probability measures is continuous from above.

**Lemma 6.14** A convex measure of risk $\varrho$ which admits a representation of the form (19) is continuous from above.

**Proof:** In view of the preceding lemma it is enough to prove that $\varrho$ has the Fatou property. To this end, let $(X_n)$ be a bounded sequence that converges almost surely to $X \in L^\infty$. Dominated convergence implies that

$$\mathbb{E}_Q[X_n] \to \mathbb{E}_Q[X] \quad \text{for all} \quad Q \in \mathcal{M}_1.$$
Hence

\[
\varrho(X) = \sup_{Q \in \mathcal{M}_1} \left( \lim_n \mathbb{E}_Q[X_n] - \alpha(Q) \right)
\leq \liminf_n \sup_{Q \in \mathcal{M}_1} (\mathbb{E}_Q[X_n] - \alpha(Q))
= \liminf_n \varrho(X_n).
\]

While “Continuity from Above” is necessary for (19) to hold, “Continuity from Below” is sufficient.

**Proposition 6.15** Let \( \varrho \) be a convex measure of risk that is continuous from below, i.e.,

\[
X_n \uparrow X \Rightarrow \varrho(X_n) \downarrow \varrho(X)
\]

and let \( \alpha \) be a representing penalty function. Then

\[
\alpha(Q) < \infty \Rightarrow Q \in \mathcal{M}_1.
\]

**Proof:** We assume with no loss of generality \( \varrho(0) = 0 \) and that \( 0 \leq X_n \leq 1 \) for all \( n \in \mathbb{N} \) and introduce the level sets

\[
\Lambda_c = \{ Q \in \mathcal{M}_1 \, : \, \alpha(Q) \leq c \} \quad (c \geq 0).
\]

Then

\[
c \geq \alpha(Q) \geq \mathbb{E}_Q[-\lambda X_n] - \varrho(\lambda X_n)
\]

so for all \( \lambda \geq 0 \) we obtain that

\[
\inf_{Q \in \Lambda_c} \mathbb{E}_Q[X_n] \geq \frac{c + \varrho(\lambda X_n)}{\lambda} \geq - \frac{c + \varrho(\lambda)}{\lambda} = - \frac{c - \lambda}{\lambda}.
\]

If we let \( \lambda \) increase to infinity, we see that

\[
\liminf_{n \to \infty} \inf_{Q \in \Lambda_c} \mathbb{E}_Q[X_n] \geq 1.
\]

For the special choice \( X_n = 1_{A_n} \) where \( \{A_n\} \) is an increasing sequence that satisfies \( A_n \uparrow \Omega \) we see that

\[
\liminf_{n \to \infty} Q[A_n] = \lim_{n \to \infty} Q[A_n] = 1
\]

whenever \( \alpha(Q) < \infty \). The latter property, however, is equivalent to \( \sigma \)-additivity. This proves the assertion. \( \square \)

We close this section with three prominent examples of risk measures that have certain continuity properties: the entropic risk measure, shortfall risk and average value at risk. All these measures require an a-priori model, i.e., a probability measure \( \mathbb{P} \) on \((\Omega, \mathcal{F})\).
Example 6.16 (Entropic Risk Measure) Consider the penalty function $\alpha : \mathcal{M}_1 \rightarrow (0, \infty]$ that is defined by

$$\alpha(Q) = \frac{1}{\beta} H(Q|P) \quad \text{where} \quad H(Q|P) = \mathbb{E}_Q[\log \frac{dQ}{dP}]$$

where $H(Q|P)$ denotes the relative entropy of $Q \in \mathcal{M}_1$ with respect to $P$. The entropic risk measure is given by

$$\varrho(X) = \sup_{Q \in \mathcal{M}_1} \left\{ \mathbb{E}_Q[-X] - \frac{1}{\beta} H(Q|P) \right\}$$

and one can show that

$$\varrho(X) = \frac{1}{\beta} \log \mathbb{E}_P[e^{-\beta X}].$$

Example 6.17 (Shortfall Risk) Let $l : \mathbb{R} \rightarrow \mathbb{R}$ be a convex loss function, i.e., a convex increasing function that is not identically zero and suppose that a position $X$ is acceptable if the expected loss $\mathbb{E}_P[l(-X)]$ is bounded from above by some interior point $x_0$ in the range of $l$. Thus the acceptance set takes the form

$$\mathcal{A} = \{ X \in \mathcal{X} : \mathbb{E}_P[l(-X)] \leq x_0 \}.$$

The associated convex risk measure $\varrho_\mathcal{A}$ is continuous from below; its penalty function can be given in closed form. The special case of shortfall risk is given by the acceptance set

$$\mathcal{A} = \{ X \in \mathcal{X} : \mathbb{E}_P[l(X^-)] \leq x_0 \}.$$

Example 6.18 (AV@R) The Average Value at Risk at level $\lambda \in (0, 1)$ of a position $X$ is given by

$$AV@R_\lambda(X) = \frac{1}{\lambda} \left( \int_0^\lambda V@R_\gamma(X) d\gamma \right).$$

Unlike $V@R$ that fails to be convex, $AV@R_\lambda$ is coherent. It is also continuous from below.

In many cases continuity from below (i.e. continuity) is too restrictive. One can show that certain tightness conditions are also sufficient for (19).

6.3 V@R, AV@R and Shortfall Risk

In this section we briefly discuss the current industry standard Value at Risk and its generalization Average Value at Risk which, in contrast to V@R is convex.
6.3.1 V@R

The V@R at level $\lambda$ can be expressed in terms of $\lambda$-quantiles. The $\lambda$-quantile of a random variable $X$ is defined as any $q \in \mathbb{R}$ that satisfies

$$P[X \leq q] \geq \lambda \quad \text{and} \quad P[X < q] \leq \lambda.$$ 

The set of all such quantiles is an interval $[q^-_X(\lambda), q^+_X(\lambda)]$ and

$$V@R_{\lambda}(X) = -q^+_X(\lambda) = q^-_X(1 - \lambda).$$

V@R is a positively homogeneous but in general not convex. Our goal is thus to look convex risk measures that come close to V@R. A first guess is to take the smallest convex risk measure on $L^\infty(\mathbb{P})$, continuous from above that dominates V@R. Such a risk measure, however, does not exists as shown by the following proposition.

