3-Manifolds

Webpage: https://www2.mathematik.hu-berlin.de/~kegemarc/SS203mfds.html

OneNote Link: https://1drv.ms/u/s!ApRQR77A3CIHiRGGsgS9jLjBZ4Gf

0. Overview

3-MFDS M = connected, orientable, closed

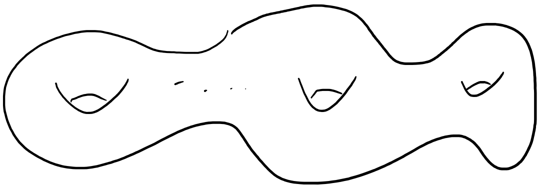
↗ (i.e. compact & $\partial M = \emptyset$)

Examples:

$S^3 = \{x \in \mathbb{R}^4 \mid |x| = 1\} \subset \mathbb{R}^4$

$S^1 \times S^2$, $S^1 \times \Sigma_g^2$

Σ_g^2 = SURFACE OF GENUS g =

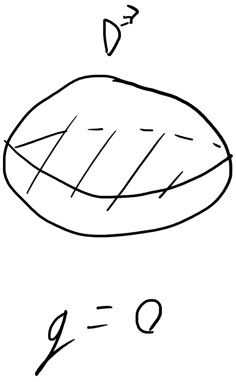


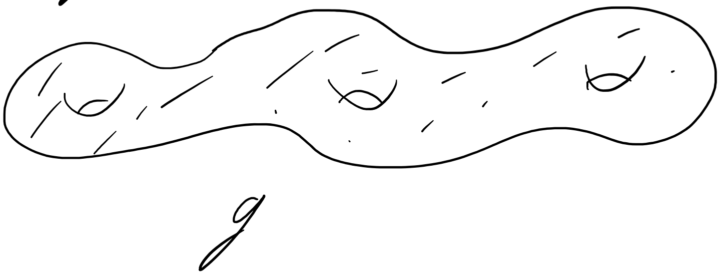
STRUCTURE THMS FOR 3-MFDS

THM:

$\forall$ 3-MFD M $\exists$ HEEGAARD SPLITTING:

$M_1, M_2 :=$ copies of the 3-DIM HANDLEBODY with $\partial M_i = \Sigma_g$





& f: $\underset{\Sigma_g}{\underset{\shortparallel}{\partial M_1}} \xrightarrow{\overset{C^0}{\cong}} \underset{\Sigma_g}{\underset{\shortparallel}{\partial M_2}}$ a homeom.

$$M = (M_1 + M_2) \big/ \underset{\partial M_1}{\underset{\rotatebox{90}{$\in$}}{p}} \sim f(p) \in \partial M_2$$

Examples:

(1) $$S^3 = \left(D^3 + D^3\right)_{\underset{\underset{\partial D^3}{\cap}}{p}\sim p}$$

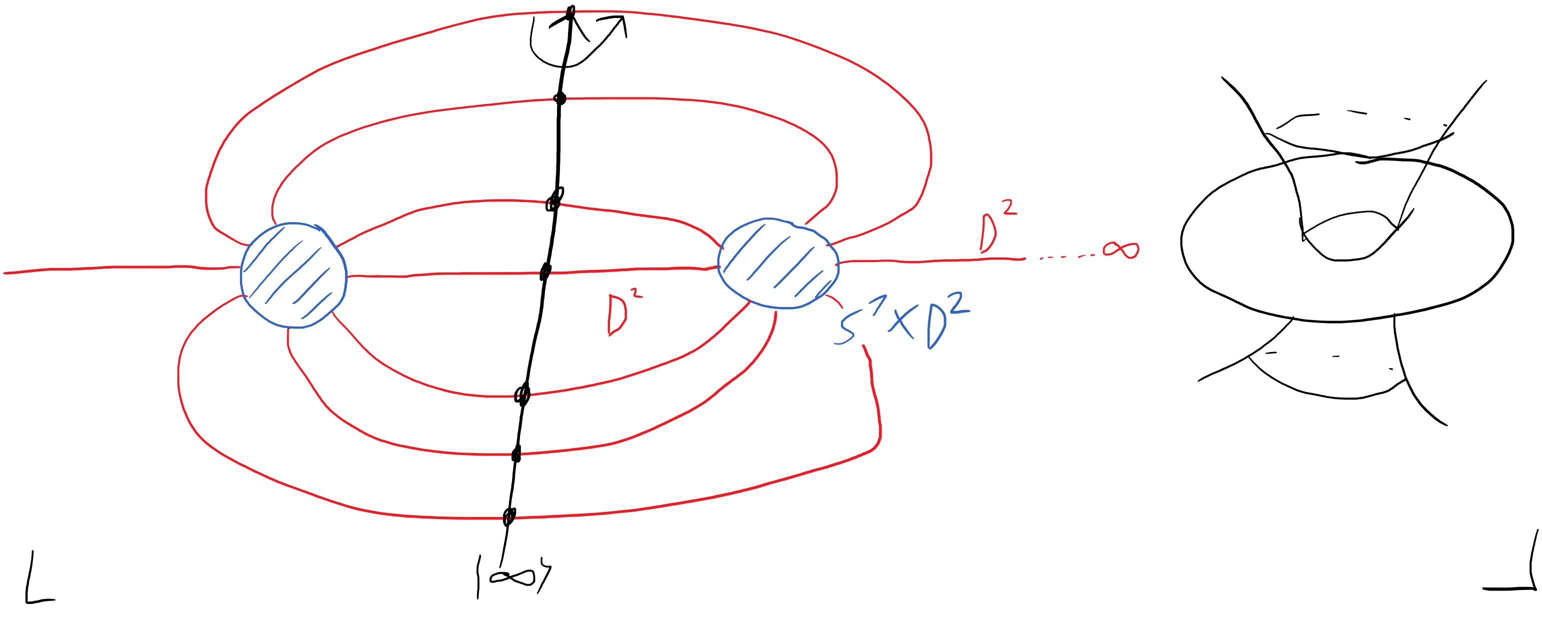
$$\left(D^2 + D^2\right)/_{\sim} = S^2$$

(2) $$S^3 = \left(S^1 \times D^2 + S^1 \times D^2\right)/_{\sim}$$

$$f(\theta_1, \theta_2) = (\theta_2, \theta_1)$$

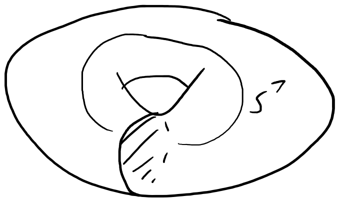
$$(\theta_1, \theta_2) \in \partial(S^1 \times D^2) = S^1 \times S^1$$

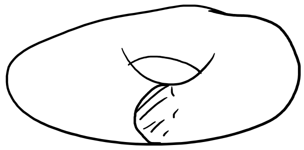
$$S^3 = \mathbb{R}^3 \cup \{\infty\}$$



(3) $S^1 \times S^2 = \left(S^1 \times D^2 + S^1 \times D^2\right)/_\sim$

$$f(\theta_1, \theta_2) = (\theta_1, \theta_2)$$





$(D^2 \quad + \quad D^2) = S^2$

THM:

∀ 3-MFD M can be obtained by SURGERY on S^3:

REMOVE: finitely many $S^1 \times D^2$ from S^3

REGLUE: via a homeom $\partial(S^1 \times D^2) \longrightarrow \partial\left(S^3 \setminus \overset{\circ}{\overline{S^1 \times D^2}}\right)$

Example:

$$\left(S^3 \setminus (D^2 \times S^1) + S^1 \times D^2\right)/_\sim = \left(S^1 \times D^2 + S^1 \times D^2\right)/_\sim = S^1 \times S^2$$

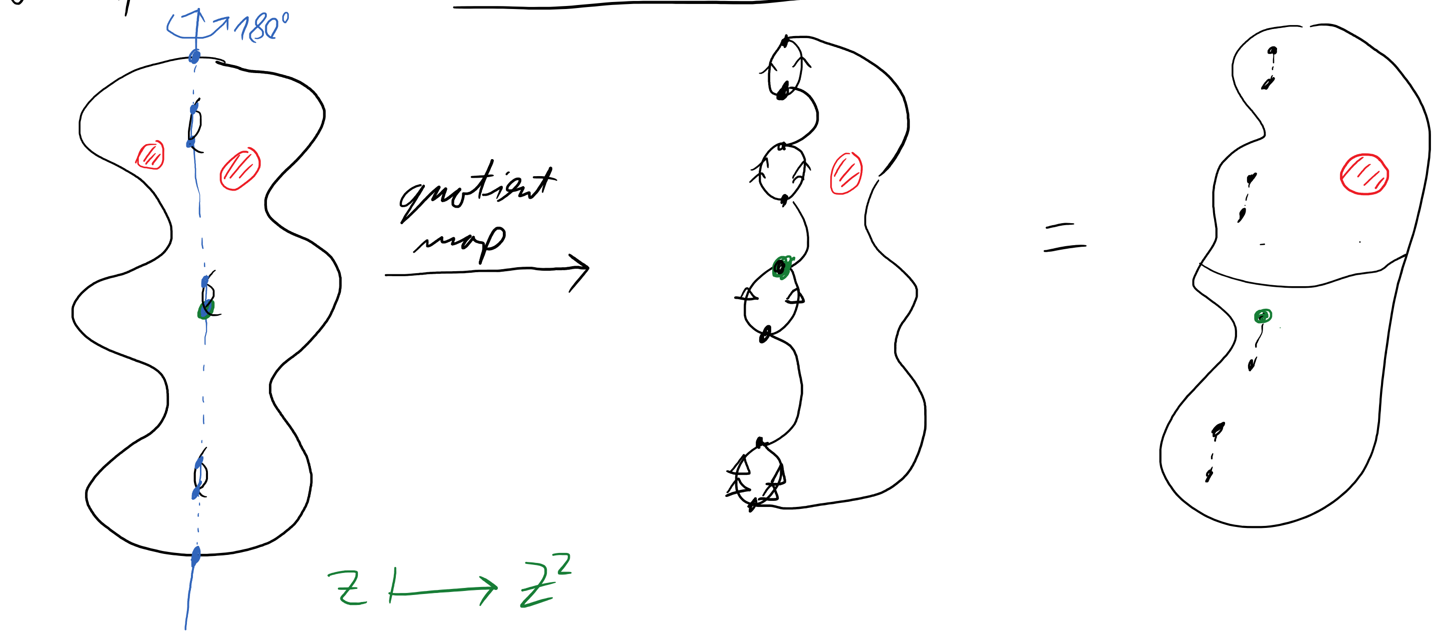
COROLLARY:

∀ 3-mfd M^3 ∃ compact 4-mfd W^4 s.t. $\partial W = M$

i.e. $\Omega_3 = 0$

THM:

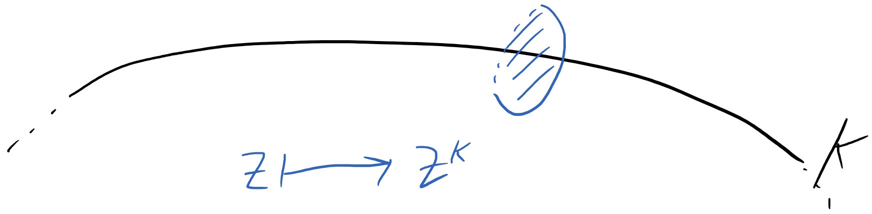
$\forall$ surface Σ $\exists$ BRANCHED COVERING $\Sigma^2 \longrightarrow S^2$



THM:

$\forall$ 3-mfd M^3 $\exists$ 3-fold BRANCHED COVERING

$M^3 \longrightarrow S^3$ branched along a knot K



TWO WAYS TO STUDY 3-MFDS:

(1) VIA THEIR 1-DIM SUBMFDS (KNOTS)

(2) " 2-DIM " (SURFACE)

Literature:

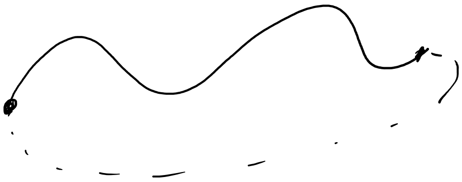
V. Prasolov and A. Sossinsky: Knots, Links, Braids and 3-Manifolds, AMS, 1997, available online here.
D. Rolfsen: Knots and Links, Publish or Perish, 1976, available online here

Exam: oral

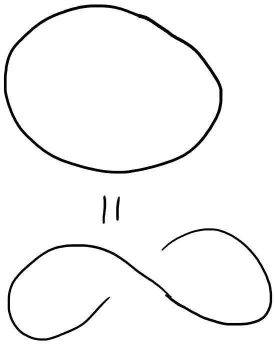
Office hour / more discussion: after the lecture / exercise

Doodle for alternative dates for lecture and exercise:
https://doodle.com/poll/pe8nfexc5vaxhiw9?utm_campaign=poll_added_participant_admin&utm_medium=email&utm_source=poll_transactional&utm_content=gotopoll-cta#table

1. Knots and Links

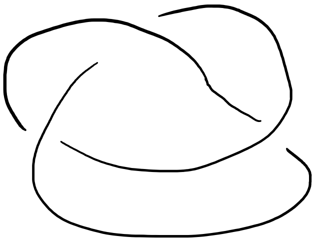


$\subset \mathbb{R}^3 \subset S^3$



UNKNOT

$\neq$



TREFOIL

$\neq$

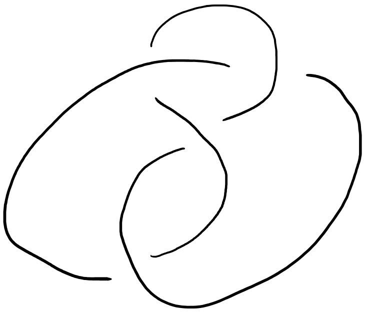
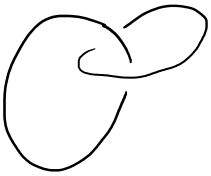
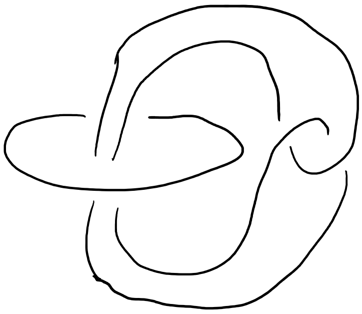


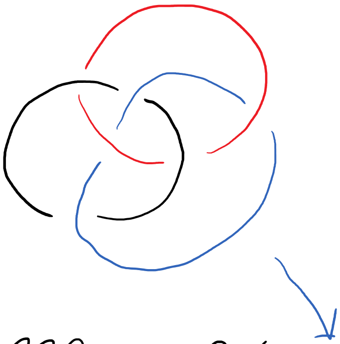
FIG 8



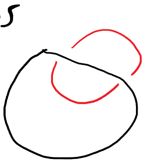
HOPF-LINK

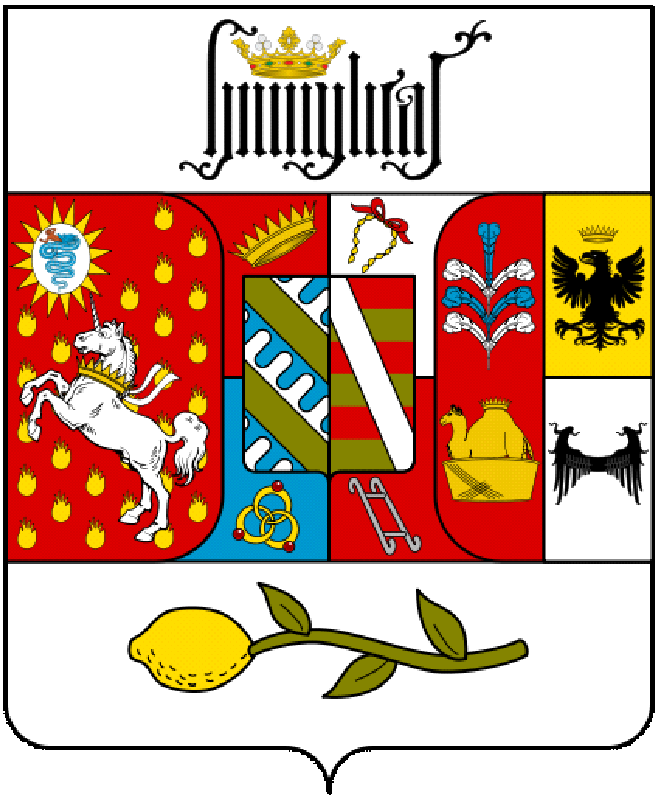


WHITEHEAD-LINK

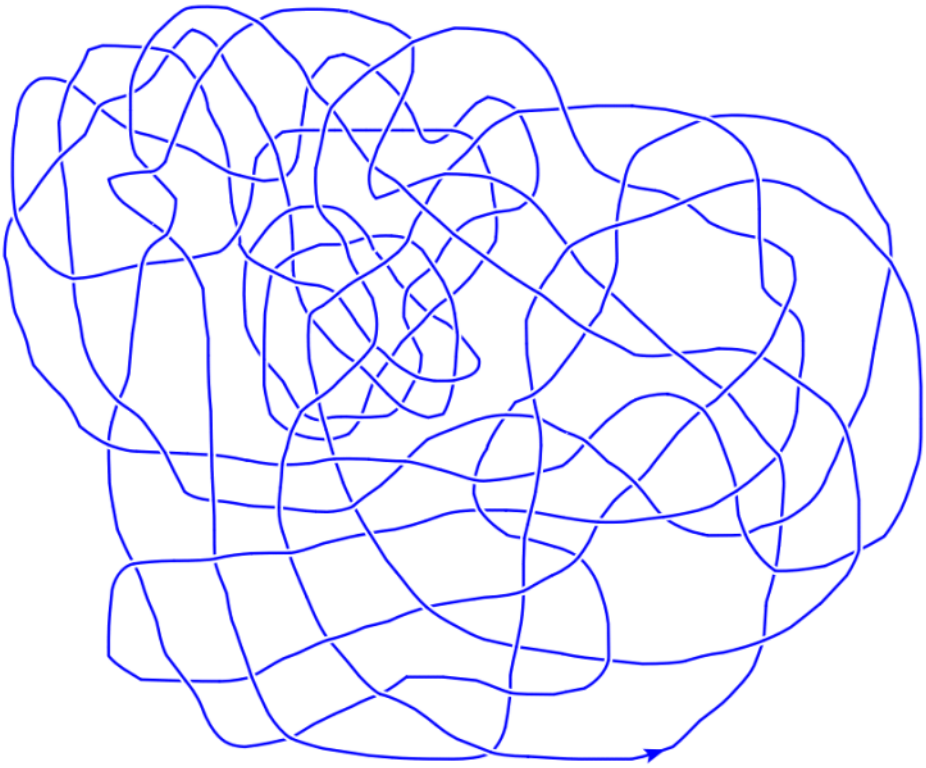


BORROMEO-RINGS





Coat of arms of the House of Borromeo. This figure is retrieved from Wikipedia (2020, April 22) created by user Flanker available online at https://de.wikipedia.org/wiki/Datei:Coat_of_arms_of_the_House_of_Borromeo.svg



HAKEN'S KNOT

GORDIAN KNOT:

https://en.wikipedia.org/wiki/Gordian_Knot

1.1. KNOT PROJECTIONS & REIDEMEISTER MOVES

* $K: S^1 \longrightarrow \mathbb{R}^3$ is a called (C^∞-/PL-/C^0-) KNOT :⟺ K is a (C^∞-/PL-/C^0-) embedding.