**Proposition 6.19** For any $X \in L^\infty$ and each $\lambda \in (0, 1)$ we have that

$$V@R_{\lambda}(X) = \min \{g(X) : g \text{ convex, continuous from above, dominates V@R}\}.$$ 

**Proof:** The idea of the proof is to construct a set of measures $Q$ such that $V@R_{\lambda}$ is dominated by the risk measure

$$g(Y) := \sup_{Q \in Q} E_Q[-X] \quad \text{and} \quad V@R_{\lambda}(X) = g(X).$$

To this end, let $q = V@R_{\lambda}(X)$ so that $P[X < q] \leq \lambda$. If $A \in \mathcal{F}$ satisfies $P[A] > \lambda$, then

$$P[A \cap \{X \geq q\}] > 0$$

so we can define a measure $Q_A$ by

$$Q_A(\cdot) = P[\cdot | A \cap \{X \geq q\}].$$

We have that

$$E_{Q_A}[-X] \leq -q = V@R_{\lambda}(X)$$

and define the set $Q$ by

$$Q := \{Q_A : P[A] > \lambda\} \quad \text{so that} \quad g(X) \leq V@R_{\lambda}(X).$$

Let us now fix $Y \in L^\infty$ and $\epsilon > 0$ and consider the set

$$A := \{Y \leq -V@R_{\lambda}(Y) + \epsilon\}.\]
Then \( \mathbb{P}[A] > \lambda \) so \( Q_A \in \mathcal{Q} \). Moreover \( Q_A(A) = 1 \) so that we obtain

\[
g(Y) \geq \mathbb{E}_{Q_A}[-Y] \geq V@R_{\lambda}(Y) - \epsilon.
\]

This shows that

\[
g(Y) \geq V@R_{\lambda}(Y) \quad \text{for all } Y \in L^\infty.
\]

\( \square \)

### 6.3.2 AV@R

A risk measure which is defined in terms of V@R but satisfies the axioms of a coherent risk measure is Average Value at Risk:

\[
AV@R_{\lambda}(X) = \frac{1}{\lambda} \int_0^\lambda V@R_{\gamma}(X) d\gamma.
\]

**Remark 6.20** Under mild technical assumptions on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) one can show that AV@R at level \( \lambda \) coincides with the worst conditional expectation at level \( \lambda \). The latter risk measure is defined as

\[
WCE_{\lambda}(X) = \sup \{ \mathbb{E}[-X|A] : \mathbb{P}[A] > \lambda \}.
\]

The following theorem yields a robust representation of AV@R in terms of bounded densities.

**Theorem 6.21** Let \( \lambda \in (0, 1) \). AV@R is a coherent risk measure that is continuous from below. It has the representation

\[
AV@R_{\lambda}(X) = \max_{Q \in \mathcal{Q}_{\lambda}} \mathbb{E}_Q[-X]
\]

where \( \mathcal{Q}_{\lambda} \) is the set of all probability measures that are absolutely continuous with respect to \( \mathbb{P} \) whose density is bounded by \( \lambda^{-1} \). The maximum is attained by the measure \( Q^* \) with density

\[
\frac{dQ^*}{d\mathbb{P}} = \frac{1}{\lambda} \left( I_{\{X < q\}} + \kappa I_{\{X = q\}} \right) \quad \text{for some } \kappa.
\]

If \( A \in \mathcal{F} \) with \( \mathbb{P}[A] \geq \lambda \), then the density of \( \mathbb{P}[-|A] \) with respect to \( \mathbb{P} \) is bounded from above by \( \frac{1}{\lambda} \). Hence the preceding theorem shows that AV@R dominates WCE. Furthermore,

\[
\mathbb{P}[-X \geq V@R_{\lambda}(X) - \epsilon] > \lambda
\]

so

\[
WCE_{\lambda}(X) \geq \mathbb{E}[-X \mid X \geq V@R_{\lambda}(X) - \epsilon]
\]
and for $\epsilon \to 0$ we obtain that
\[
WCE_\lambda(X) \geq \mathbb{E}[-X] - X \geq V\@R_\lambda(X)] \geq V\@R_\lambda(X).
\]
In particular, we have shown that Average Value at Risk dominates the Worst Conditional Expectation.

### 6.3.3 Shortfall Risk

We close this section with a brief discussion of the shortfall risk. To this end, we introduce a convex, increasing and not identically constant loss function $l : \mathbb{R} \to \mathbb{R}$ and define, for an interior point $x_0$ of its range, an acceptance set
\[
\mathcal{A} := \{X \in \mathcal{X} : \mathbb{E}[l(-X)] \leq x_0\}.
\]
The associated convex risk measure $\rho$ is called shortfall risk. Its minimal penalty function is concentrated on the set of probability measures on $(\Omega, \mathcal{F})$ and can be given in closed form.

**Theorem 6.22** The minimal penalty function of shortfall is given by
\[
\alpha_{\min}(Q) = \inf_{\lambda > 0} \frac{1}{\lambda} (x_0 + \mathbb{E}[l^*(\lambda dQ/d\mathbb{P})]) \quad (Q \in \mathcal{M}_1)
\]
where $l^*$ is the Legendre-Fenchel transform of $l$, i.e.,
\[
l^*(z) := \sup_{x \in \mathbb{R}} (xz - l(x)).
\]
The special choice $l(x) = e^{\beta x}$ corresponds to the entropic risk measure. The case
\[
l(x) = \frac{1}{p} x^p \quad (p > 1, x \geq 0)
\]
yields
\[
l^*(z) = \frac{1}{q} z^q \text{ if } z \geq 0 \quad \text{and} \quad l^*(z) = +\infty \text{ otherwise}
\]
where $q = \frac{p}{p-1}$. If $Q << \mathbb{P}$ with density $\varphi$, then $\alpha_{\min}(Q) = \infty$ if $\varphi \notin L^q(\mathbb{P})$. Otherwise the infimum in the representation of $\alpha_{\min}$ is attained for
\[
\alpha_{\min}(Q) = (px_0)^{1/p} \mathbb{E} \left[ (\frac{dQ}{d\mathbb{P}}) q \right]^{1/q}.
\]
6.4 Law Invariance

We argued above that there is no smallest convex risk measure that dominates \( \text{V@R} \). The situation is different if we restrict ourselves to convex risk measures that dominate \( \text{V@R} \) and only depend on the distribution of a financial position. Risk measures that depend only the on the law of a random variable are called law invariant.

**Definition 6.23** A risk measure \( \varrho \) on \( X = L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \) is called law invariant if \( \varrho(X) = \varrho(Y) \) if \( X \) and \( Y \) have the same distribution under \( \mathbb{P} \).

The following theorem shows that AV@R can be viewed as a basis for the set of law invariant risk measures.

**Theorem 6.24** A convex risk measure \( \varrho \) is law invariant and continuous from above if and only if

\[
\varrho(X) = \sup_{\mu \in \mathcal{M}_1((0,1])} \left\{ \int_0^1 \text{AV@R}_\lambda(X) \mu(d\lambda) - \beta_{\text{min}}(\mu) \right\}
\]

where

\[
\beta_{\text{min}}(\mu) = \sup_{X \in \mathcal{A}_p} \int_0^1 \text{AV@R}_\lambda(X) \mu(d\lambda).
\]

For a coherent risk measure the preceding theorem takes the following form:

**Corollary 6.25** A coherent risk measure \( \varrho \) is law invariant and continuous from above if and only if

\[
\varrho(X) = \sup_{\mu \in \mathcal{M}_1((0,1])} \int_0^1 \text{AV@R}_\lambda(X) \mu(d\lambda).
\]

We proved above that a smallest convex risk measure that dominates \( \text{V@R} \) does not exist. The following theorem shows that within the class of law invariant risk measures a smallest convex risk measure dominating \( \text{V@R} \) exists: AV@R.

**Theorem 6.26** Suppose that the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) has no atoms. Then AV@R is the smallest law invariant convex risk measure that is continuous from above and dominates \( \text{V@R} \).

7 Indifference valuation and optimal derivative design

So far we considered the problem of measuring and quantifying risk. Now we go a step further and consider the problem of transferring risk. To this end, we shall first give a brief introduction into the theory of indifference valuation. In a subsequent section we discuss an equilibrium approach to evaluating non-financial (non-hedgeable) risk.
7 INDIFFERENCE VALUATION AND OPTIMAL DERIVATIVE DESIGN

7.1 A simple model of optimal derivative design

Following the framework of Barrieu & ElKaroui (2002), let us consider a model with two risk averse agents, each being characterized by an exponential utility function, denoted $U_B$ and $U_I$, respectively:

$$U_i(z) = -\exp(-\gamma_i z) \quad (i = B, I).$$

We think of agent $B$ as a bank and of agent $I$ as an investor. The bank is assumed to be exposed to some non-financial (e.g. weather) risk $\Theta$, and wants to hedge its position by selling a derivative security written on $\Theta$ to the investor. It has to choose the optimal structure of this contract according to its utility while the investor must find an interest in doing such a transaction.

7.1.1 Model characteristics and optimal claims

We assume that $\Theta$ is the only source of risk, that it is not tradable, and that the bank is endowed with a random income $X(\Theta)$. Its goal is to design a contract $F(\theta)$ and sell it at a price $\pi$ to the investor such that

$$\mathbb{E}[U_B(X(\theta) - F(\Theta) + \pi)] \rightarrow \max$$

subject to

$$\mathbb{E}[U_I(F(\Theta)) - \pi] \geq 1.$$

The optimization problem is equivalent to

$$\mathbb{E} \left[ \exp \left( -\gamma_B (X(\theta) - F(\Theta) + \pi) \right) \right] \rightarrow \min \quad \text{s.t.} \quad \mathbb{E} \left[ \exp \left( -\gamma_I (F(\Theta) - \pi) \right) \right] \leq 1.$$

From this we see that the two-dimensional optimization problem of contracts-price pairs $(F, \pi)$ can be reduced to a one-dimensional optimization problem over contracts $F$. In fact, for any given contract $F$ the bank will charge no less than

$$\pi^*(F) = -\frac{\log \mathbb{E}[-\gamma_I F]}{\gamma_I}.$$

This price is called the investor’s indifference price. It makes the investor indifferent between owning the contract at a price $\pi^*$ and not owning it. The following proposition shows that in an economy with exponential utility maximizing agents, the agents share their total endowment $X$ according to their degrees of risk tolerance.

**Proposition 7.1** The optimal contract $F^*$ is given by

$$F^* = \frac{\gamma_B}{\gamma_B + \gamma_I} X(\Theta).$$
The dependence of the optimal structure $F^*$ on the bank exposure $X$ is proportional and does not depend on the distribution of $\Theta$. The proportionality factor is $\frac{\gamma_B}{\gamma_B + \gamma_I}$, which may be considered as the relative risk aversion of the bank. Thus, the more relatively risk averse the bank, the greater the part of its exposure $X$ it wants to hedge.

7.1.2 Bayesian Heterogeneity

A priori, there is no reason why the two agents should have the same Bayesian view of the world. The preceding results can easily be extended to the case where the two agents differ in their beliefs, through two ‘prior’ probability measures $\mathbb{P}_B$ and $\mathbb{P}_I$. Both these probability measures are assumed to be equivalent to the real world measure $\mathbb{P}$. In this case the optimization problem can be rewritten as

$$
\mathbb{E}_{\mathbb{P}_B}[\exp (-\gamma_B (X(\theta) - F(\Theta) + \pi))] \rightarrow \min \quad \text{s.t.} \quad \mathbb{E}_{\mathbb{P}_I}[\exp (-\gamma_I (F(\Theta) - \pi))] \leq 1.
$$

Let us now denote by $\varphi$ the logarithm of the density of $\mathbb{P}_B$ with respect to $\mathbb{P}_I$. Then the bank’s optimization problem can be restated as

$$
\mathbb{E}_{\mathbb{P}_I}[\exp (-\gamma_B (X(\theta) - F(\Theta) + \pi)) e^{\varphi}] \rightarrow \min
$$

while the constraint at the investor’s end remains unchanged. As a result, we obtain the same optimization problem as before if we change the endowment $X$ to $X - \frac{1}{\gamma_B} \varphi$.