* $K_0, K_1: S^1 \longrightarrow \mathbb{R}^3$ are called ISOTOPIC ($K_0 \sim K_1$)

:⟺ ∃ (C^∞-/PL-/C^0-) map $F: S^1 \times \underset{\substack{\shortparallel \\ I}}{[0,1]} \longrightarrow \mathbb{R}^3$ s.t.

$F(\cdot, 0) = K_0$

$F(\cdot, 1) = K_1$

$F(\cdot, t)$ is a (C^∞-/PL-/C^0-) knot.

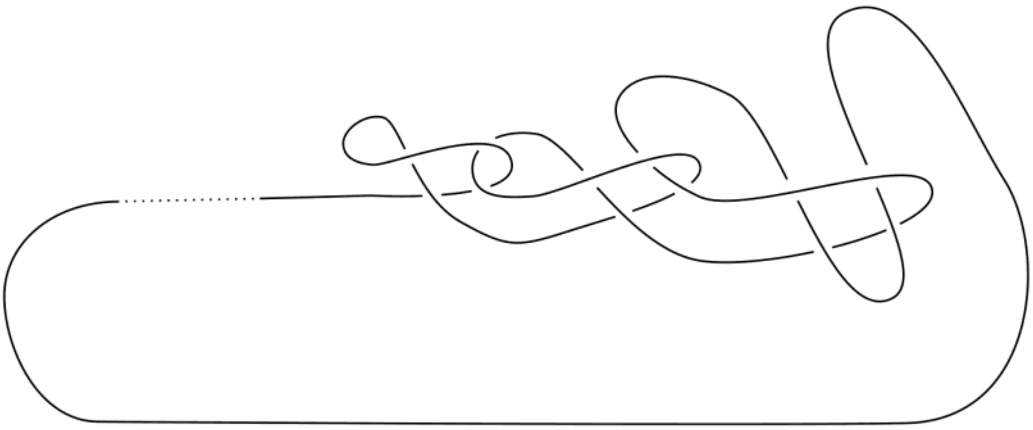
Lemma 1:

$$\{\text{PL-knots}\}/_\sim = \{C^\infty\text{-knots}\}/_\sim =: \text{TAME KNOTS}$$

Proof: **G. Burde, M. Heusener and H. Zieschang:** Knots, De Gruyter, 2013, available online here.

PROP. 1.10 □

Remark: ∃ WILD KNOTS



FOX'S WILD ARC

An example of an isotopy of wild knots: https://i.stack.imgur.com/lptcS.gif
By Jim Belk, see https://math.stackexchange.com/questions/1336275/which-two-knots-are-isotopic-but-not-ambient-isotopic

NOTATION: KNOT := isotopy class of a TAME KNOT.

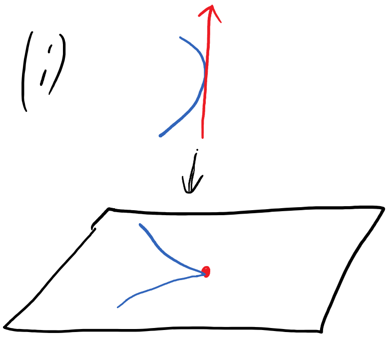
SLOGAN: think C^∞, prove PL

Lemma 2:

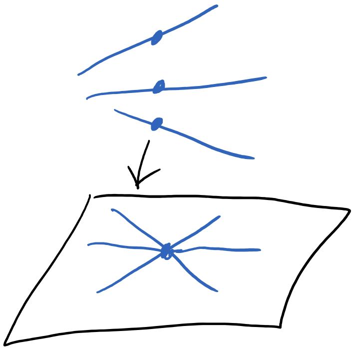
∀ knot K ∃ REGULAR PROJECTION:

a projection $D: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ s.t. the following does NOT happen:

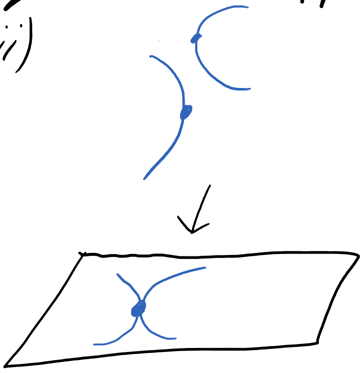
(i)



(ii)



(iii)



Proof: sheet 1 ▨

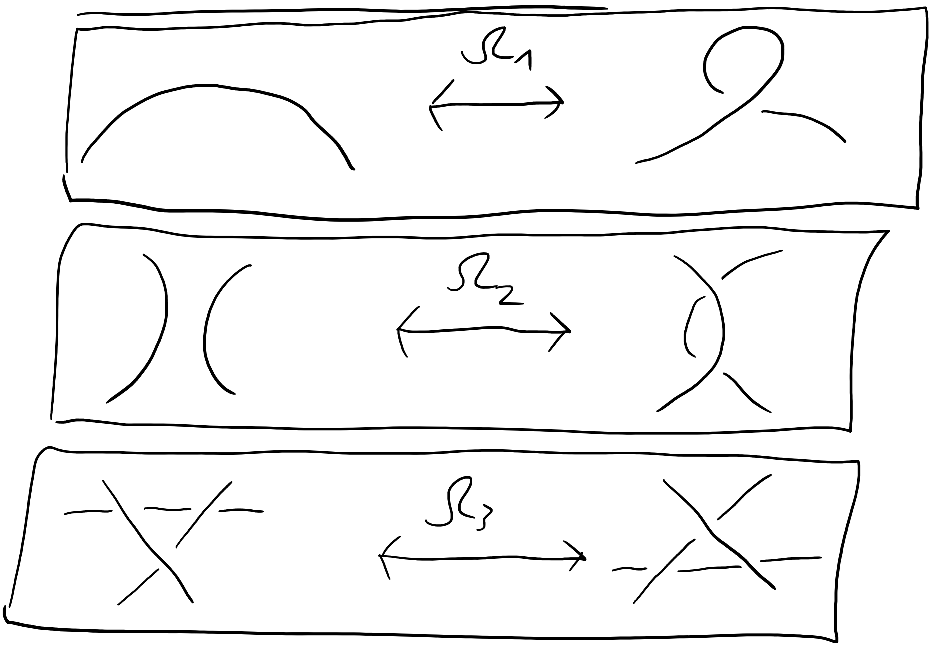
KNOT-DIAGRAM

$D_K := \mathrm{Im}(D \circ K)$ together with under/overcrossing inf.

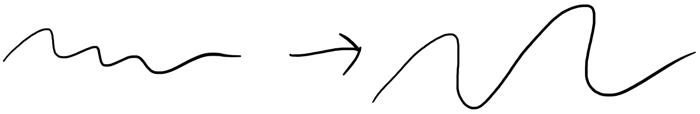
THM 3 (REIDEMEISTER 1932)

$K_1 \sim K_2 \iff D_{K_1}$ can be transformed to D_{K_2} by finitely many

REIDEMEISTER-MOVES

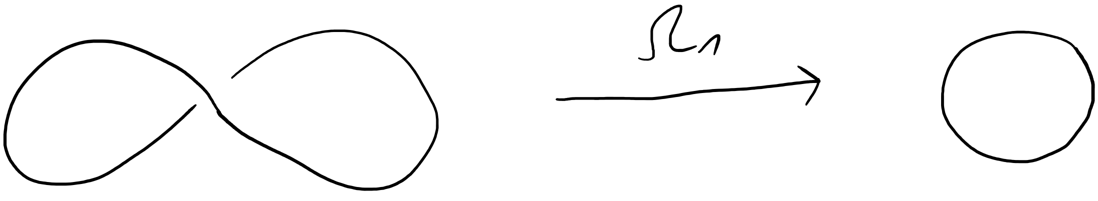


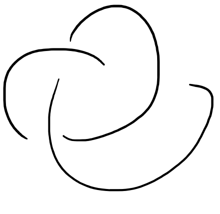
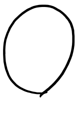
& planar isotopies



Proof: Sheet 1 ▨

Example:



WHY IS  $\not\sim$  ?

1.2. KNOT INVARIANTS

Knot $K \longrightarrow i(K)$ = number, group, polynomial

s.t. $K_1 \sim K_2 \Rightarrow i(K_1) = i(K_2)$

Ex: CROSSING NUMBER:

$c(K) := \min\{\#\text{ crossings in } D_K \mid D_K \text{ reg proj of } K\}$

$c(O) = 0$

$c($  $) \leq 3$ Why $c($  $) \neq 0$

⟶ HARD TO COMPUTE

KNOT TABLES: (are usually ordered by c)

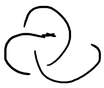
http://katlas.math.toronto.edu/wiki/Main_Page
https://knotinfo.math.indiana.edu/
https://en.wikipedia.org/wiki/List_of_prime_knots

c	0	1	2	3	4	5	6	7	8	..
#K	1	0	0	1	1	2	3	7	21	..

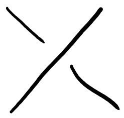
↑ (prime)
(up to mirroring)

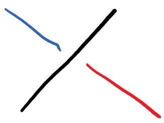
17 crossings ⇒ ~ 350 MILLION KNOTS (BURTON)

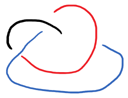
A002863 = Number of prime knots with n crossings
https://oeis.org/A002863

Lemma 4:  $\not\sim O$

Proof sketch: via 3-COLORABILITY of a knot (see sheet 1)





O is NOT 3-COLO. BUT  is ▨

1.3 KNOT POLYNOMIALS:

THM 5 (JONES, FIELDS MEDAL 1990)

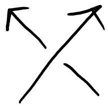
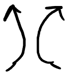
$$\exists !\ V : \{\text{oriented links}\}/\text{isotopy} \longrightarrow \{\text{polynomials in } q^{\pm 1/2}\} = \mathbb{Z}[q^{1/2}, q^{-1/2}]$$

called JONES POLYNOMIAL def by

* $V(O) = 1$

* SKEIN RELATION:

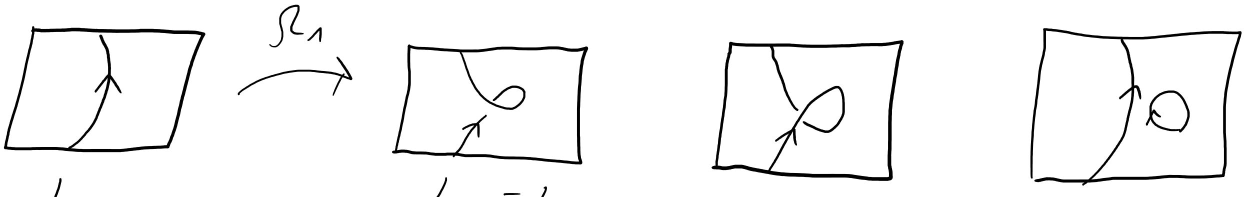
$$q^{-1} V(L_+) - q V(L_-) = (q^{1/2} - q^{-1/2}) V(L_0)$$

where $L_+ =$  $L_- =$  $L_0 =$ 

Ex:

(0) $V(L \sqcup O) = -(q^{-1/2} + q^{1/2}) V(L)$ ⌝

⌜



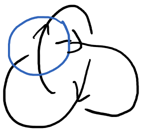
L $\quad$ $L_- = L$ $\quad$ $L_+ = L$ $\quad$ $L_0 = L \sqcup O$

(SKEIN-REL)

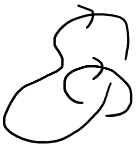
$$\Rightarrow q^{-1} V(L) - q V(L) = (q^{1/2} - q^{-1/2}) V(L \sqcup O)$$

$$\Rightarrow V(L \sqcup O) = \frac{q^{-1} - q}{q^{1/2} - q^{-1/2}} V(L) = -(q^{-1/2} + q^{1/2}) V(L)$$

⌞ ⌟

(1)  $= L_-$

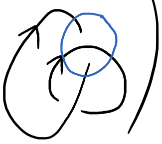
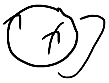
SKEIN REL

$\Rightarrow q^{-1} V($  $) - q V($  $) = (q^{1/2} - q^{-1/2}) V($  $)$

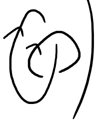
$= L_+ = 0$, $= 1$

$= L_-$

$= L_0 =$ Hopf-link

(2) $-q V($  $) + q^{-1} \underbrace{V(}_{-(q^{-1/2}+q^{1/2})}$  $) = (q^{1/2} - q^{-1/2}) \underbrace{V(}_{=1}$  $)$

L_-

$\Rightarrow V($  $) = -q^{-2}(q^{-1/2} + q^{1/2}) - q^{-1}(q^{1/2} - q^{-1/2})$

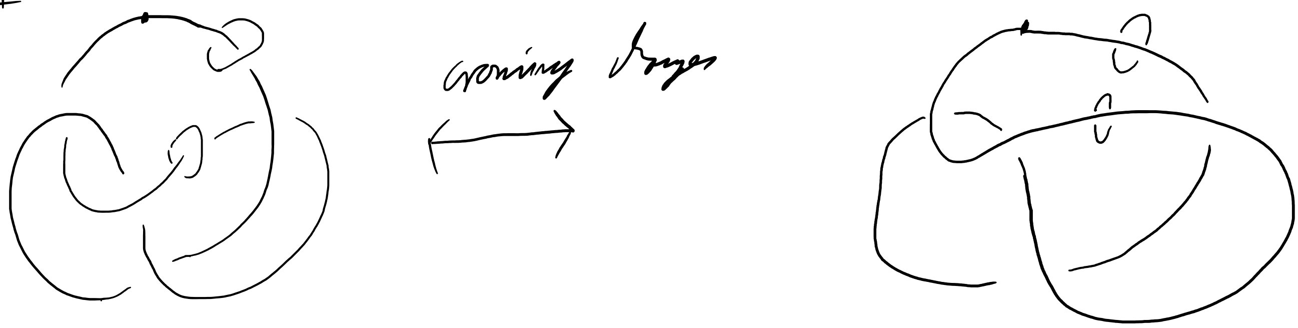
$= -q^{-5/2} - q^{-1/2}$

$\Rightarrow V($  $) = q^{-1} + q^{-3} - q^{-4}$

Lemma 6: $V(L)$ can always be computed via skein relation

Lemma 7: $\forall$ m-component link L $\exists$ finite sequence of crossing changes that transforms L into $\underbrace{O\cdots O}_{m}$

Proof:



▨

Proof of L.6:

Let L_1 be a diagram of an m-comp. link L with n crossings

$$\overset{L.7}{\Rightarrow} \ \exists \text{ finite sequence } L_1 \longrightarrow L_2 \longrightarrow \cdots \longrightarrow L_k \sim \underbrace{O\cdots O}_{m}$$

crossing change

$$\overset{Ex\,(0)}{\Rightarrow} \quad V(L_k) = (-q^{-1/2} - q^{1/2})^{m-1}$$

$\forall\, i=1,\dots,k-1:$

$$q^{-1}\, V(L_i) - q V(L_{i+1}) = (q^{1/2} - q^{-1/2})\, V(L_i')$$

$$\text{or} \quad q^{-1}\, V(L_{i+1}) - q V(L_i) = (q^{1/2} - q^{-1/2})\, V(L_i')$$

where L_i' has $n-1$ crossings.

induction on n $\Rightarrow$ claim

▨

KAUFFMAN POLYNOMIAL

Lemma 8:

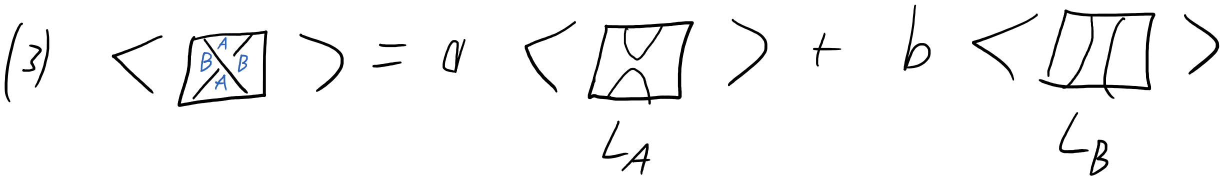
∀ link diagram D_L

⇒ ∃! polynomial $\langle D_L \rangle$ in a, b, c s.t.

$D_L \sqcup O := \boxed{D_L} \quad \bigcirc$

(1) $\langle \bigcirc \rangle = 1$

(2) $\langle D_L \sqcup O \rangle = c \langle D_L \rangle$



Proof: we choose an ordering 1,…,n of the crossings of D_L

A STATE s of D_L assigns the value A or B to every crossing

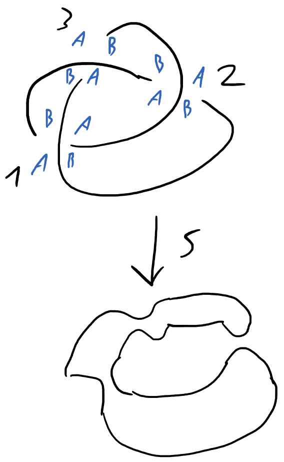
(→ 2^n possibilities)

$\alpha(s) :=$ # (crossings in state A)

$\beta(s) :=$ # (" B)

$\gamma(s) :=$ # (circles after the simplification according to s)

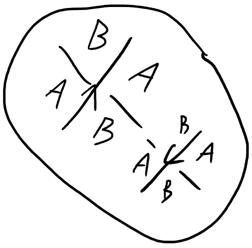
Ex:



s = (A, B, A)

$\alpha(s) = 2$ $\beta(s) = 1$

⇒ $\gamma(s) = 1$



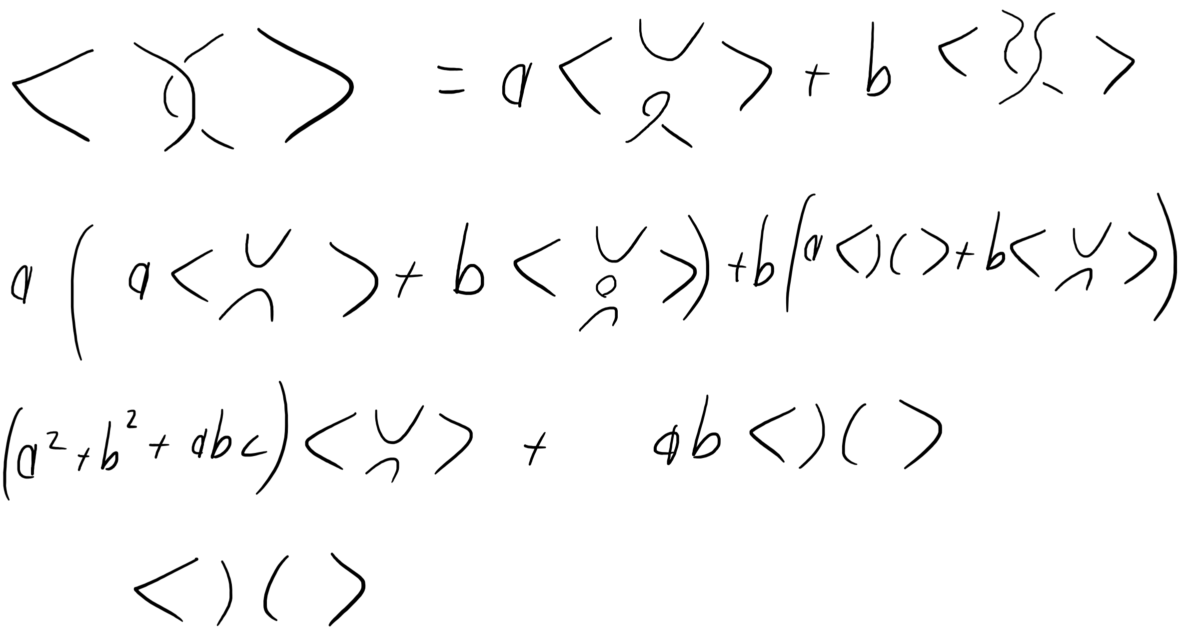
(1),(2)&(3) ⇒ $\langle D_L \rangle = \sum_s a^{\alpha(s)} b^{\beta(s)} c^{\gamma(s)-1}$ (*)