7.2 Introducing a financial market

We are now going to assume that the agents have also the opportunity to invest on a financial market. We shall adopt a simplified approach and assume that both agents have previously determined the structure of their ‘market portfolios’, i.e. the investment strategy they adopt on the financial market. They will not change these structures when taking into account the source of risk $\Theta$.

8 Optimal risk transfer in principal agent games

The notion of utility indifference valuation assumes a high degree of market asymmetry. For indifference valuation to be a pricing rather than valuation principle, the demand for a financial security must come from identical agents with known preferences and negligible market impact while the supply must come from a single principal. When the demand comes from heterogeneous individuals with hidden characteristics, indifference arguments do not always yield an appropriate pricing scheme. In this section we move away from the assumption of investor homogeneity and allow for heterogeneous agents. Our discussion is based on Horst & Moreno (2007).
8.1 The Microeconomic Setup

We consider an economy with a single principal whose income $W$ is exposed to non-hedgeable risk factors rising from, e.g., climate or weather phenomena. The random variable $W$ is defined on a standard, non-atomic, probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and it is square integrable:

$$W \in L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

The principal’s goal is to lay off parts of her risk with individual agents. The agents have heterogeneous mean-variance preferences\(^2\) and are indexed by their coefficients of risk aversion $\theta \in \Theta$. Given a contingent claim $Y \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ an agent of type $\theta$ enjoys the utility

$$U(\theta, Y) = \mathbb{E}[Y] - \theta \text{Var}[Y].$$

Types are private information. The principal knows the distribution $\mu$ of types but not the realizations of the random variables $\theta$. We assume that the agents are risk averse and that the risk aversion coefficients are bounded away from zero. More precisely,

$$\Theta = [a, 1] \quad \text{for some } a > 0.$$  

The principal offers a derivative security $X(\theta)$ written on her random income for any type $\theta$. The set of all such securities is denoted by

$$\mathcal{X} := \{X = \{X(\theta)\}_{\theta \in \Theta} \mid X \in L^2(\mathbb{R} \times \Theta, \mathbb{P} \otimes \mu), X \text{ is } \sigma(W) \times \mathcal{B}(\Theta) \text{ measurable}\}.$$  

We refer to a list of securities $\{X(\theta)\}$ as a contract. A catalogue is a contract along with prices $\pi(\theta)$ for every available derivative $X(\theta)$. For a given catalogue $(X, \pi)$ the optimal net utility of the agent of type $\theta$ is given by

$$v(\theta) = \sup_{\theta' \in \Theta} \{U(\theta, X(\theta')) - \pi(\theta')\}. $$

Remark 8.1 No assumption will be made on the sign of $\pi(\theta)$; our model contemplates both the case where the principal takes additional risk in exchange of financial compensation and the one where she pays the agents to take part of her risk.

A catalogue $(X, \pi)$ will be called incentive compatible (IC) if the agent’s interests are best served by revealing her type. This means that her optimal utility is achieved by the security $X(\theta)$:

$$U(\theta, X(\theta)) - \pi(\theta) \geq U(\theta, X(\theta')) - \pi(\theta') \quad \text{for all } \theta, \theta' \in \Theta.$$  

\(^2\)Our analysis carries over to preferences of mean-variance type with random initial endowment as in \(\ast\); the assumption of simple mean-variance preferences is made for notational convenience.
We assume that each agent has some outside option ("no trade") that yields a utility of zero. A catalogue is thus called individually rational (IR) if it yields at least the reservation utility for all agents, i.e., if

\[ U(\theta, X(\theta)) - \pi(\theta) \geq 0 \quad \text{for all} \quad \theta \in \Theta. \tag{25} \]

**Remark 8.2** By offering only incentive compatible contracts, the principal forces the agents to reveal their type. Offering contracts where the IR constraint is binding allows the principal to exclude undesirable agents from participating in the market. It can be shown that under certain conditions, the interests of the Principal are better served by keeping agents of "lower types" to their reservation utility; Rochet and Choné have shown that in higher dimensions this is always the case.

If the principal issues the catalogue \((X, \pi)\), she receives a cash amount of \(\int_\Theta \pi(\theta) \, d\mu(\theta)\) and is subject to the additional liability \(\int_\Theta X(\theta) \mu(d\theta)\). She evaluates the risk associated with her overall position

\[ W + \int_\Theta (\pi(\theta) - X(\theta)) \, d\mu(\theta) \]

via a coherent and law-invariant risk measure \(\varrho : L^2(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R} \cup \{\infty\}\) that has the Fatou property, i.e., is lower semi continuous:

\[ \lim_{n \to \infty} \mathbb{E}[|X_n - X|^2] = 0 \quad \text{implies} \quad \liminf_{n \to \infty} \varrho(X_n) \leq \varrho(X). \]

It turns out that such risk measures can be represented as robust mixtures of Average Value at Risk.\(^3\) The principal’s risk associated with the catalogue \((X, \pi)\) is given by

\[ \varrho \left( W + \int_\Theta (\pi(\theta) - X(\theta)) \, d\mu(\theta) \right). \tag{26} \]

Her goal is to devise contracts \((X, \pi)\) that minimize (26) subject to the incentive compatibility and individual rationality condition:

\[ \inf \left\{ \varrho \left( W + \int_\Theta (\pi(\theta) - X(\theta)) \, d\mu(\theta) \right) \mid X \in \mathcal{X}, X \text{ is IC and IR} \right\}. \tag{27} \]

We are now ready to state the main result of this paper. The proof requires some preparation and will be carried out in the following section.

**Theorem 8.3** If \(\varrho\) is a coherent and law invariant risk measure on \(L^2(\mathbb{P})\) and if \(\varrho\) has the Fatou property, then the principal’s optimization problem has a solution.

For notational convenience we establish our main result for the spacial case \(d\mu(\theta) = d\theta\).

\(^3\)We review properties of coherent risk measures on \(L^p\) spaces in the appendix and refer to the textbook by Föllmer and Schied and the paper of Jouini, Schachermayer and Touzi for detailed discussion of law invariant risk measures.
8.2 Proof of the Main Theorem

Let \((X, \pi)\) be a catalogue. In order to prove our main result it will be convenient to assume that the principal offers any square integrable contingent claim and to view the agents’ optimization problem as optimization problems over the set \(L^2(\mathbb{P})\). This can be achieved by identifying the price list \(\{\pi(\theta)\}\) with the pricing scheme \(\pi : L^2(\mathbb{P}) \to \mathbb{R}\) that assigns the value \(\pi(\theta)\) to an available claim \(X(\theta)\) and the value \(E[Y]\) to any other claim \(Y \in L^2\). In terms of this pricing scheme the value function \(v\) defined in (23) satisfies

\[
v(\theta) = \sup_{Y \in L^2(\mathbb{P})} \{U(\theta, Y) - \pi(Y)\}.
\]

(28)

for any individually rational catalogue. For the remainder of this section we shall work with the value function of the (28). It is \(U\)-convex in the sense of the following definition; it actually turns out to be convex and non-increasing as we shall prove in Proposition 8.5 below.

**Definition 8.4** Let two spaces \(A\) and \(B\) and a function \(U : A \times B \to \mathbb{R}\) be given.

(i) The function \(f : A \to \mathbb{R}\) is called \(U\)-convex if there exists a function \(p : B \to \mathbb{R}\) such that

\[
f(a) = \sup_{b \in B} \{U(a, b) - p(b)\}.
\]

(ii) For a given function \(p : B \to \mathbb{R}\) the \(U\)-conjugate \(p^U(a)\) of \(p\) is defined by

\[
p^U(a) = \sup_{b \in B} \{U(a, b) - p(b)\}.
\]

(iii) The \(U\)-subdifferential of \(p\) at \(b\) is given by the set

\[
\partial_U p(b) := \{a \in A \mid p^U(a) = U(a, b) - p(b)\}.
\]

(iv) If \(a \in \partial_U p(b)\), then \(a\) is called a \(U\)-subgradient of \(p(b)\).

Our goal is to identify the class of \(IC\) and \(IR\) catalogues with a class of convex and non-increasing functions on the type space. To this end, we first recall the link between incentive compatible contracts and \(U\)-convex functions from Rochet and Choné and Carlier, Ekeland and Touzi.

**Proposition 8.5** (? , ?) If a catalogue \((X, \pi)\) is incentive compatible, then the function \(v\) defined by (23) is proper and \(U\)-convex and \(X(\theta) \in \partial_U v(\theta)\). Conversely, any proper, \(U\)-convex function induces an incentive compatible catalogue.
Proof: Incentive compatibility of a catalogue \((X, \pi)\) means that
\[
U(\theta, X(\theta)) - \pi(\theta) \geq U(\theta, X(\theta')) - \pi(\theta') \quad \text{for all } \theta, \theta' \in \Theta,
\]
so \(v(\theta) = U(\theta, X(\theta)) - \pi(\theta)\) is U-convex and \(X(\theta) \in \partial_U v(\theta)\). Conversely, for a proper, U-convex function \(v\) and \(X(\theta) \in \partial_U v(\theta)\) let
\[
\pi(\theta) := U(\theta, X(\theta)) - v(\theta).
\]
By the definition of the U-subdifferential, the catalogue \((X, \pi)\) is incentive compatible. 

The following lemma is key. It shows that the U-convex function \(v\) is convex and non-increasing and that any convex and non-increasing function is U-convex, i.e., it allows a representation of the form (28). This allows us to rephrase the principal’s problem as an optimization problem over a compact set of convex functions.

**Lemma 8.6** (i) Suppose that the value function \(v\) as defined by (28) is proper. Then \(v\) is convex and non-increasing. Any optimal claim \(X^*(\theta)\) is a U-subgradient of \(v(\theta)\) and almost surely
\[
-\text{Var}[X^*(\theta)] = v'(\theta).
\]

(ii) If \(\bar{v} : \Theta \rightarrow \mathbb{R}_+\) is proper, convex and non-increasing, then \(\bar{v}\) is U-convex, i.e., there exists a map \(\bar{\pi} : L^2(\mathbb{P}) \rightarrow \mathbb{R}\) such that
\[
\bar{v}(\theta) = \sup_{Y \in L^2(\mathbb{P})} \{U(\theta, Y) - \bar{\pi}(Y)\}.
\]
Furthermore, any optimal claim \(\bar{X}(\theta)\) belongs to the U-subdifferential of \(\bar{v}(\theta)\) and satisfies
\[
-\text{Var}[\bar{X}(\theta)] = \bar{v}'(\theta).
\]

Proof:

(i) Let \(v\) be a proper, U-convex function. Its U-conjugate is:
\[
v^U(Y) = \sup_{\theta \in \Theta} \{\mathbb{E}[Y] - \theta \text{Var}[Y] - v(\theta)\}
= \mathbb{E}[Y] + \sup_{\theta \in \Theta} \{\theta (-\text{Var}[Y]) - v(\theta)\}
= \mathbb{E}[Y] + v^*(-\text{Var}[Y]),
\]
where \( v^* \) denotes the convex conjugate of \( v \). As a \( U \)-convex function, the map \( v \) is characterized by the fact that \( v = (v^U)^U \). Thus

\[
v(\theta) = (v^U)^U(\theta) \\
= \sup_{Y \in L^2(\mathbb{P})} \{ U(\theta, Y) - \mathbb{E}[Y] - v^*(-\text{Var}[Y]) \} \\
= \sup_{Y \in L^2(\mathbb{P})} \{ \mathbb{E}[Y] - \theta \text{Var}[Y] - \mathbb{E}[Y] - v^*(-\text{Var}[Y]) \} \\
= \sup_{y \leq 0} \{ \theta \cdot y - v^*(y) \}
\]

where the last equality uses the fact that the agents’ consumption set contains claims of any variance. We deduce from the preceding representation that \( v \) is non-increasing. Furthermore \( v = (v^*)^* \) so \( v \) is convex. To characterize \( \partial_U v(\theta) \) we proceed as follows:

\[
\partial_U v(\theta) = \{ Y \in L^2 : v(\theta) = U(\theta, X) - v^U(Y) \} \\
= \{ Y \in L^2 : v(\theta) = \mathbb{E}[Y] - \theta \text{Var}[Y] - v^U(Y) \} \\
= \{ Y \in L^2 : v(\theta) = \mathbb{E}[Y] - \theta \text{Var}[Y] - \mathbb{E}[Y] - v^*(-\text{Var}[Y]) \} \\
= \{ Y \in L^2 : v(\theta) = \theta(-\text{Var}[Y]) - v^*(-\text{Var}[Y]) \} \\
= \{ Y \in L^2 : \text{Var}[Y] \in \partial v(\theta) \}
\]