⇒ $\langle D_L \rangle$ is unique & (*) def $\langle D_L \rangle$ ▨

GOAL: upgrade $\langle D_L \rangle$ to a knot invariant,

i.e. **WE WANT** $\langle D_L \rangle$ does **NOT** change under Reidemeister moves

* $\boxed{\Omega_2}$

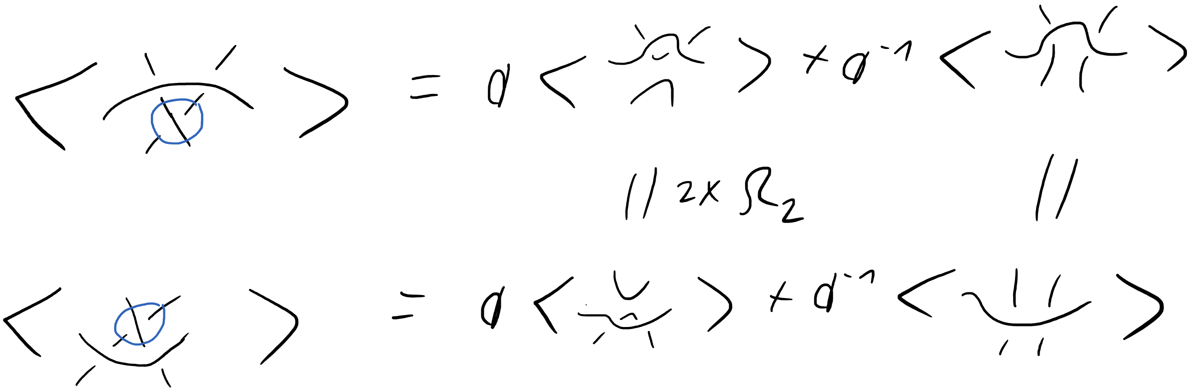


$(=)\ a^2+b^2+abc=0$ & $ab=1$

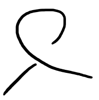
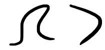
$(=)\ \langle \cdot \rangle$ is inv. under Ω_2

Define: $b := a^{-1}$ & $c := -a^2-b^2 = -a^2-a^{-2}$

* $\boxed{\Omega_3}$



$=)\ \langle \cdot \rangle$ is inv. under Ω_3

$\boxed{\Omega_1}$ $\langle$  $\rangle = a \langle$  $\rangle + a^{-1} \langle$  $\rangle$

$= \left(a(-a^2-a^{-2}) + a^{-1} \right) \langle$  $\rangle$

$= -a^3 \langle$  $\rangle$

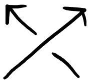
similarly $\langle$  $\rangle = \cdots = -a^{-3} \langle$  $\rangle$

$\Rightarrow$ $\langle \cdot \rangle$ is NOT a link invariant

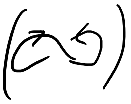
Def: Let D_L be an oriented link diagram.

The WRITHE is

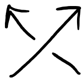
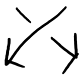
$$w(D_L) = \sum_i \varepsilon_i \qquad \text{(over all crossings)}$$

where  & 

$\varepsilon = +1$ $\qquad$ $\varepsilon = -1$

Ex: $w($  $) = +3$; $w($  $) = 0$; $w($  $) = -1$

Rem: $w(L) = w(-L)$

$\varepsilon = +1$ $\qquad$ $\varepsilon = +1$

The KAUFFMAN POLYNOMIAL

THM 9 Let D_L be a link diag of an oriented link.

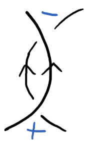
Then

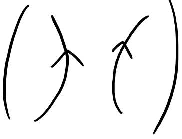
$$X(L) := (-a)^{-3w(D_L)} \langle D_L \rangle$$

depends only on L, i.e. is a link invariant.

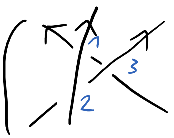
Proof:

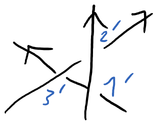
$\boxed{\Omega_2}$ $\langle D_L \rangle$ is invariant under Ω_2

$w\Big($ 

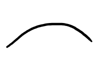
$\Big) = w\Big($  $\Big)$

$\boxed{\Omega_3}$ $\langle D_L \rangle$ is invariant under Ω_3

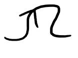
$w\Big($ 

$\Big) = w\Big($ 

$\Big)$

$\boxed{\Omega_1}$ $\langle$  $\rangle = -a^{-3} \langle$  $\rangle$

$w\Big($  $\Big) = w\Big($  $\Big) - 1$

$X\Big($  $\Big) = -a^{-3}(-a)^{3} X\Big($  $\Big) = X\Big($  $\Big)$

The same for 

▨

Corollary 10:

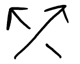
$$V(L)\,(q^{\pm 1/2}) = X(L)\,(a = q^{-1/4})$$

Proof:

$$-a^4 X(L_+) + a^{-4} X(L_-) = (a^2 - a^{-2})\, X(L_0)$$

$$w(L_\pm) = w(L_0) \pm 1$$

$\Rightarrow -a^4 X(L_+) + a^{-4} X(L_-)$

$= -a^4 (-a)^{-3w(L_+)} \langle$  $\rangle + a^{-4} (-a)^{-3w(L_-)} \langle$  $\rangle$

$= -a^4 (-a)^{-3w(L_0)} (-a)^3 \Big[a^{-1} \langle$  $\rangle + a \langle \underbrace{)\,(}_{L_0} \rangle \Big]$

$+ a^{-4} (-a)^{-3w(L_0)} (-a)^3 \Big[a \langle$  $\rangle + a^{-1} \langle \underbrace{)\,(}_{L_0} \rangle \Big]$

$= a^2 X(L_0) - a^{-2} X(L_0)$

▨

2. Manifolds and handle decompositions

2.1. TOP-PL-DIFF

Def: * M^n is a (TOP) MANIFOLD of DIMENSION n :(=)

(1) M is a top HAUSDORFF SPACE

(2) M has a COUNTABLE BASIS

⌈ $\exists \{B_i\}$ countable, $B_i \subset M$ open s.t.

$$\forall U \subset M \text{ open} : U = \bigcup_{i \in I} B_i$$

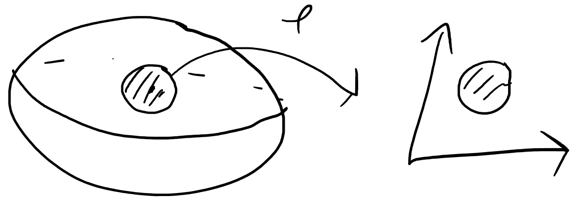
Ex: $\mathbb{R}^n$ $\{B_i = B_r(x) \mid r \in \mathbb{Q}_+, x \in \mathbb{Q}^n\}$ ⌋

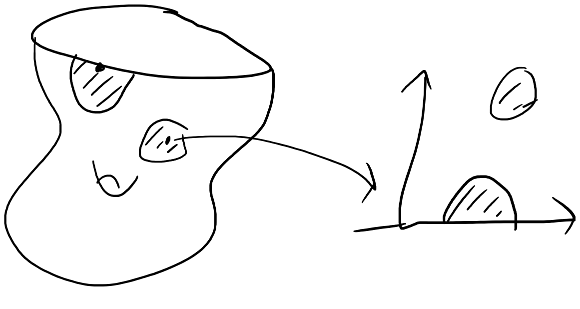
(3) $\forall p \in M \; \exists \, p \in U \subset M$ open

$$\exists \, \varphi : U \xrightarrow{\overset{C^0}{\cong}} V \subset \mathbb{R}^n \text{ open}$$

φ = CHART, φ^{-1} = PARAMETRIZATION

[REPLACE: $\mathbb{R}^n$ by $\mathbb{R}^n_+ := \{x_n \geq 0\}$ → MFDS WITH BOUNDARY]





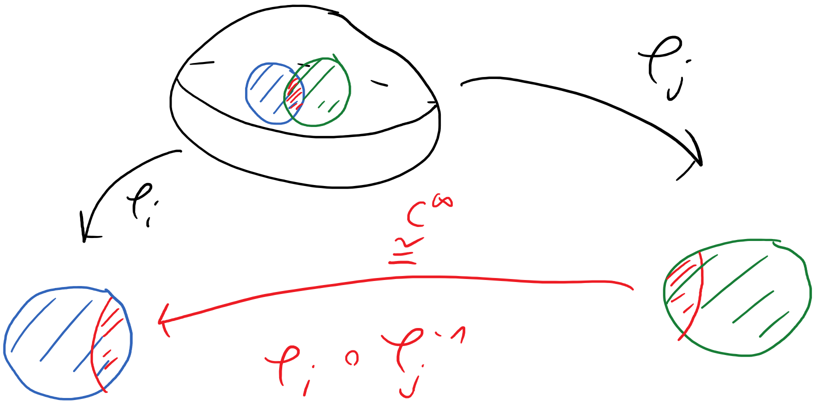
* An <u>ATLAS</u> $\mathcal{A}$ of M is a family of charts

$$\{(U_i, \varphi_i)_{i\in I}\} \text{ s.t. } M = \bigcup_{i\in I} U_i$$

* (U_i, φ_i), (U_j, φ_j) are <u>COMPATIBLE</u> :⟺

$$\varphi_i \circ \varphi_j^{-1} : \underset{\underset{\mathbb{R}^n}{\cap}}{\phi_j(U_i \cap U_j)} \xrightarrow[\overset{\cong}{}]{C^\infty} \underset{\underset{\mathbb{R}^n}{\cap}}{\phi_i(U_i \cap U_j)}$$

is a diffeo.



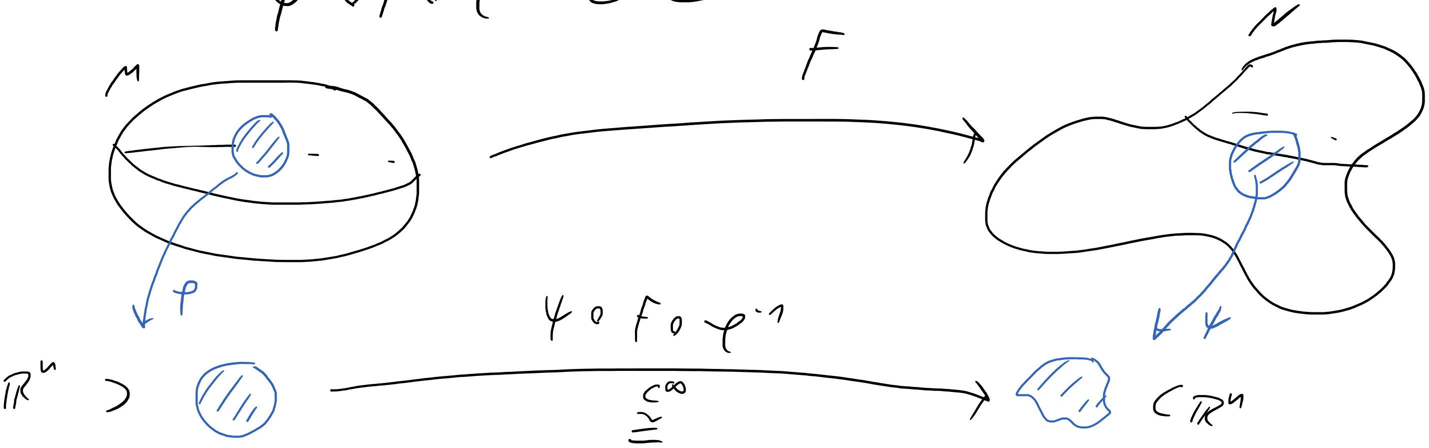
* $\mathcal{A}_1$ & $\mathcal{A}_2$ are <u>EQUIVALENT</u> :⟺
all charts in $\mathcal{A}_1$ & $\mathcal{A}_2$ are compatible

* The equivalence class of an atlas (in which all charts are compatible) is called <u>SMOOTH STRUCTURE</u>.

* $F: M \longrightarrow N$ is called a DIFFEOMORPHISM $:\Leftrightarrow$

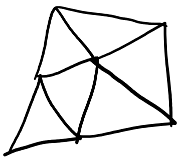
- F is a homeom
- $\forall$ charts (U,φ) of M & (V,ψ) of N :

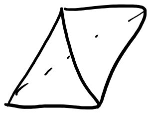
$$\psi \circ F \circ \varphi^{-1} \in C^\infty$$



Remark: * If we replace C^∞ by PL we get the class of PL-mfds

* PL mfds have triangulation





We have:

$$DIFF \subset PL \subset TOP$$

↑ (WHITEHEAD)

i.g. $DIFF \neq PL \neq TOP$

Ex: $n=4$ $TOP \neq PL = DIFF$

$*$ $n=1,2,3$: $TOP = PL = DIFF$ (MOISE 1953)

POINCARÉ CONJECTURE (1903)

Let M^n be closed & $M \simeq S^n$

$\boxed{?}$ $\Downarrow$ C^0-PC

$M \overset{C^0}{\cong} S^n$

YES!

$n=1,2$ ✓

$n=3$ PERELMAN 2003

$n=4$ FREEDMAN 1981

$n \geq 5$ SMALE 1960

↑

HANDLE DECOMPOSITIONS

$\Downarrow$ C^∞-PC

$M \overset{C^\infty}{\cong} S^n$

$n=1,2,3$ YES!

$n=7$ NO!

(MILNOR 1956)

$n \geq 5$ well understood

$\boxed{n=4}$ $\boxed{?}$ OPEN

CONSTRUCTION OF EXOTIC 7-SPHERES

COMPLEX NUMBERS

$\mathbb{C} \cong \mathbb{R}^2 \ni \begin{pmatrix} a \\ b \end{pmatrix} \hat{=} a+ib$, multip def by $i^2=-1$

QUATERNIONS:

$\mathbb{H} \cong \mathbb{R}^4 \ni \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \hat{=} a+ib+jc+kd$, mult def by $i^2=j^2=k^2=ijk=-1$

$S^3 \subset \mathbb{H}$ gets a group str

$$E^7 := D^4 \times S^3 \cup_\varphi D^4 \times S^3$$

$$S^3 \times S^3 \ni (z,w) \longmapsto (z, z^2 w z^{-1}) \in S^3 \times S^3$$

is a closed <u>smooth</u> 7-mfd.

* $E^7 \overset{C^0}{\cong}$ [E^7 has a Morse fct with 2 crit pts

→ E^7 = 0-handle $\cup$ 7-handle

$= D^7 \cup D^7 \overset{C^0}{\cong} S^7$

[ALEXANDER-TRICK]]

* $E^7 \overset{C^\infty}{\not\cong} S^7$

⌜ $\nexists$ $E \xrightarrow{\overset{C^\infty}{\cong}} -E :=$ E with reversed or. ⌝

⌞ → HARDER! ⌟

OTHER CONSTRUCTIONS

$$E^7_k = \left\{ z=(z_1,\dots,z_5) \in \mathbb{C}^5 \;\middle|\; \begin{array}{c} z_1^2+z_2^2+z_3^2+z_4^3+z_5^{6k-1}=0 \\ |z|^2 = \varepsilon \text{ for } \varepsilon>0 \\ \text{sufficiently small} \end{array} \right\}$$

BRIESKORN MFDS

$k=1,\dots,28$ → get all 28 smooth str on S^7

Further reading on the Poincare conjecture:
https://nilesjohnson.net/seven-manifolds.html
https://en.wikipedia.org/wiki/Poincar%C3%A9_conjecture
https://en.wikipedia.org/wiki/Generalized_Poincar%C3%A9_conjecture
https://en.wikipedia.org/wiki/Exotic_sphere
https://www.semanticscholar.org/paper/On-Manifolds-Homeomorphic-to-the-7-Sphere-Milnor/621f403ad244bca225bdf367215119202175e3d7

K-HANDLE IN DIM n : $D^k \times D^{n-k} \overset{C^0}{\cong} D^n$

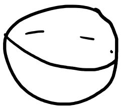
attached to an n-MFD M via an

embedding $\partial D^k \times D^{n-k} \hookrightarrow \partial M$

$\boxed{n=3}$

0-HANDLE :

$\{0\} \times D^3$

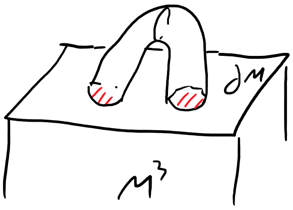


$\partial\{0\} \times D^3 = \emptyset$

1-HANDLE :

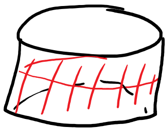
$D^1 \times D^2$

$\partial D^1 \times D^2$

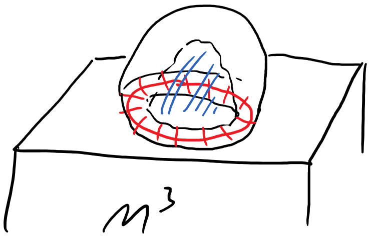


2-HANDLE

$D^2 \times D^1$

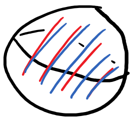


$\partial D^2 \times D^1$



3-HANDLE

$D^3 \times \{0\}$



$\partial D^3 \times \{0\}$

2.2. HANDLE DECOMPOSITIONS

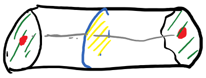
Def: An n-dim $\underset{\nearrow \text{INDEX}}{k}$-HANDLE h_k is a copy of $D^k \times D^{n-k}$

ATTACHED to a smooth mfd M^n via an embedding

$$\varphi : \partial D^k \times D^{n-k} \hookrightarrow \partial M$$

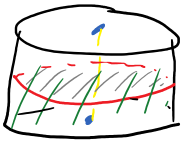
Ex: 3-dim 1-handle:

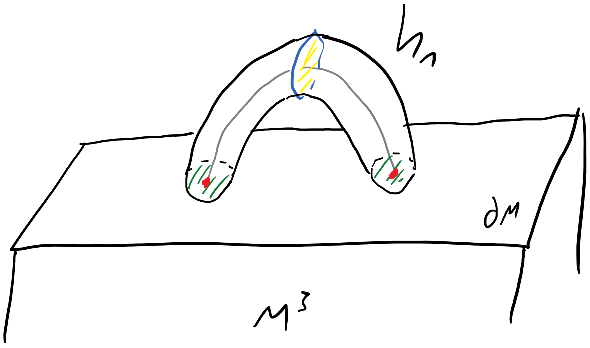
$D^1 \times D^2$

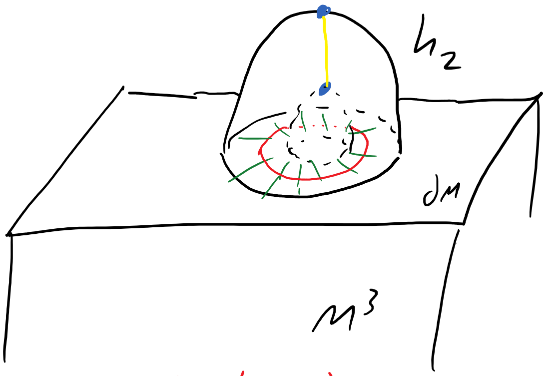


3-dim 2-handle:

$D^2 \times D^1$







ATTACHING SPHERE $A_k = \partial D^k \times \{0\} \equiv \varphi(\partial D^k \times \{0\})$

BELT SPHERE $B_k = \{0\} \times \partial D^{n-k}$

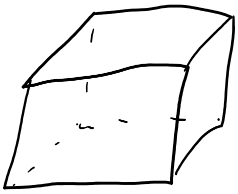
ATTACHING REGION $= \partial D^k \times D^{n-k} \equiv \varphi(\partial D^k \times D^{n-k})$

CORE $= D^k \times \{0\}$

CO CORE $= \{0\} \times D^{n-k}$

Remark: we see $M \cup h_k$ as a smooth mfd.