The convexity of \( v \) implies it is a.e. differentiable so we may write

\[
\partial_U v(\theta) := \{ Y \in L^2 : v'(\theta) = -\text{Var}[Y] \}.
\]

(ii) Let us now consider a proper, non-negative, convex and non-increasing function \( \bar{v} : \Theta \to \mathbb{R} \). There exists a map \( f : \mathbb{R} \to \mathbb{R} \) such that

\[
\bar{v}(\theta) = \sup_{y \leq 0} \{ \theta \cdot y - f(y) \}.
\]

Since \( \bar{v} \) is non-increasing there exists a random variable \( Y(\theta) \in L^2(\mathbb{P}) \) such that

\[
-\text{Var}[Y(\theta)] \in \partial \bar{v}(\theta)
\]

and the definition of the subgradient yields

\[
\bar{v}(\theta) = \sup_{Y \in L^2} \{ \theta(-\text{Var}[Y]) - f(-\text{Var}[Y]) \}.
\]

With the pricing scheme on \( L^2(\mathbb{P}) \) defined by

\[
\bar{\pi}(Y) := -\mathbb{E}[Y] - f(-\text{Var}[Y])
\]

this yields

\[
\bar{v}(\theta) = \sup_{Y \in L^2} \{ U(\theta, Y) - \bar{\pi}(Y) \}.
\]

The characterization of the subdifferential follows by analogy to part (i).
The preceding lemma along with Proposition 8.5 shows that any convex, non-negative and non-increasing function \( v \) on \( \Theta \) induces an incentive compatible catalogue \((X, \pi)\) via

\[
X(\theta) \in \partial_U v(\theta) \quad \text{and} \quad \pi(\theta) = U(\theta, X(\theta)) - v(\theta).
\]

Here we may with no loss of generality assume that \( \mathbb{E}[X(\theta)] = 0 \). In terms of the principal’s choice of \( v \) her income is given by

\[
I(v) = \int_{\Theta} \left( \theta v'(\theta) - v(\theta) \right) d\theta.
\]

Since \( v \geq 0 \) is decreasing and non-negative the principal will only consider functions that satisfy the normalization constraint

\[
v(1) = 0.
\]

We denote the class of all convex, non-increasing and non-negative real-valued functions on \( \Theta \) that satisfy the preceding condition by \( \mathcal{C} \):

\[
\mathcal{C} = \{ v : \Theta \to \mathbb{R} \mid v \text{ is convex, non-increasing, non-negative and } v(1) = 0. \}
\]

Conversely, we can associate with any \( \text{IC} \) and \( \text{IR} \) catalogue \((X, \pi)\) a non-negative \( U \)-convex function of the form (28) where the contract satisfies the variance constraint

\[
-\text{Var}[X(\theta)] = v'(\theta).
\]

In view of the preceding lemma this function is convex and non-increasing so after normalization we may assume that \( v \) belongs to the class \( \mathcal{C} \). We therefore have the following alternative formulation of the principal’s problem.

**Theorem 8.7** The principal’s optimization problem allows the following alternative formulation:

\[
\inf \left\{ \rho \left( W - \int_{\Theta} X(\theta) d\theta \right) - I(v) \mid v \in \mathcal{C}, \mathbb{E}[X(\theta)] = 0, -\text{Var}[X(\theta)] = v'(\theta) \right\}.
\]

In terms of our alternative formulation we can now prove a preliminary result. It states that a principal with no initial endowment will not issue any contracts.

**Lemma 8.8** If the principal has no initial endowment, i.e., if \( W = 0 \), then \((v, X) = (0, 0)\) solves her optimization problem.
Proof: Since \( \varrho \) is a coherent, law invariant risk measure on \( L^2(\mathbb{P}) \) that has the Fatou property it satisfies
\[
\varrho(Y) \geq -\mathbb{E}[Y] \quad \text{for all } Y \in L^2(\mathbb{P}). \tag{29}
\]
For a given function \( v \in \mathcal{C} \) the normalization constraint \( \mathbb{E}[X(\theta)] = 0 \) therefore yields
\[
\varrho \left( -\int_{\Theta} X(\theta) d\theta \right) - I(v) \geq \mathbb{E} \left[ \int_{\Theta} X(\theta) d\theta \right] - I(v) = -I(v).
\]
Since \( v \) is non-negative and non-increasing \( -I(v) \geq 0 \). Taking the infimum in the preceding inequality shows that \( v \equiv 0 \) and hence \( X(\theta) \equiv 0 \) is an optimal solution. \( \Box \)

In the general case we approach the principal’s problem in two steps. We start by fixing a function \( v \) from the class \( \mathcal{C} \) and minimize the associated risk
\[
\varrho \left( W - \int_{\Theta} X(\theta) d\theta \right)
\]
subject to the moment conditions \( \mathbb{E}[X(\theta)] = 0 \) and \(-\text{Var}[X(\theta)] = v'(\theta)\). To this end, we shall first prove the existence of optimal contracts \( X_v \) for a relaxed optimization where the variance constraint is replaced by the weaker condition
\[
\text{Var}[X(\theta)] \leq -v'(\theta).
\]
In a subsequent step we show that based on \( X_v \) the principal can transfer risk exposures among the agents in such a way that (i) the aggregate risk remains unaltered; (ii) the variance constraint becomes binding. We assume with no loss of generality that \( v \) does not have a jump at \( \theta = a \).

For a given \( v \in \mathcal{C} \) let us consider the convex set of derivative securities
\[
\mathcal{X}_v := \left\{ X \in \mathcal{X} \mid E[X(\theta)] = 0, \text{Var}[X(\theta)] \leq -v'(\theta) \mu - \text{a.e.} \right\}. \tag{30}
\]

Lemma 8.9 (i) All functions \( v \in \mathcal{C} \) that are acceptable for the principal are uniformly bounded.

(ii) Under the conditions of (i) the set \( \mathcal{X}_v \) is closed and bounded in \( L^2(\mathbb{P} \otimes \mu) \). More precisely,
\[
\|X\|^2 \leq v(a) \quad \text{for all } X \in \mathcal{X}_v.
\]

Proof:

(i) If \( v \) is acceptable for the principal, then any \( X \in \mathcal{X}_v \) satisfies
\[
\varrho \left( W - \int_{\Theta} X(\theta) d\theta \right) - I(v) \leq \varrho(W).
\]
From (29) and that fact that $\mathbb{E}[X(\theta)] = 0$ we deduce that

$$-\mathbb{E}[W] - I(v) \leq \varrho \left( W - \int_{\Theta} X(\theta) d\theta \right) - I(v) \leq \varrho(W)$$

so

$$-I(v) \leq \mathbb{E}[W] + \varrho(W) =: K.$$ Integrating by parts twice and using that $v$ is non-increasing and $v(1) = 0$ we see that

$$K \geq -I(v) = av(a) + 2 \int_{a}^{1} v(\theta) d\theta \geq av(a).$$

This proves the assertion because $a > 0$.

(ii) For $X \in \mathcal{X}_v$ we deduce from the normalization constraint $v(1) = 0$ that

$$\|X\|_2^2 = \int \int X^2(\theta, \omega) d\mathbb{P} d\theta \leq -\int v'(\theta) d\theta \leq v(a)$$

so the assertion follows from part (i).

The risk measure $\varrho$ is a convex functional on the reflexive Hilbert space $L^2$. The set $X_v$ of contingent claims is a closed, convex and bounded subset of $L^2$. Thus, a general result from the theory of convex optimization (see, for instance, Proposition 2.1.2 of Ekeland & Téman ?) yields the following proposition.

**Proposition 8.10** If the function $v$ is proper and acceptable for the principal, then there exists a contract $\{X_{v}(\theta)\}$ such that

$$\inf_{X \in \mathcal{X}_v} \varrho \left( W - \int_{\Theta} X(\theta) d\theta \right) = \varrho \left( W - \int_{\Theta} X_{v}(\theta) d\theta \right).$$

The solution to the relaxed risk minimization problem is unique if $\varrho$ is strictly convex.

The contract $X_v$ along with the pricing scheme associated with $v$ does not yield an incentive compatible catalogue unless the variance constraints happen to be binding. However, as we are now going to show, based on $X_v$ the principal can find a redistribution of risk among the agents such that the resulting contract satisfies our IC condition.

### 8.2.1 Redistributing risk exposures among agents

Let

$$\partial \mathcal{X}_v = \{ X \in \mathcal{X}_v \mid E[X(\theta)] = 0, \ Var[X(\theta)] = -v'(\theta), \mu - a.e. \}$$
be the set of all contracts from the class $\mathcal{X}_v$ where the variance constraint is binding. Clearly,

$$\varrho \left( W - \int_\Theta X_v(\theta) d\theta \right) \leq \inf_{X \in \partial \mathcal{X}_v} \varrho \left( W - \int_\Theta X(\theta) d\theta \right).$$

Let us then introduce the set of types

$$\Theta_v := \{ \theta \in \Theta \mid \text{Var}[X_v(\theta)] < -v'(\theta) \},$$

for whom the variance constraint is not binding. If $\mu(\Theta_v) = 0$, then $X_v$ yields an incentive compatible contract. Otherwise, we consider a random variable $\tilde{Y} \in \mathcal{X}_v$, fix some type $\theta \in \Theta$ and define

$$Y := \frac{\tilde{Y}(\theta)}{\sqrt{\text{Var}[\tilde{Y}(\theta)]}}.$$  \hfill (31)

We may with no loss of generality assume that $Y$ is well defined for otherwise the status quo is optimal for the principal and her risk minimization problem is void. The purpose of introducing $Y$ is to offer a set of structured products $Z_v$ based on $X_v$, such that $Z_v$ together with the pricing scheme associated with $v$ yields an incentive compatible catalogue. To this end, we choose constants $\tilde{\alpha}(\theta)$ for $\theta \in \Theta_v$ such that

$$\text{Var}[X_v(\theta) + \tilde{\alpha}(\theta) Y] = -v'(\theta).$$

This equation holds for

$$\tilde{\alpha}_\pm(\theta) = -\text{Cov}[X_v(\theta), Y] \pm \sqrt{\text{Cov}^2[X_v(\theta), Y] - v'(\theta) - \text{Var}[X_v(\theta)]}.$$  

For a type $\theta \in \Theta_v$ the variance constraint is not binding. Hence $-v'(\theta) - \text{Var}[X_v(\theta)] > 0$ so that $\alpha_+(\theta) > 0$ and $\alpha_-(\theta) < 0$. An application of Jensen’s inequality together with the fact that $\|X_v\|_2$ is bounded shows that $\alpha_\pm$ are $\mu$-integrable functions. Thus there exists a threshold type $\theta^* \in \Theta$ such that

$$\int_{\Theta_v \cap (a, \theta^*)} \alpha_+(\theta) d\theta + \int_{\Theta_v \cap (\theta^*, 1]} \alpha_-(\theta) d\theta = 0.$$  

In terms of $\theta^*$ let us now define a function

$$\alpha(\theta) := \begin{cases} \tilde{\alpha}_+(\theta), & \text{if } \theta \leq \theta^* \\ \tilde{\alpha}_-(\theta), & \text{if } \theta > \theta^* \end{cases}$$

and a contract

$$Z_v := X_v + \alpha Y \in \partial \mathcal{X}_v.$$  \hfill (32)

Since $\int \alpha d\theta = 0$ the aggregate risks associated with $X_v$ and $Z_v$ are equal. As a result, the contract $Z_v$ solves the risk minimization problem

$$\inf_{X \in \partial \mathcal{X}_v} \varrho \left( W + \int_\Theta X(\theta) \mu(d\theta) \right).$$  \hfill (33)
Remark 8.11 In Section ?? we shall consider a situation where the principal restricts itself to a class of contracts for which the random variable $X_v$ can be expressed in terms of the function $v$. In general such a representation will not be possible since $v$ only imposes a restriction on the contracts’ second moments.