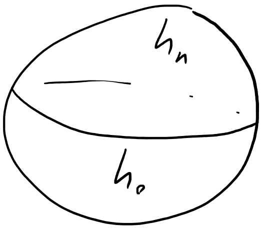
∂I^n



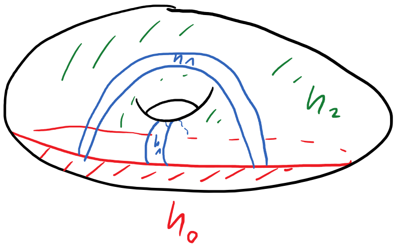
$\overset{C^0}{\cong} S^{n-1}$

Examples:

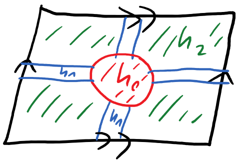
(1) $S^n = D^n \cup D^n = h_0 \cup h_n$



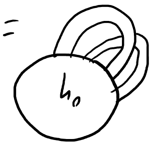
(2) $T^2 =$



$=$

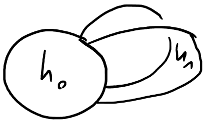


$=$

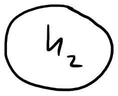


$\cup$

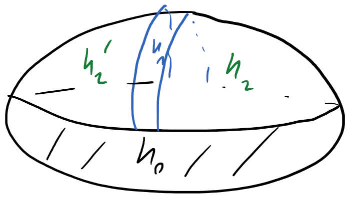
(3) $\mathbb{RP}^2 =$



$\cup$



(4) $S^2 =$



Lemma 1: $\varphi_i : \partial D^k \times D^{n-k} \hookrightarrow \partial M$ for $i = 1, 2$

φ_1 is isotopic to φ_2 $\Rightarrow$ $M \cup_{\varphi_1} h_k \overset{C^\infty}{\cong} M \cup_{\varphi_2} h_k$

Proof: ISOTOPY EXTENSION THEOREM:

M, N compact & $h : I \times N \longrightarrow M$ isotopy

$\Rightarrow \exists\, H : I \times M \longrightarrow M$ smooth s.t.

- $H_0 = \mathrm{id}_M$
- H_t is a diffeo of M
- $h_t = H_t \circ h_0$

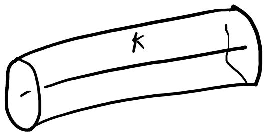
AMBIENT ISOTOPY

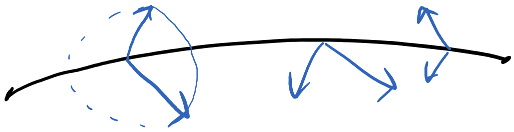
$$\begin{tikzcd}
M_1 \arrow[d, dashed, "C^\infty"', "\cong"] & := M \cup_{\varphi_1} h_k /\sim & \varphi_1(p) \arrow[d, "H_1"', "\cong"] & p \arrow[l, mapsto] \arrow[d, "\cong"', "\mathrm{id}"] \arrow[dl, phantom, "\circlearrowleft"] \\
M_2 & := M \cup_{\varphi_2} h_k /\sim & \varphi_2(p) & p \arrow[l, mapsto]
\end{tikzcd}$$

▨

Remark: The isotopy class of $\varphi : \partial D^k \times D^{n-k} \hookrightarrow \partial M$ is determined by $\varphi_0 : \partial D^k \times \{0\} \hookrightarrow \partial M$ together with a FRAMING of $\varphi_0(S^{k-1}) =: K \subset \partial M$, i.e. a map $K \longrightarrow GL_{n-k}(\mathbb{R})$

$n=4$ $k=2$ $\partial D^2 \times D^2$





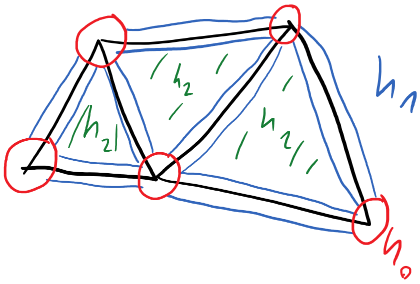
Thm 2 (SMALE, 1960)

$\forall$ smooth compact mf M $\exists$ a handle decomp of M

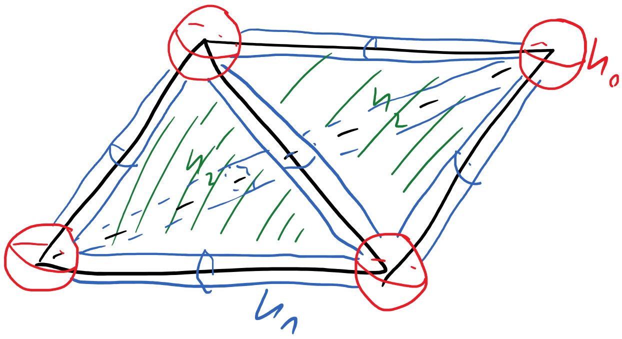
Proof: PL:

* Let T be a triangulation of M

$n=2$



$n=3$

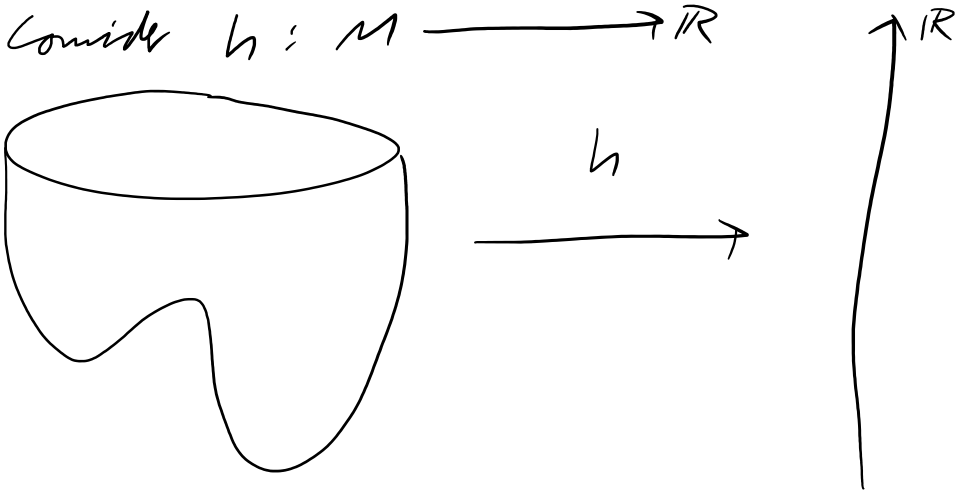


regular nbhd of a k-simplex $\cong$ k-handle

(2) C^∞: MORSE THEORY:

Choose an embedding of $M \subset \mathbb{R}^N$ (WHITNEY)

Consider $h: M \longrightarrow \mathbb{R}$



h MORSE $:\Longleftrightarrow$ $\forall$ CRITICAL POINT $p \in M$ (i.e. $\nabla_p h = 0$)

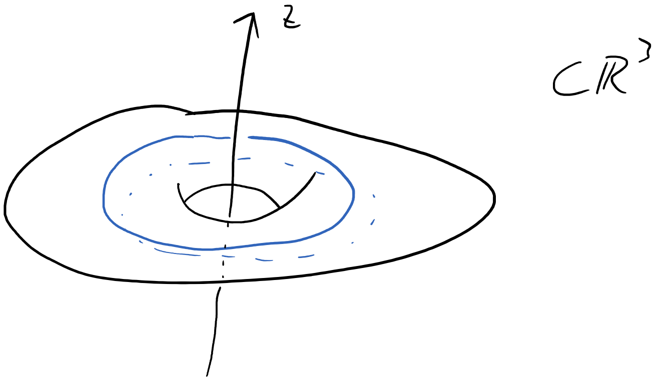
$\Longrightarrow \det(H_p h) \neq 0$

$\Longleftrightarrow \forall p \in \mathrm{Crit}(h)$ $\exists$ coord $(x_1, \dots, x_n)$ s.t.

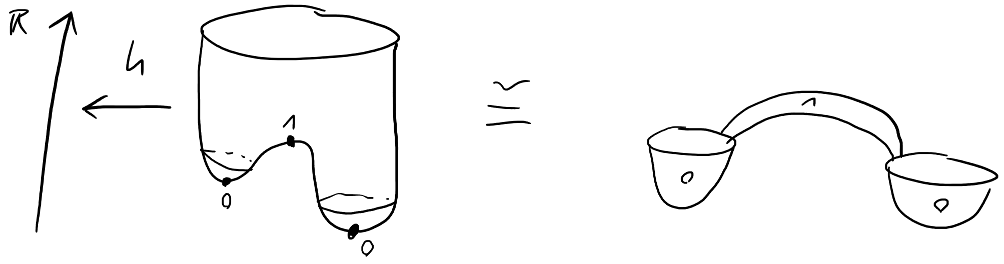
$$h: (x_1, \dots, x_n) \longmapsto -\sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2$$

$\Longleftrightarrow$ h generic

$\Longrightarrow \forall M \ \exists h: M \longrightarrow \mathbb{R}$ Morse



Observation! critical point of h $\longleftrightarrow$ k-handle



∎

Lemma 3 for $l \leq k$:

$$(M \cup h_k) \cup h_l \overset{C^\infty}{\cong} (M \cup h_l) \cup h_k$$

Proof sketch:

let $A_l = S^{l-1} \subset \partial(M \cup h_k)$ the attaching sphere of h_l

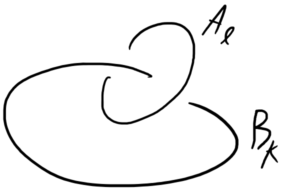
& $B_k = S^{n-k-1}$ the belt sphere of h_k
$\subset \partial(M \cup h_k)$

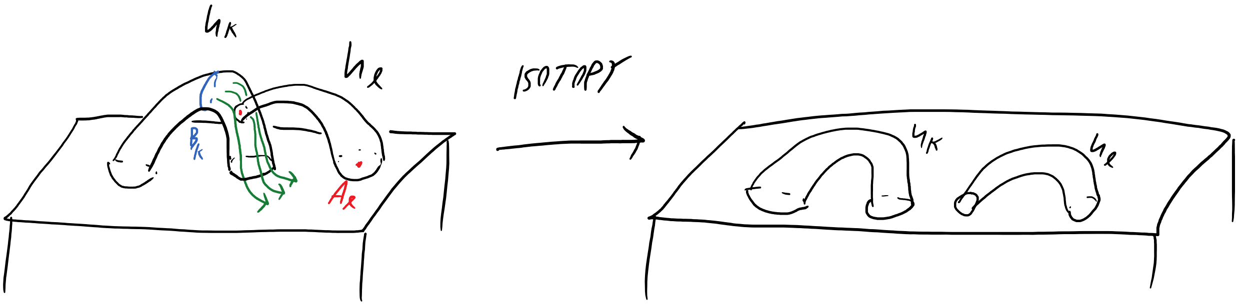
$l \leq k$

$\Rightarrow \dim(A_l) + \dim(B_k) = l-1 + n-k-1 < n-1 = \dim(\partial(M \cup h_k))$

transversality thm

$\Rightarrow A_l \cap B_k = \emptyset$

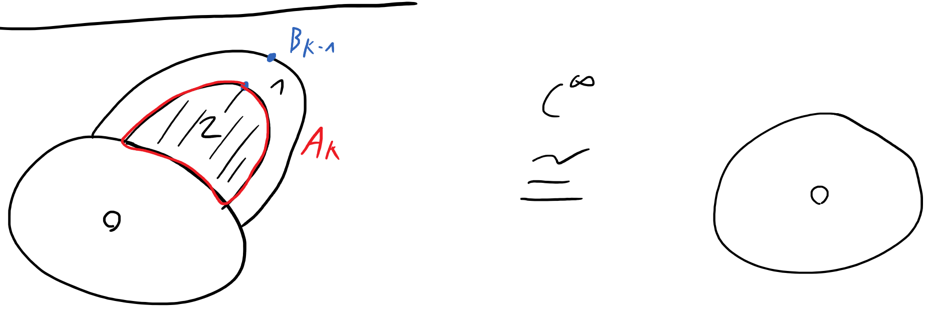


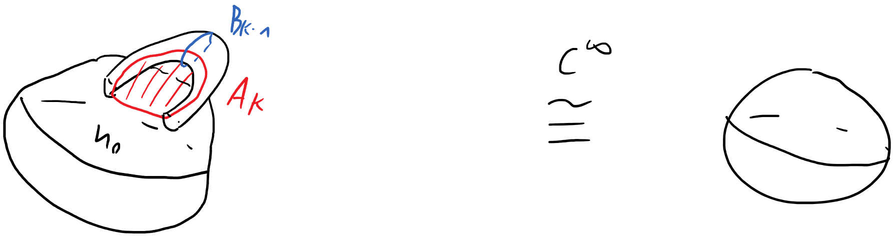


□

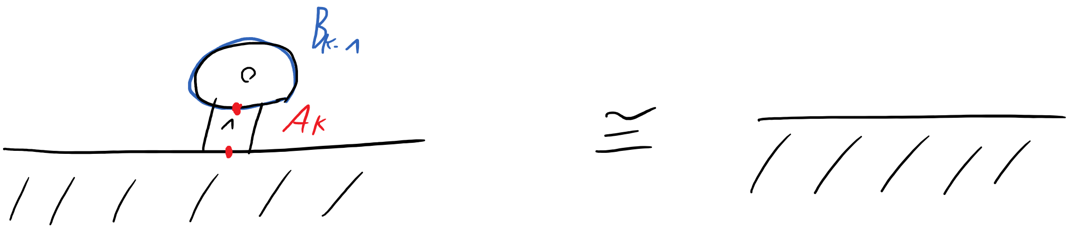
HANDLE CANCELLATION:

Ex:





$A_k \pitchfork B_{k-1} = \{pt\}$



Lemma 4:

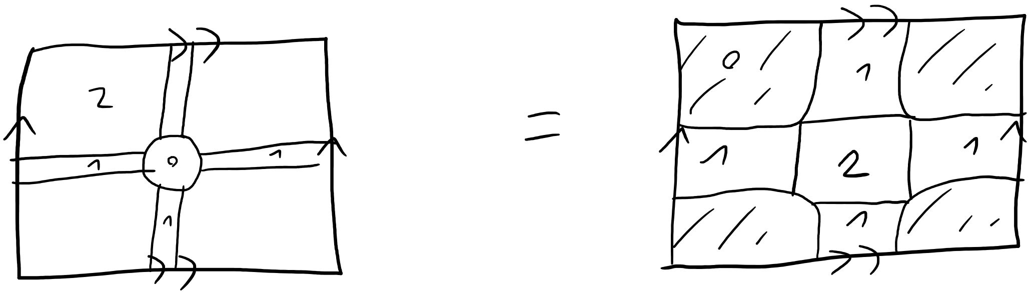
If $A_k \pitchfork B_{k-1} = \{pt\}$

$$\Rightarrow \quad M \cup h_{k-1} \cup h_k \overset{C^\infty}{\cong} M$$ ▨

DUAL HANDLE DECOMPOSITION

Observation k-handle $h_k = D^k \times D^{n-k} = D^{n-k} \times D^k = (n-k)$-handle h_{n-k}

cocore of h_k = core of h_{n-k}



„put handlebody upside down" $\hat{=}$ Morse fct $h \longrightarrow 1-h$

Lemma 5:

M^n connected, closed, smooth

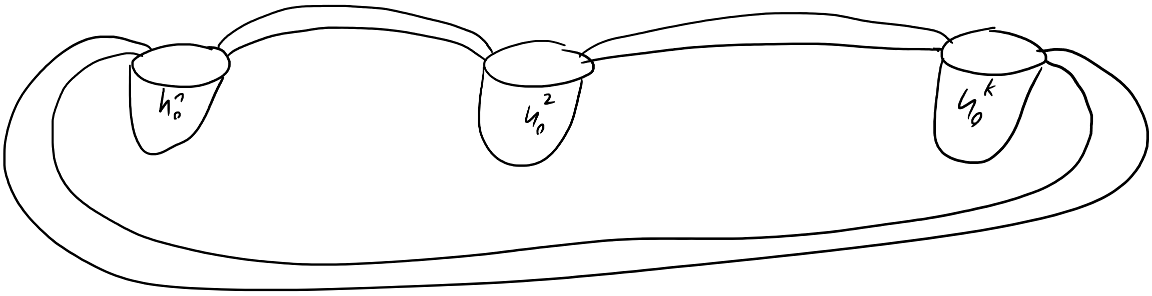
$\Rightarrow$ $\exists$ handle decomp of M with

- exactly one 0-handle
- " n-handle

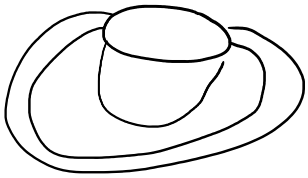
Proof: Take a handle decomp of M.

- M closed $\Rightarrow$ $\exists$ at least one 0-handle h_0
- Let $h_0^1, \dots, h_0^k$ the 0-handles

M connected $\Rightarrow$ h_0^i are connected by 1-handles



$\cong$

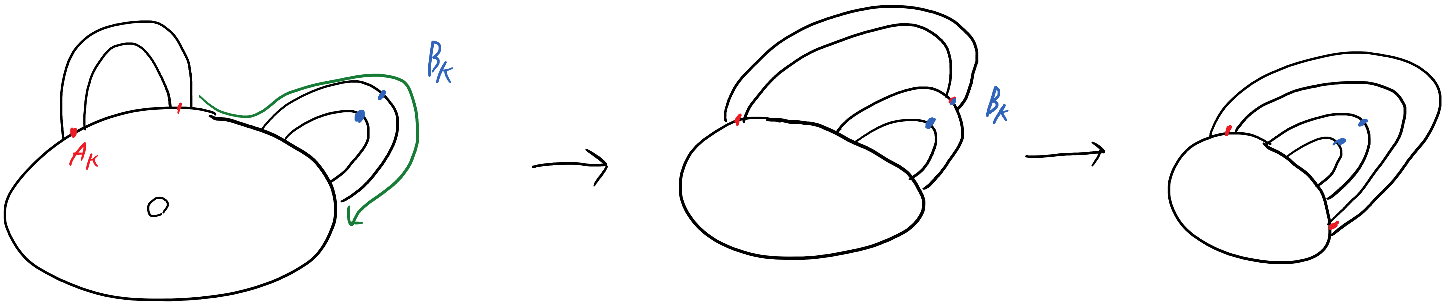


- after handle cancellation $\Rightarrow$ $\exists!$ 0-handle
- dual handle decomp. $\Rightarrow$ $\exists!$ n-handle

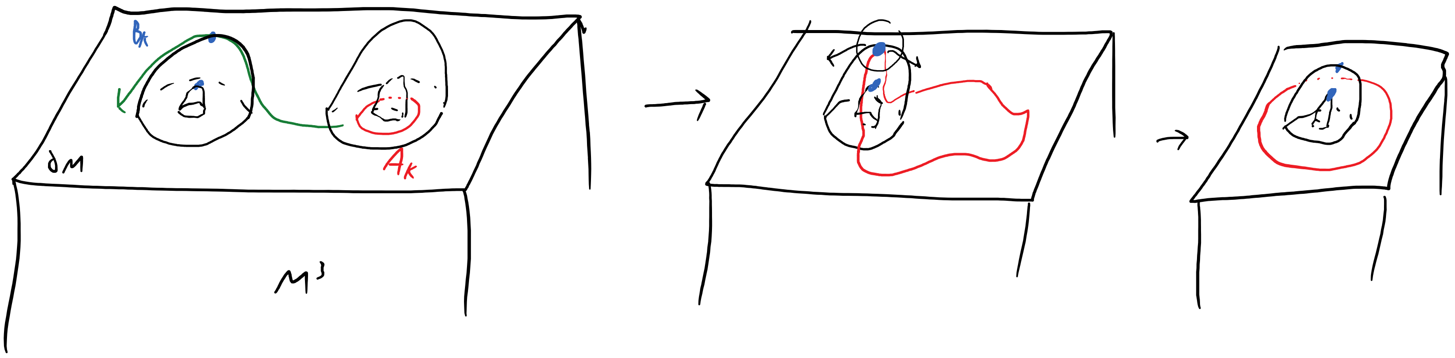
▨

HANDLE SLIDES!

Examples: $\boxed{n=2}$ $\boxed{k=1}$



$\boxed{n=3}$ $\boxed{k=2}$



Let h_k^1, h_k^2, $0<k<n$ be two k-handles attached to ∂M

A <u>HANDLE SLIDE</u> of h^1 over h^2 is the isotopy of A^1 in $\partial(M\cup h^2)$ through B^2

$\dim(A^1)+\dim(B^2) = k-1 + n-k-1 = n-2 = \dim(\partial(M\cup h^2))-1$

Thm 6: (CERF, 1970)

* Two handle decompositions (ordered by increasing index) of a compact mfd M are related by finitely many handle slides & introducing / removing cancelling pairs.

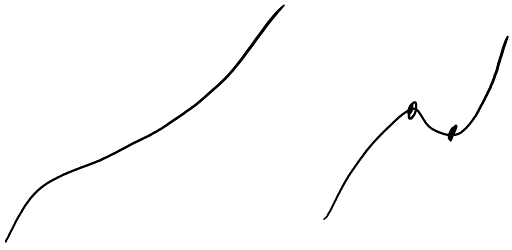
* If the handle decomp have unique 0- & n-handle we do **NOT** need to introduce cancelling 0-1 or $(n-1)$-n pairs.

Proof idea:

$$\begin{tikzcd} \text{handle decomp}_1 \arrow[d, leftrightarrow] & \text{handle decomp}_2 \arrow[d, leftrightarrow] \\ \text{Morse fct } h_1 : M \to \mathbb{R} & \text{Morse fct } h_2 : M \to \mathbb{R} \end{tikzcd}$$

Analyse $h_t : M \longrightarrow \mathbb{R}$ a generic homotopy connecting h_1 & h_2.

∎

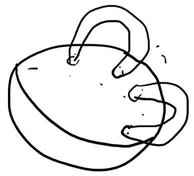


3. Heegaard splittings

3.1. EXISTENCE OF HEEGAARD SPLITTINGS

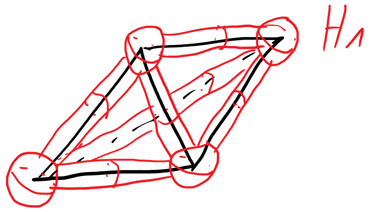
GOAL: ∀ closed, or. 3-mfd M ∃ a HEEGAARD SPLITTING:

$M = H_1 \cup H_2$ $\qquad H_i =$



IDEA: Let T be a triangulation of M

$H_1 :=$ reg nbhd of vertices ∪ edges in T



$H_2 := M \setminus H_1$

PROBLEM: This is WRONG if M is NOT or.

Let M^3 be a connected, closed, orientable 3-mfd with a handle decomposition

$$M = \underbrace{h_0 \cup h_1^1 \cup \dots \cup h_1^{g_1}}_{H_1} \cup \underbrace{h_2^1 \cup \dots \cup h_2^{g_2} \cup h_3}_{H_2}$$

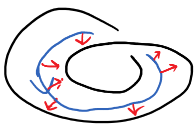
Def: A smooth mfd M^n is called ORIENTABLE :⟺

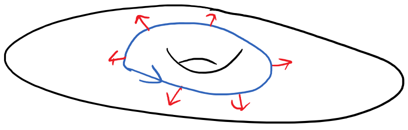
∃ Atlas $A = \{(U_i, \varphi_i)\}$ on M s.t.

$\forall p \in M \ \forall i,j : \det(D_p(\varphi_j \circ \varphi_i^{-1})) > 0$

$\boxed{\text{„∄ loop in M interchanging „left" \& „right" "}}$

Ex:





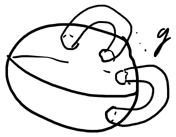
Lemma 1:

M^n smooth, orientable & compact with a handle decomposition

$$\Rightarrow M_1 := \{0\text{-handle}\} \cup \{1\text{-handles}\} \overset{C^\infty}{\cong} \natural_g S^1 \times D^{n-1} \qquad (n \geq 3)$$

↗ 1-HANDLE BODY OF GENUS g

$\natural_g S^1 \times D^{n-1} =$

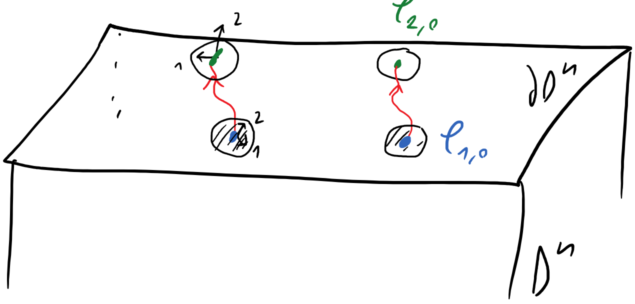


Proof: We show: $\forall \varphi_1, \varphi_2 : \partial D^1 \times D^{n-1} \hookrightarrow \partial D^n$

$$D^n \cup_{\varphi_1} h^1 \overset{C^\infty}{\cong} D^n \cup_{\varphi_2} h^1$$

* Any embedding $\varphi_{i,0} : \underset{S^0}{\underset{\|}{\partial D^1}} \times \{0\} \hookrightarrow \partial D^n$

are isotopic $(n \geq 3)$

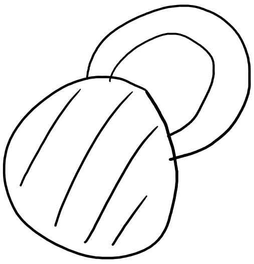


* Framings of $K := \varphi_0(\partial D^1 \times \{0\}) \subset \partial D^n$ are homotopy classes of maps $K = S^0 \longrightarrow GL_{n-1}(\mathbb{R})$

$\Rightarrow \{\text{framings of } K\} = \pi_0(GL_{n-1}(\mathbb{R})) = \text{con. comp of } GL_{n-1}(\mathbb{R}) = \mathbb{Z}_2$

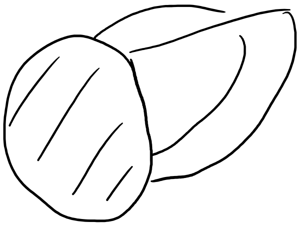
M orientable $\Rightarrow \exists!$ framing of K along which to attach h_1

▨

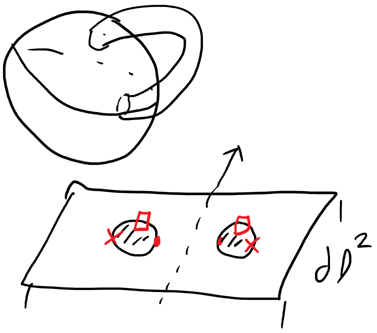


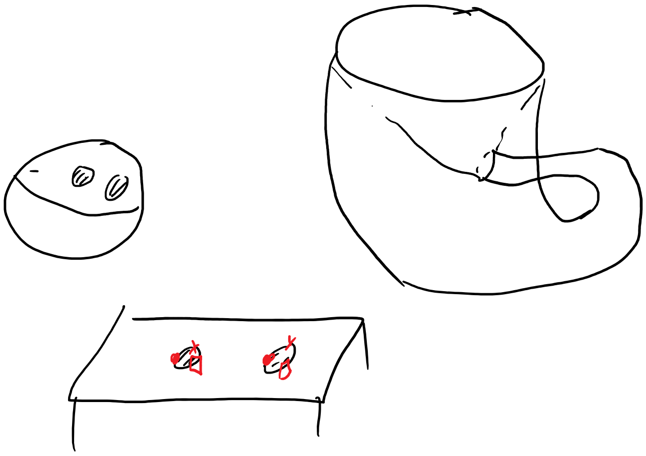
1-HANDLEBODY

H_1 $S^1 \times D^1$



MÖBIUS STRIP





Lemma 2 H_1 & H_2 are 1-handlebodies of the same genus.

Proof: * L1 $\Rightarrow$ H_1 is a 1-handlebody

$\Rightarrow$ $\partial H_1 = \Sigma_{g_1}$

* dual handle decomp of M & L_1

$\Rightarrow$ H_2 is a 1-handlebody

$\Rightarrow$ $\partial H_2 = \Sigma_{g_2}$

* $\Sigma_{g_1} = \partial H_1 \cong \partial H_2 = \Sigma_{g_2}$

$\Rightarrow$ $g_1 = g_2$ ▨

Def: A decomposition of M^3 into two 1-handlebodies of the same genus

$$M = H_1 \cup_\partial H_2$$

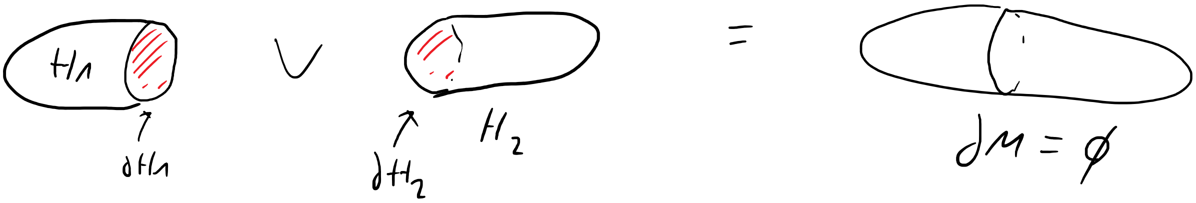
is called a HEEGAARD SPLITTING.

Corollary 3

∀ closed, orientable 3-mfd M ∃ Heegaard splitting

Remark: If M has a Heegaard splitting ⇒ M closed & orientable ⌝

⌜ * closed



* orientable:

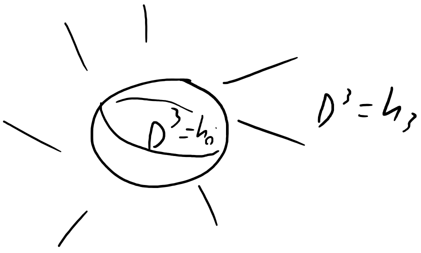
H_1, H_2 oriented ⇒ $H_1 \cup_\varphi H_2$

* φ is or reversing ✓

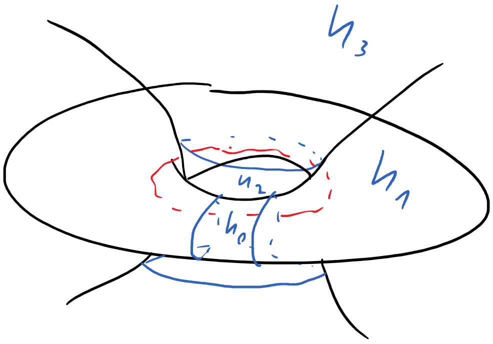
* φ is or preserving → change or of H_2

⌞ ⌟

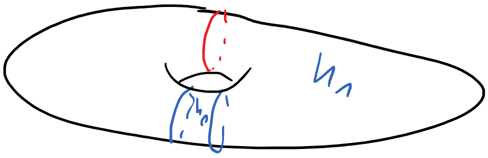
Ex: (1) $S^3 = D^3 \cup D^3$

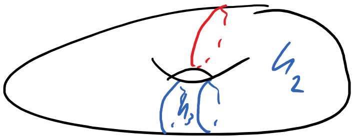


(2) $S^3 = S^1 \times D^2 \cup D^2 \times S^1$



(3) $S^1 \times S^2 = S^1 \times D^2 \cup S^1 \times D^2 = S^1 \times (D^2 \cup D^2)$



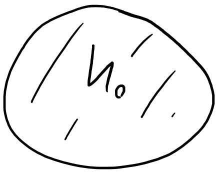


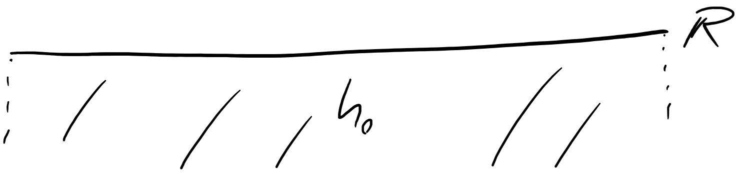
3.2. „KIRBY CALCULUS" OF SURFACES

Let F^2 be closed, orientable with a handle decomposition

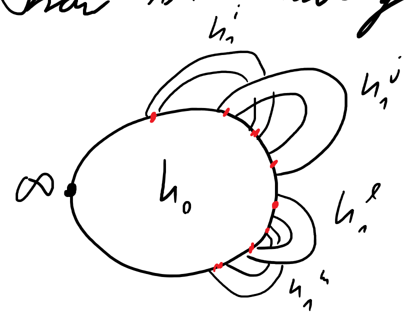
$$h_0 \cup h_1^1 \cup \dots \cup h_1^k \cup h_2$$

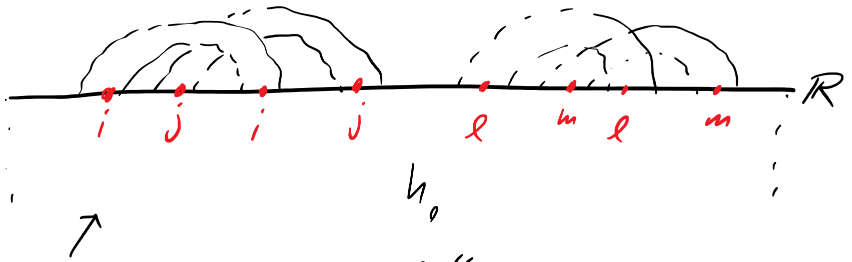
Consider: $\partial h_0 = \partial D^2 = S^1 = \mathbb{R} \cup \{\infty\}$





Draw the attaching spheres of the 1-handles h_1^i on $\mathbb{R} \subset \partial h_0$





„KIRBY DIAGRAM OF F"

Lemma 4 (ALEXANDER TRICK)

$$\forall f: \partial D^n \xrightarrow{\overset{C^0}{\cong}} \partial D^n \Rightarrow \exists F: D^n \xrightarrow{\overset{C^0}{\cong}} D^n \text{ s.t.}$$

$$F|_{\partial D^n} = f$$

Remark: for $n=1,2,3$ also true for C^∞

Proof: $F: D^n \longrightarrow D^n$

$$F(t \cdot x) := t \cdot f(x)$$ is cont. at 0.

$x \in \partial D^n$, $t \in [0,1]$

∎

Corollary 5:

A Kirby diagram of F determines the handledecomposition of F & thus also F.

Proof: * 1-handles are det by a notation L 1

* attaching map of a 2-handle: $\varphi : \underset{\overset{||}{S^1}}{\partial D^2} \times \{0\} \hookrightarrow \partial F_1$

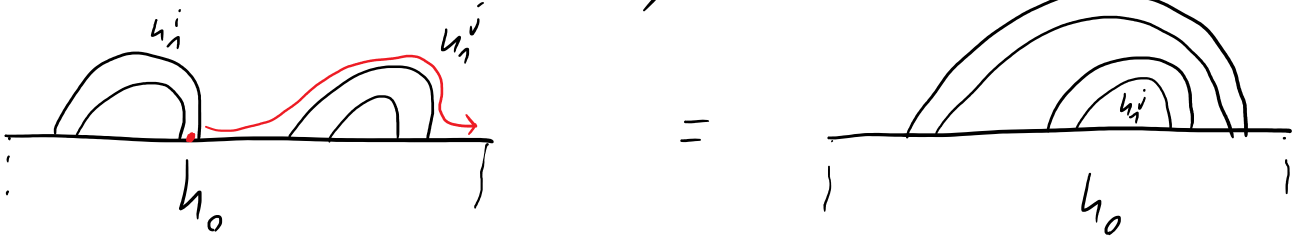
F closed $\Rightarrow \partial F_1 = S^3$

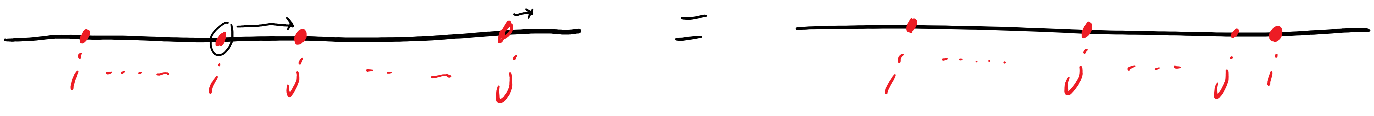
$$\begin{tikzcd}
F_1 \arrow[dd, bend right, "\cong"', "\mathrm{id}"] & \cup_{\varphi_1} & h_2 \arrow[dd, bend left, "\cong"] \\
\varphi_1(p) & p \in \partial h_2 \arrow[l, mapsto, "\varphi_1"'] \arrow[d, phantom, "\circlearrowleft"] & \\
\varphi_2(p) & p \in \partial h_2 \arrow[l, mapsto, "\varphi_2"] & \\
F_1 & \cup_{\varphi_2} & h_2
\end{tikzcd}$$

$\varphi_2^{-1} \circ \varphi_1$ extends to a homeo $h_2 \to h_2$ by L.4.