8.2.2 Minimizing the overall risk

In order to finish the proof of our main result it remains to show that the minimization problem

$$\inf_{v \in C} \left\{ \varrho \left( W - \int_{\Theta} Z_v(\theta) \mu(d\theta) \right) - I(v) \right\}$$

has a solution and the infimum is obtained. To this end, we consider a minimizing sequence $\{v_n\} \subset C$. The functions in $C$ are locally Lipschitz continuous because they are convex. In fact they are uniformly locally Lipschitz: by Lemma 8.9 (i) the functions $v \in C$ are uniformly bounded and non-increasing so all the elements of $\partial v(\theta)$ are uniformly bounded on compact sets of types. As a result, $\{v_n\}$ is a sequence of uniformly bounded and uniformly equicontinuous functions when restricted to compact subsets of $\Theta$. Thus there exists a function $\bar{v} \in C$ such that, passing to a subsequence if necessary,

$$\lim_{n \to \infty} v_n = \bar{v} \text{ uniformly on compact sets.}$$

A standard $3\epsilon$-argument shows that the convergence properties of the sequence $\{v_n\}$ carry over to the derivatives so that

$$\lim_{n \to \infty} v'_n = \bar{v}' \text{ almost surely uniformly on compact sets.}$$

Since $-\theta v'_n(\theta) + v_n(\theta) \geq 0$ it follows from Fatou’s lemma that $-I(\bar{v}) \leq \lim inf_{n \to \infty} -I(v_n)$ so

$$\lim inf_{n \to \infty} \left\{ \varrho \left( W - \int_{\Theta} Z_{v_n}(\theta) \mu(d\theta) \right) - I(v_n) \right\}$$

$$\geq \lim inf_{n \to \infty} \varrho \left( W - \int_{\Theta} Z_{v_n}(\theta) \mu(d\theta) \right) + \lim inf_{n \to \infty} -I(v_n)$$

$$\geq \lim inf_{n \to \infty} \varrho \left( W - \int_{\Theta} Z_{v_n}(\theta) \mu(d\theta) \right) - I(\bar{v})$$

and it remains to analyze the associated risk process. For this, we first observe that for $Z_{v_n} \in \partial X_{v_n}$ Fubini’s theorem yields

$$\|Z_{v_n}\|_2^2 = \int \int Z_{v_n}^2 dP d\theta = -\int v'_n(\theta) d\theta = v_n(a). \quad (34)$$

Since all the functions in $C$ are uniformly bounded, we see that the contracts $Z_{v_n}$ are contained in an $L^2$ bounded, convex set. Hence there exists a square integrable random variable $Z$ such
that, after passing to a subsequence if necessary,

\[ w - \lim_{n \to \infty} Z_n = Z \quad (35) \]

Let \( Z_\bar{v} \in X_\bar{v} \). Convergence of the functions \( v_n \) implies \( \|v_n\|_2 \to \|Z_\bar{v}\|_2 \). Thus (35) yields \( \|Z\|_2 = \|Z_\bar{v}\|_2 \) along with convergence of aggregate risks:

\[ \|Z\|_2 = \|Z_\bar{v}\|_2 \quad \text{and} \quad \int_\Theta Z_n(\theta, \omega) d\theta \to \int_\Theta Z(\theta, \omega) d\theta \quad \text{weakly in } L^2(\mathbb{P}). \]

By Corollary I.2.2 in Ekeland and Témam (1976)?, a lower semi-continuous convex function \( f : X \to \mathbb{R} \) remains, so with respect to the weak topology \( \sigma(X, X^*) \), the Fatou property of the risk measure \( \varrho \) guarantees that

\[ \varrho \left( W - \int_\Theta Z_\bar{v}(\theta) \mu(d\theta) \right) \leq \varrho \left( W - \int_\Theta Z(\theta) \mu(d\theta) \right) \]

\[ \leq \liminf_{n \to \infty} \varrho \left( W - \int_\Theta Z(\theta) \mu(d\theta) \right). \]

### A Equivalent measures and dominated convergence

Let \((\Omega, \mathcal{F})\) be a measure space. Two probability distributions \( \mathbb{P} \) and \( \mathbb{Q} \) on \((\Omega, \mathcal{F})\) are called *equivalent* if

\[ \mathbb{P}[A] = 0 \iff \mathbb{Q}[A] = 0 \quad (A \in \mathcal{F}) \]

The measure \( \mathbb{Q} \) is called *absolutely continuous* with respect to \( \mathbb{P} \) denoted \( \mathbb{Q} << \mathbb{P} \) if

\[ \mathbb{P}[A] = 0 \Rightarrow \mathbb{Q}[A] = 0 \quad (A \in \mathcal{F}). \]

**Theorem A.1 (Radon-Nikodym)** If \( \mathbb{Q} << \mathbb{P} \) then there exists a density \( \varphi := \frac{d\mathbb{Q}}{d\mathbb{P}} > 0 \) such that:

\[ \mathbb{Q}(A) = \int_A \varphi \ d\mathbb{P}. \]

A standard measure-theoretic argument shows how expectations of a non-negative random variable \( X \) with respect to \( \mathbb{P} \) and \( \mathbb{Q} \) are related:

\[ \mathbb{E}_\mathbb{Q}[X] = \mathbb{E}_\mathbb{P}[\varphi X] \]

Furthermore,

\[ \mathbb{Q}[\{\varphi > 0\}] = \int_{\{\varphi > 0\}} \varphi \ d\mathbb{P} = 1 \]

so we have \( \mathbb{Q}\)-a.s. that \( \varphi > 0 \). Thus, if \( \mathbb{Q} \) and \( \mathbb{P} \) are equivalent, then

\[ \varphi^{-1} = \frac{d\mathbb{P}}{d\mathbb{Q}}. \]
Lemma A.2 (Dominated convergence) Let $X_n$ ($n \in \mathbb{N}$) and $Y$ be random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. If

$$|X_n| \leq Y, \quad \mathbb{E}[Y < \infty] \quad \text{and} \quad X_n \xrightarrow{a.s.} X \quad (n \to \infty)$$

then

$$\mathbb{E}|X| < \infty, \quad \mathbb{E}[X_n] \to \mathbb{E}[X] \quad \text{and} \quad \mathbb{E}|X_n - X| \to 0 \quad (n \to \infty).$$

References