∎

HANDLE SLIDE: („KIRBY MOVE")



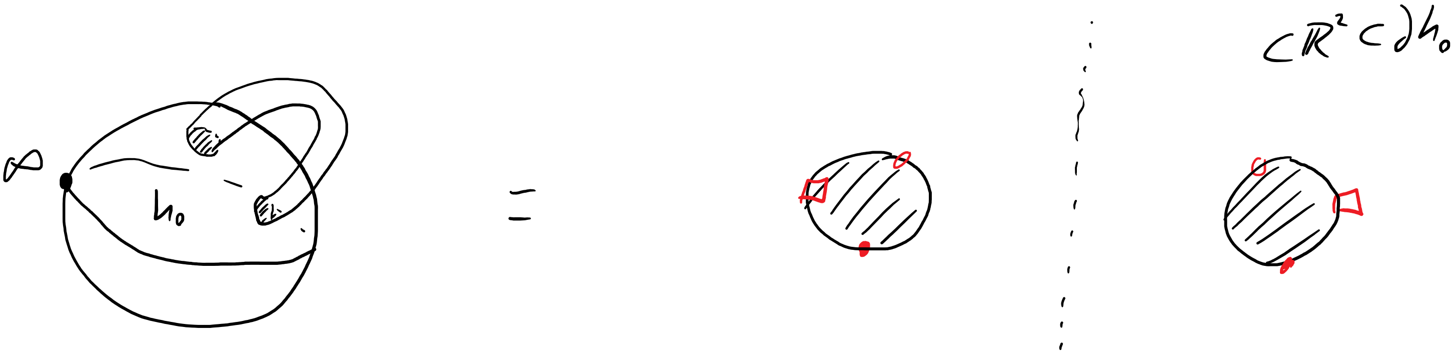


3.3. HEEGAARD DIAGRAMS:

Let $M = \underbrace{h_0 \cup h_1^1 \cup \dots \cup h_1^g}_{H_1} \cup \underbrace{h_2^1 \cup \dots \cup h_2^g \cup h_3}_{H_2}$

be a Heegaard splitting of M

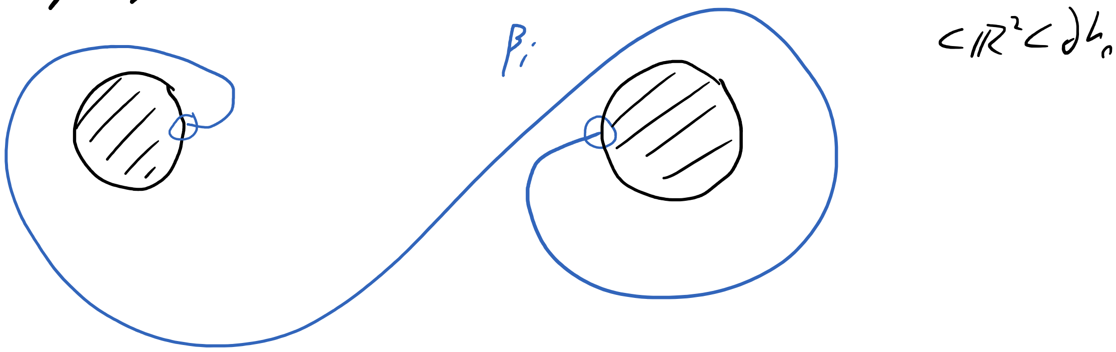
- Consider $\partial h_0 = \partial D^3 = S^2 = \mathbb{R}^2 \cup \{\infty\}$
- Attaching region of 1-handle: $D^2 \sqcup D^2 \subset \mathbb{R}^2 \subset \partial h_0$



identify D^2's via $(x,y) \longmapsto (-x,y)$

- Attaching a 1-handle to h_0: gluing two disks $D^2 \sqcup D^2 \subset \mathbb{R}^2 \subset \partial h_0$ via an or. reversing diffeo.
- Attaching sphere of a 2-handle: $S^1 \subset \partial(h_0 \cup 1\text{-handles})$

i.e. arcs $\beta_i \subset \mathbb{R}^2$ with endpoints on ∂D^2, the attaching regions of 1-handles



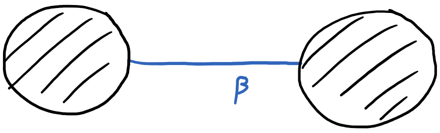
Def: $\mathbb{R}^2$ together with attaching regions $(D^2 \sqcup D^2)$'s of 1-handles & the attaching spheres β_i of the 2-handles is called (PLANAR) HEEGAARD DIAGRAMM

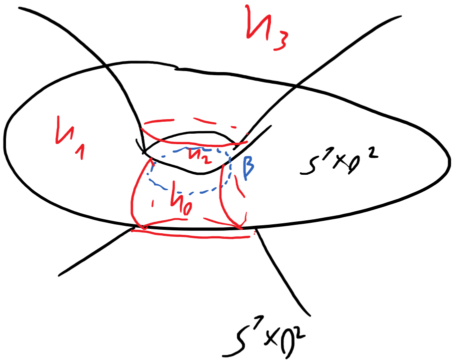
Rem: Sometimes $(\partial H_1, \beta_i)$ is called Heegaard diagr.

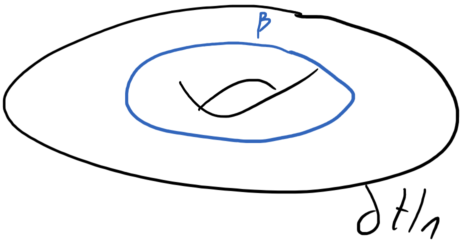
Ex: (1) $S^3 = h_0 \cup h_3 \hat{=} \emptyset \subset \mathbb{R}^2$

(2) $S^3 = S^1 \times D^2 \cup D^2 \times S^1 \hat{=}$

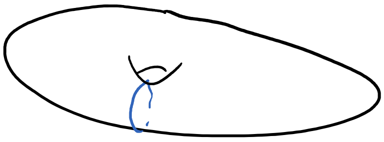
$h_0 \cup h_1 \cup h_2 \cup h_3$

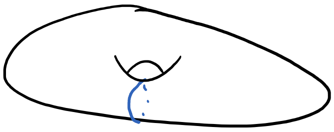




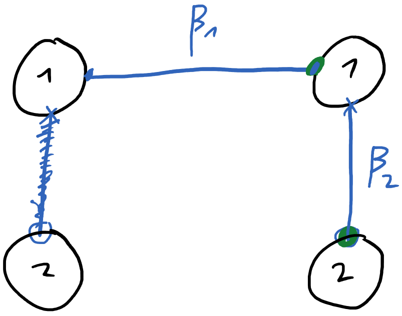


(3) $S^1 \times S^2 = S^1 \times D^2 \cup S^1 \times D^2 \hat{=}$





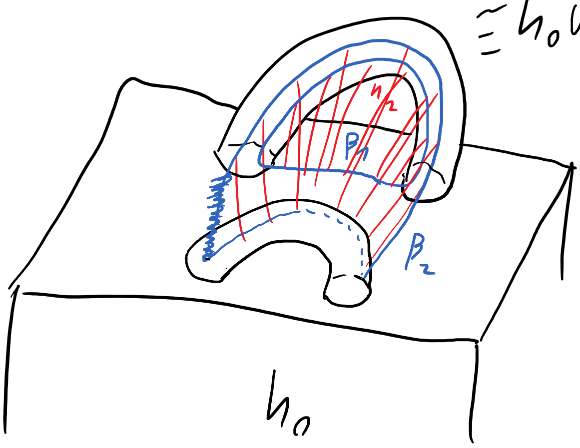
(4)



S^3

$\|$

$\cong h_0 \cup h_3$



Exercise 4.
Which 3-manifold is presented by the following planar Heegaard diagram?

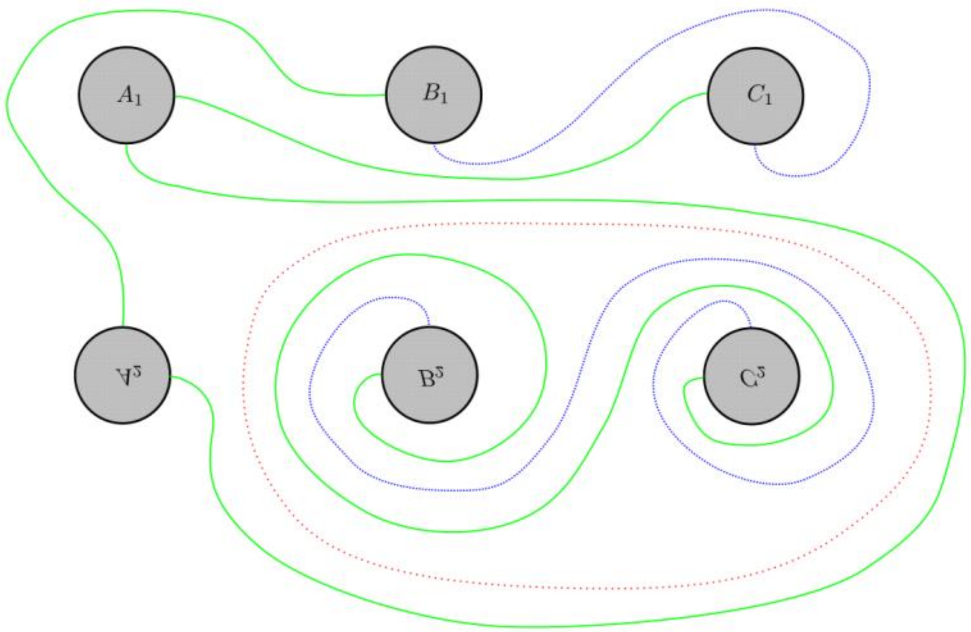


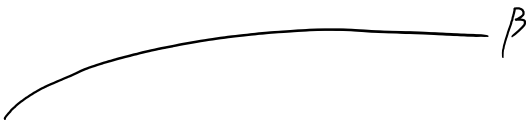
Abbildung 1: The attaching disks of the 1-handles are pairwise identified via a reflection along the horizontal middle line in this planar Heegaard diagram.

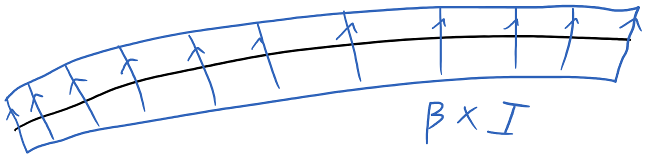
Thm 6: A Heegaard diagram describes a unique handle decomposition of a unique 3-mfd.

Proof:

- $\leq 1 \Rightarrow$ Heegaard diagram describes $M_1 = H_1$
- attaching map of a 2-handle: $\varphi : \partial D^2 \times D^1 \hookrightarrow \partial M_1$

we know $\varphi_0(\partial D^2 \times \{0\}) = \beta \subset \partial M_1 \leftarrow$ 2-DIM





$\{\text{framings of } \beta\} = \{\beta = S^1 \longrightarrow GL_1(\mathbb{R}) = \mathbb{R}\setminus\{0\}\} = \mathbb{Z}_2$

$\Rightarrow$ Heegaard diagram det. M_2

* attaching map of a 3-handle: $\varphi : \underset{\underset{S^2}{\parallel}}{\partial D^3 \times \{0\}} \hookrightarrow \partial M_2$

M closed $\Rightarrow \partial M_2 = S^2$

$$\begin{tikzcd}
M_2 \arrow[ddd, bend right, "\mathrm{id}"', "\cong"] & \cup_{\varphi_1} & h_3 \arrow[ddd, bend left, "\cong"] \\
\varphi_1(p) & & p \in \partial h_3 \arrow[ll, mapsto] \\
\varphi_2(p) & & p \in \partial h_3 \arrow[ll, mapsto] \\
M_2 & \cup_{\varphi_2} & h_3
\end{tikzcd}$$

$\varphi_2^{-1} \circ \varphi_1$ extends to a homeo $h_3 \to h_3$ by L.4

▨

3.3. LENS SPACES:

Def: The HEEGAARD GENUS of M^3 is

$$g(M^3) := \min \left\{ g(\Sigma) \;\middle|\; \begin{array}{l} \Sigma \text{ is a Heegaard surface} \\ \text{in Heegaard splitting} \\ \text{of } M = H_1 \cup_\Sigma H_2 \end{array} \right\}$$

Lemma 7: $g(M^3) = 0 \Leftrightarrow M = S^3$

Proof:

$$\begin{tikzcd}
M = & D^3 \arrow[ddd, bend right, "\cong", "\substack{\text{(L.4) extension} \\ \text{of } \varphi^{-1} \text{ to } D^3}"'] & \cup_\varphi & D^3 \arrow[ddd, bend left, "\cong"', "\mathrm{id}"] \\
& \varphi(p) & & p \in \partial D^3 \arrow[ll, mapsto] \\
& p & & p \in \partial D^3 \arrow[ll, mapsto] \\
S^3 = & D^3 & \cup_{\mathrm{id}} & D^3
\end{tikzcd}$$

▨

$\Rightarrow \quad g(S^1 \times S^2) = 1$

Def: Let p, q be coprime integers. We define the **LENS SPACE**

$$L(p,q) = S^3 \Big/ {\scriptstyle (z_1,z_2) \sim \left(e^{2\pi i/p} z_1,\, e^{2\pi i q/p} z_2\right)}, \quad S^3 \subset \mathbb{C}^2, \text{ for } p \neq 0$$

$$\& \quad L(0,q) := L(0,1) := S^1 \times S^2$$

$\Rightarrow$ $L(p,q)$ = Quotient of S^3 under free group action of $\mathbb{Z}_p$

$\Rightarrow$ $L(p,q)$ is a 3-mfd

Ex: * $L(1,q) = S^3/\langle 1 \rangle = S^3$

* $L(2,1) = S^3 \Big/ {\scriptstyle (z_1,z_2) \sim (-z_1,-z_2)} \cong \mathbb{R}P^3$

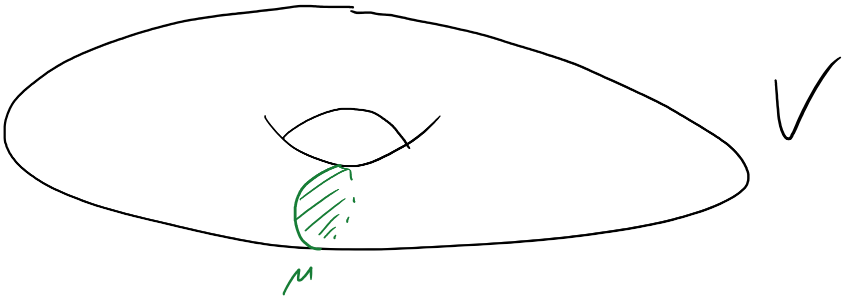
Thm 8: $g(M^3) = 1 \Leftrightarrow M$ is a lens space $(\neq S^3)$

OPEN QUESTION: which 3-mfds have $g = 2$?

Def: Let V be an oriented solid torus $(\cong S^1 \times D^2)$

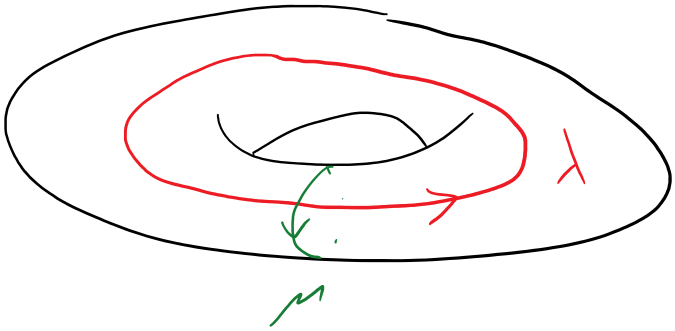
$\exists$ two distinguished isotopy classes of simple closed curves on ∂V:

* The **MERIDIAN** μ: the unique non-trivial simple closed curve on ∂V, that is trivial in V



* The <u>LONGITUDES</u> λ : non-trivial simple closed curve in ∂V intersecting μ transversally in a single point.

* μ, λ oriented, s.t. (μ,λ) represents the pos or of ∂V



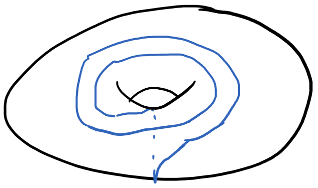
<u>Remark</u>: * λ is <u>NOT</u> unique

* For a fixed identification $V \cong S^1 \times D^2$ there is a preferred choice $\lambda := S^1 \times \{pt\}$ $(\mu := \{pt\} \times \partial D^2)$

<u>Lemma 9</u>:

(1) Every simple closed curve c on $\partial V = T^2$ is isotopic to exactly one curve of the form:

$$p\mu + q\lambda \quad \text{for } p, q \text{ coprime}$$



$2\lambda - \mu$

(2)
$$\mathrm{Homeo}(T^2)/\text{isotopy} \xrightarrow[\text{iso}]{\cong} GL_2(\mathbb{Z})$$

$$\left(h : T^2 \overset{C^0}{\xrightarrow{\cong}} T^2\right) \longmapsto \left(h_* : \underset{\mathbb{Z}^2\langle\mu,\lambda\rangle}{\underset{\shortparallel}{\pi_1(T^2)}} \longrightarrow \underset{\mathbb{Z}^2\langle\mu,\lambda\rangle}{\underset{\shortparallel}{\pi_1(T^2)}}\right)$$

<u>Proof</u>: see <u>ROLFSEN</u> chapter 2

c.f. chapter 4 of lecture ▨

Proof of T.8:

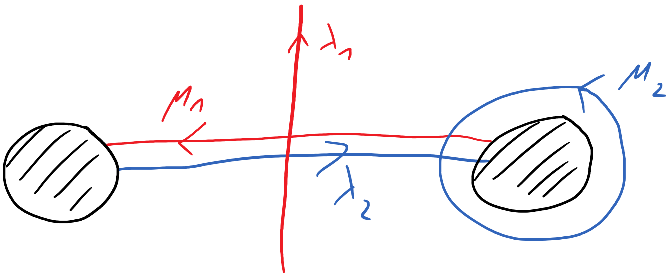
"$\Leftarrow$" * Let $S^3 \subset \mathbb{C}$

$$S^3 = \underbrace{\{ |z_1| \le 1/2 \}}_{=: V_1 \cong S^1 \times D^2} \cup \underbrace{\{ |z_1| \ge 1/2 \}}_{=: V_2 \cong S^1 \times D^2}$$

is a Heegaard splitting of S^3 of genus 1

$$\left[(z_1, z_2) = (r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2}) = (r_1 e^{i\varphi_1}, \sqrt{1-r_1^2}\, e^{i\varphi_2}) \overset{\cong}{\longmapsto} (r e^{i\varphi_1}, e^{i\varphi_2}) \right]$$

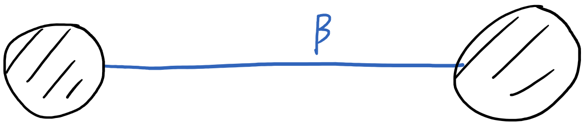
(from inside)



$\mu_1 \longmapsto -\lambda_2$

$\lambda_1 \longmapsto -\mu_2$

$S^3 =$



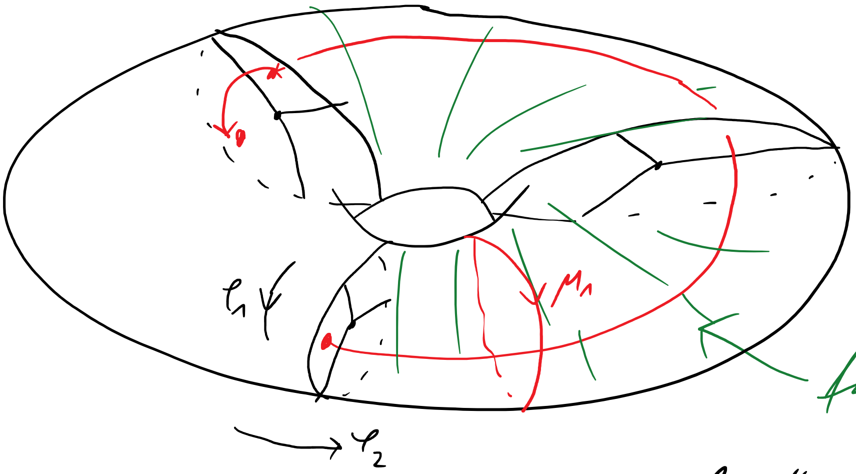
---

* Consider the $\mathbb{Z}_p$-action on S^3 s.t. $L(p,q) = S^3/\mathbb{Z}_p$

i.e. $(z_1, z_2) \sim (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$

W.L.O.G. $q < p$ (otherwise replace q by $q - np$)

$V_1 =$



$p = 3$

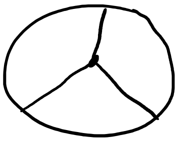
$q = 2$

1 step in D^2-factor

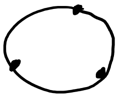
q steps in S^1-factor

fundamental region

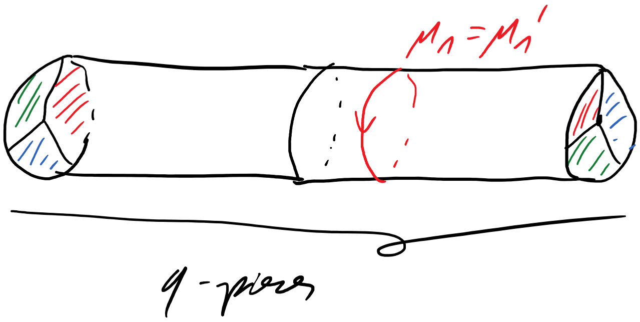
- divide D^2-factor into p „pizza slices"



- " S^1- " p pieces

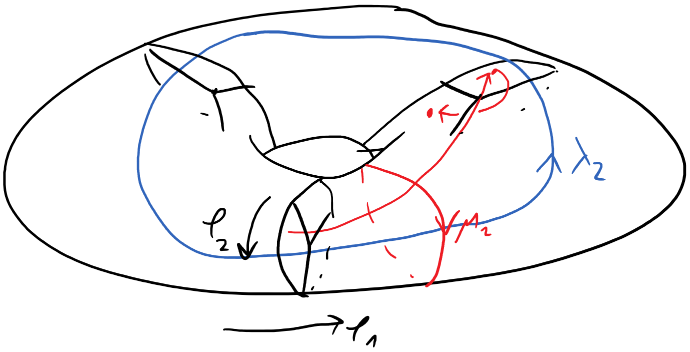


Consider a fundamental region:



$\Rightarrow V_1' := V_1/\mathbb{Z}_p \cong S^1 \times D^2$

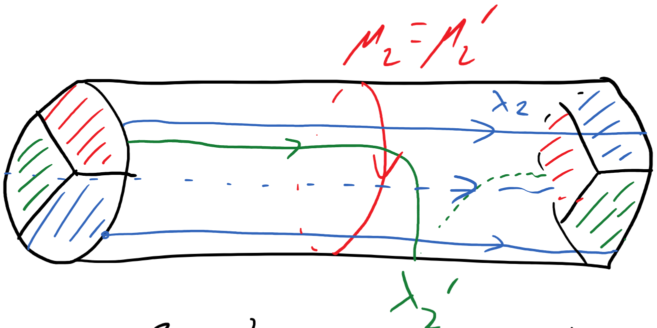
$V_2 =$



1-step in S^1-direction

q-steps in D^2-direction

fundamental region:



$\Rightarrow V_2' := V_2/\mathbb{Z}_p \cong S^1 \times D^2$

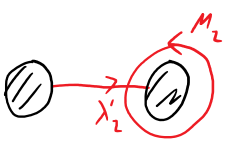
$\lambda_2 = p\lambda_2' - q\mu_2'$

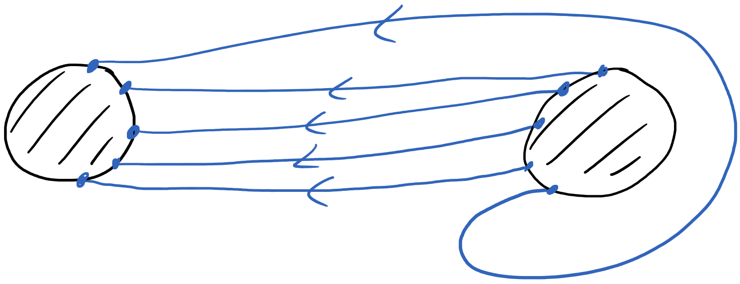
$$\boxed{\mu_1' = \mu_1 \longmapsto -\lambda_2 = -p\lambda_2' + q\mu_2'}$$

correct

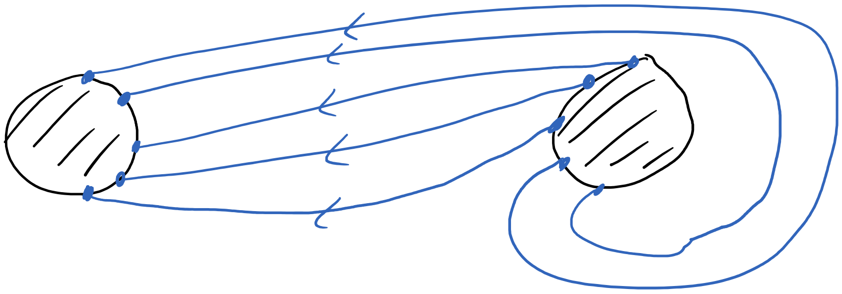
$\Rightarrow$ $L(p,q)$ has Heegaard diagram $(\partial V_2', -p\lambda_2' + q\mu_2')$

Ex: $*$ $L(5,1)$





$*$ $L(5,2)$



„$\Rightarrow$" Let M be a mfd with $g(M) = 1$

$\Rightarrow$ $M = V_1 \cup_\varphi V_2$

where $\varphi : \partial V_1 \xrightarrow{\cong_{C^0}} \partial V_2$ is an or. reversing homeom.

L.9.(2) $\Rightarrow$ φ is isotopic to

$$\mu_1 \longmapsto q\mu_2 - p\lambda_2$$
$$\lambda_1 \longmapsto s\mu_2 + r\lambda_2$$

where $\det \begin{pmatrix} q & -p \\ s & r \end{pmatrix} = -1$

$\Rightarrow$ $M \cong L(p,q)$

▨

3.5. HANDLE SLIDES & STABILIZATIONS

We have shown:

$$\{\text{Heegaard diagrams}\} \overset{T.6.}{\twoheadrightarrow} \{\text{Heegaard splittings}\}$$

$$\{\text{Heegaard splittings}\} \overset{C.3}{\twoheadrightarrow} \{\text{3-mfds}\}$$

$\boxed{Q}$ what are the „tricks"?

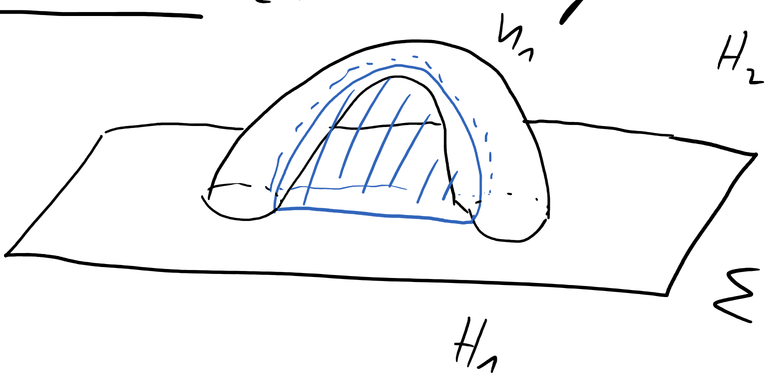
HANDLE CANCELLATION:

$$M^3 = \underbrace{(h_0 \cup h_1^1 \cup \dots \cup h_1^g)}_{H_1} \underset{\overset{\shortparallel}{\partial H_1 = \partial H_2}}{\cup_{\Sigma}} \underbrace{(h_2^1 \cup \dots \cup h_2^g \cup h_3)}_{H_2}$$

$$= \underbrace{(h_0 \cup h_1^1 \cup \dots \cup h_1^{g+1})}_{= H_1'} \cup_{\Sigma'} \underbrace{(h_2^1 \cup \dots \cup h_2^{g+1} \cup h_3)}_{= H_2'}$$

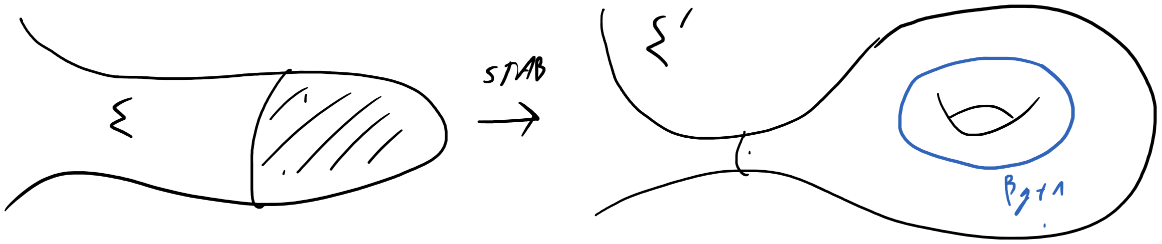
s.t. h_1^{g+1} & h_2^{g+1} cancel each other

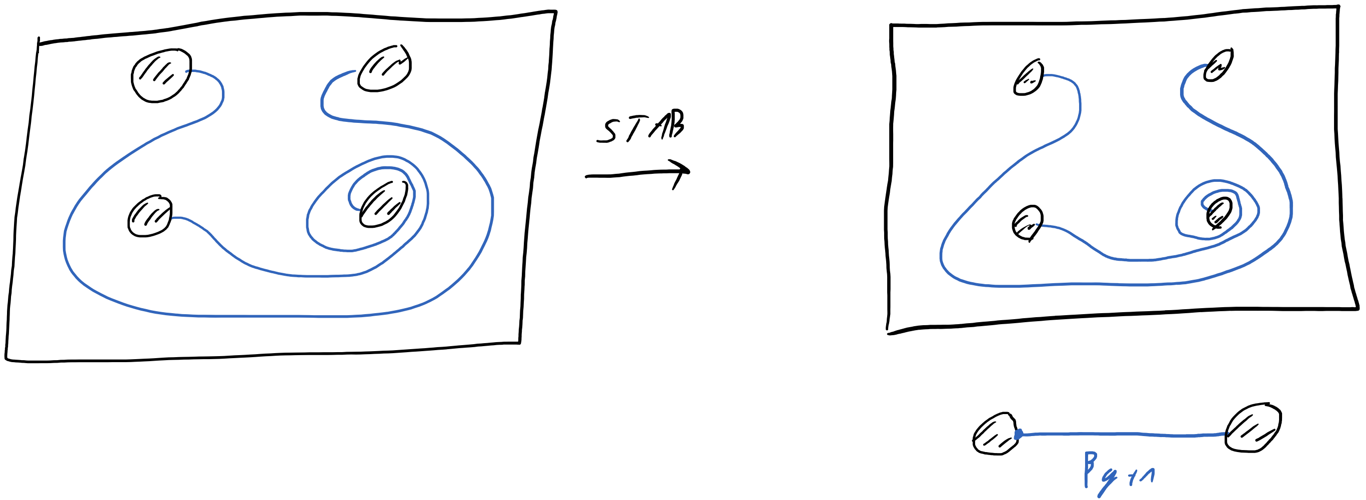
STABILIZATION :$\Leftrightarrow$ introducing a cancelling 1/2-handle pair



$$g(\Sigma') = g(\Sigma) + 1$$

In a Heegaard diagram:





Thm 10 (WALDHAUSEN)

$\forall\, g \geq k \geq 0$ $\exists!$ (up to isotopy) genus-g-Heegaard splitting of $\#_k S^1\times S^2$ $\quad(\#_0 S^1\times S^2 = S^3)$ ▨

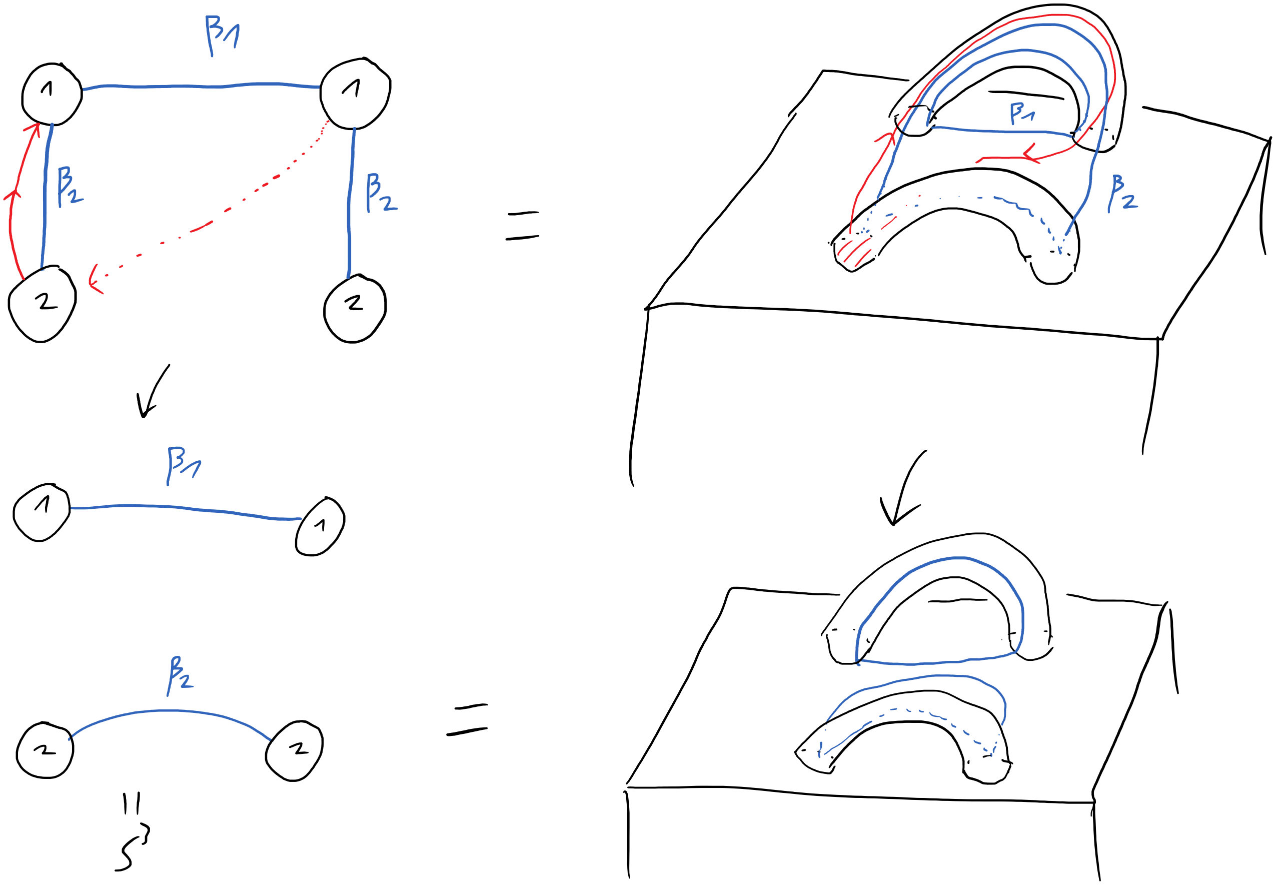
Remark: * similar results for $L(p,q)$, T^3, T^2-bundles over S^1, ...

* T10 is for general 3-mfd WRONG, for example on $L(p,q)\# L(p',q')$

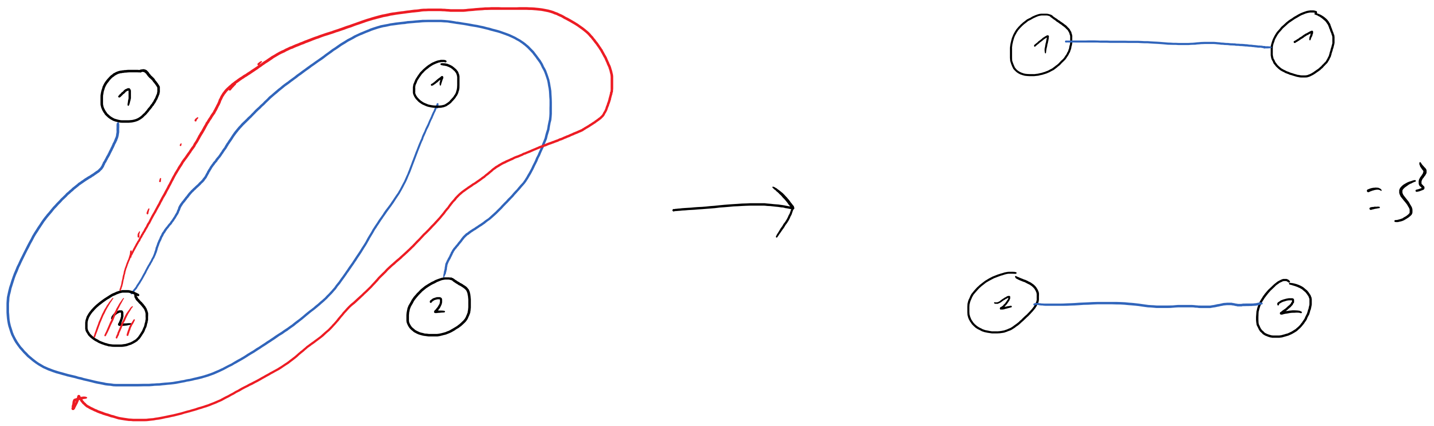
For examples of non-isotopic Heegaard splittings of the same genus, see https://www2.mathematik.hu-berlin.de/~kegemarc/Kirby/Hausarbeit%20F.Frede.pdf

HANDLE SLIDES:

1-HANDLE SLIDES

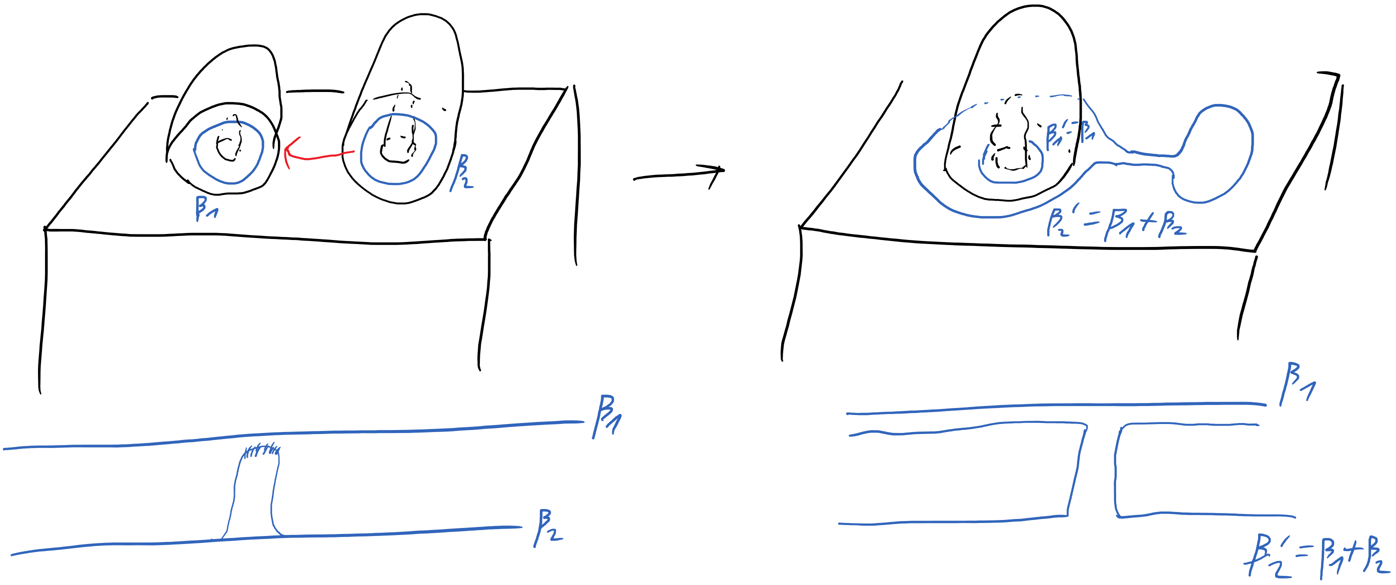


Ex: ISOTOPY OF 1-HANDLES:

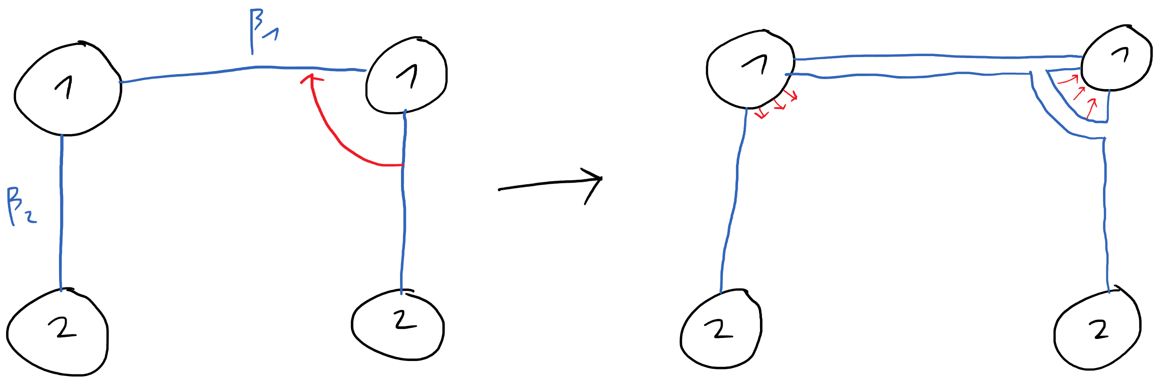


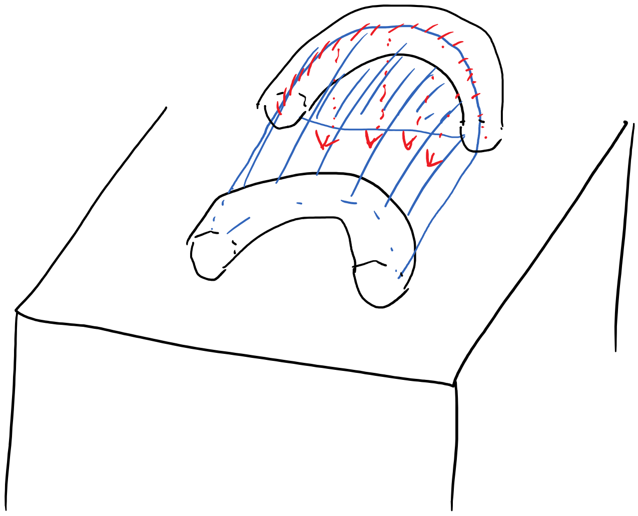
→ NOT a handle slide

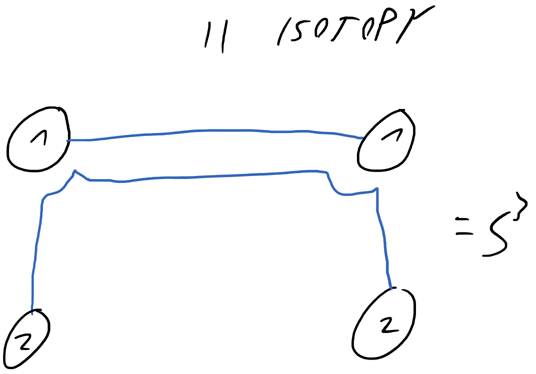
2-HANDLE SLIDES:



Ex:







Thm 11:

(1) $$\{\text{Heegaard diagrams}\}\big/\{\text{1- \& 2-handle slides}\} \xrightarrow{\overset{1:1}{\cong}} \{\text{Heeg. splittings}\}$$

(JOHANNSON)

(2) $$\{\text{Heegaard splittings}\}\big/\{\text{stab.}\} \xrightarrow{\overset{1:1}{\cong}} \{\text{3-mfds}\}$$

(REIDEMEISTER-SINGER 1935)

Proof: follows from T. 2.6. (CERF) ▨

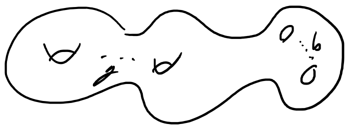
4. THE MAPPING CLASS GROUP OF SURFACES

Motivation: let $\varphi, \varphi' : \partial H_1 \xrightarrow{\overset{C^0}{\cong}} \partial H_2$ be isotopic homeo. $(\varphi \sim \varphi')$

$$\Rightarrow H_1 \cup_\varphi H_2 \overset{C^0}{\cong} H_1 \cup_{\varphi'} H_2$$

Let F be a connected, oriented, compact surface

$$\text{(i.e. } F \overset{C^0}{\cong} \Sigma_{g,b} := \Sigma_g \setminus \bigsqcup_{i=1}^{b} \mathring{D}^2 =$$



Def: * $\mathrm{Homeo}^+(F) := \{ \varphi : F \to F$ orient. pres. homeo s.t. $\varphi|_{\partial F} = id \}$

* The MAPPING CLASS GROUP of F is

$$MCG(F) := \mathrm{Homeo}^+(F) / \text{isotopy}$$

Lemma 1: $MCG(F)$ is a group

Proof: $N(F) := \{ \varphi \in \mathrm{Homeo}^+(F)$ s.t. $\varphi \sim id_F \}$

$N(F) \lhd \mathrm{Homeo}^+(F)$

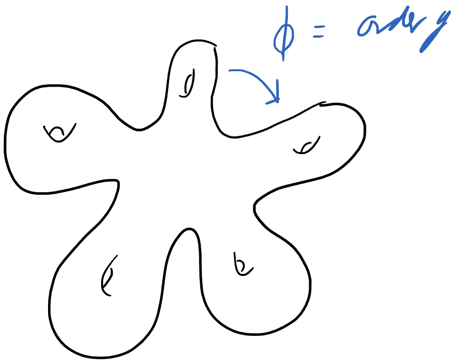
⌜ Let $\varphi \in \mathrm{Homeo}^+(F)$ & $n \in N(F)$ ⌝

⌞ $\Rightarrow \varphi n \varphi^{-1} \sim \varphi \varphi^{-1} = id_F \quad \Rightarrow \varphi n \varphi^{-1} \in N(F)$ ⌟

$$MCG(F) = \mathrm{Homeo}^+(F) / \text{isotopy} = \mathrm{Homeo}^+(F) / N(F)$$

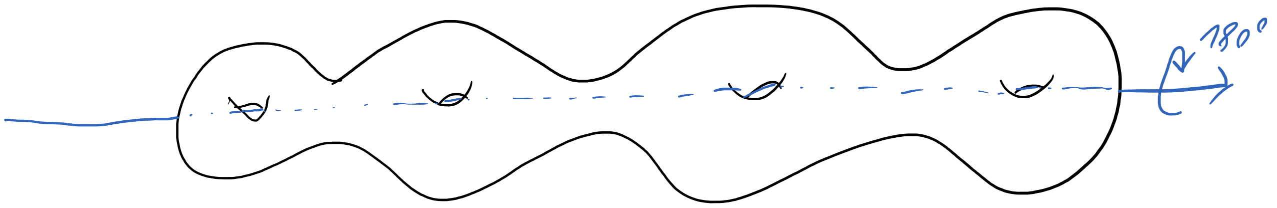
▨

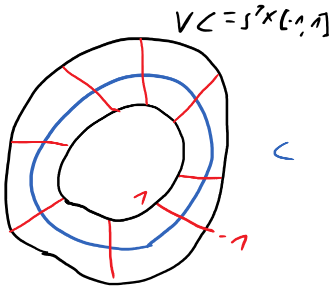
Ex: (1)



(2) HYPERELLIPTIC INVOLUTION

ι = order 2





Def: Let c be a simple closed curve on F

$$\Rightarrow \quad Vc \overset{C^0 \text{ or. pres.}}{\cong} S^1\times[-1,1] \quad \text{s.t.}$$

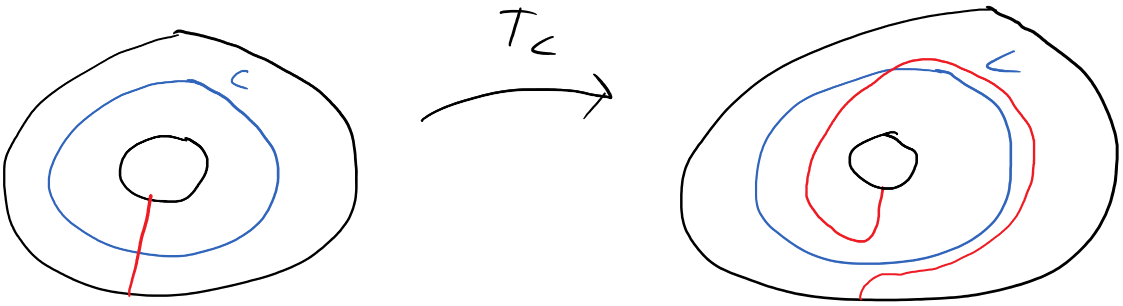
$$c \mathrel{\hat{=}} S^1\times\{0\} \quad , \quad S^1 = \mathbb{R}/2\pi\mathbb{Z}$$

* A (RIGHT-HANDED) DEHN TWIST $T_c \in \mathrm{Homeo}^+(F)$ along c

is def. by

$$T_c\big|_{F\setminus \mathring{V}c} = \mathrm{Id}_{F\setminus \mathring{V}c}$$

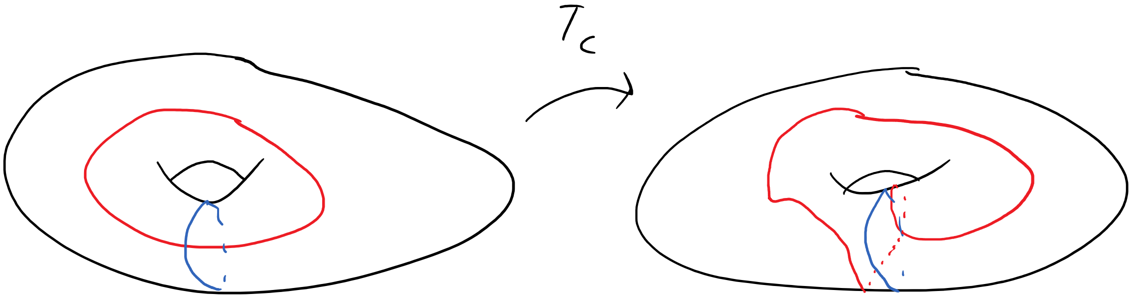
$$T_c\big|_{Vc\cong S^1\times[-1,1]} \quad (\theta,t) \longmapsto (\theta+\pi(t+1),\, t)$$



* T_c^{-1} is a LEFT-HANDED Dehn twist

Rem: This dep on the or. of F, but NOT on or. of c

Ex:

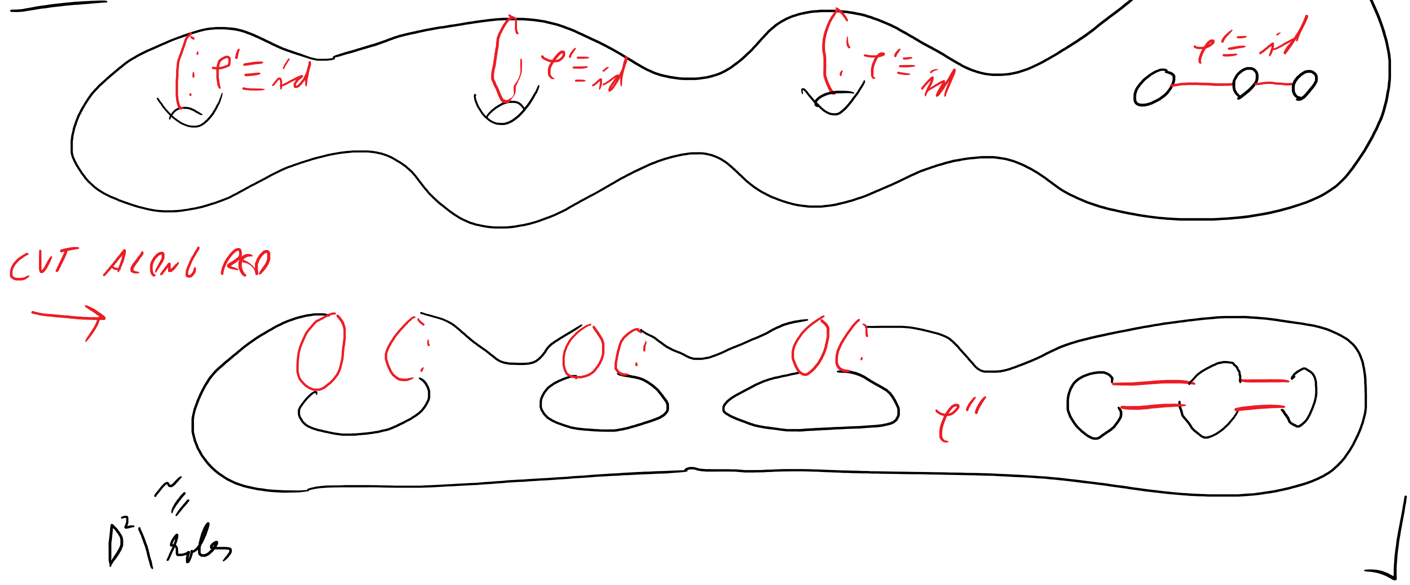


4.1. GENERATORS:

Thm 2 (DEHN, LICKORISH)

MCG(F) is generated by Dehn twist

IDEA: $\varphi \in \mathrm{Homeo}^+(F)$

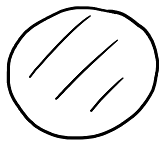


Def: $\varphi \in \mathrm{Homeo}^+(F)$ is called ADMISSIBLE :$\Leftrightarrow$

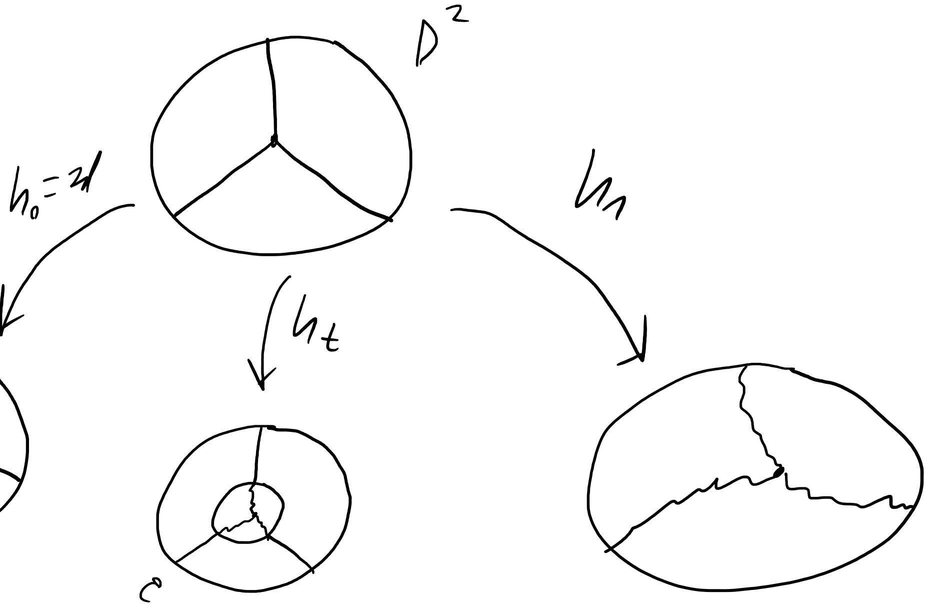
$\varphi \sim$ comp of Dehn twist

Lemma 3 (ALEXANDER TRICK)

$MCG(D^2) = 1$ import.: $T_{\partial D^2} \sim id$



Proof:



let $h: D^2 \xrightarrow{\overset{c}{\cong}} \underset{\overset{\cap}{\mathbb{R}^2 = \mathbb{C}}}{D^2}$ with $h|_{\partial D^2} = id$

we extend h to $h: \mathbb{R}^2 \xrightarrow{\cong} \mathbb{R}^2$ by

$z \longmapsto z$ for $|z| \geq 1$

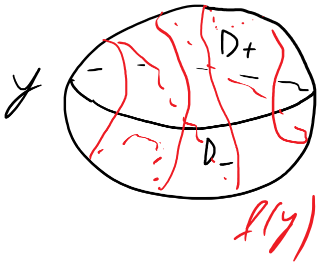
Define $h_t(z) = \begin{cases} t\, h(\frac{z}{t}) & ; t \neq 0 \\ z & ; t = 0 \end{cases}$

h_t is an isotopy from $h_0 = id$ to $h_1 = h$ s.t.

$h_t(z) = z$ for $|z| = 1$ ∎

Lemma 3' $MCG(S^2) = 0$

Proof: $S^2 = D_+ \cup D_-$



$\partial D_+ = y = \partial D_-$

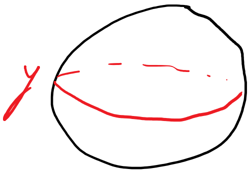
Let $f : S^2 \longrightarrow S^2$ be an or. pr. homeom.

Consider $f(y)$:

[BAER: $c_1, c_2 \subset F^2$ s.c.c. : c_1 is homotopic to c_2 $\Rightarrow$ c_1 is isotopic to c_2]

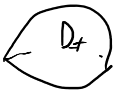
$\Rightarrow$ $f(y)$ is isotopic to y

$\Rightarrow$ After isotopy: $f(y) = y$



$f|_y = id$

↓ CUT ALONG y



$f_+ \in MCG(D_+) = 0$

$f_- \in MCG(D_-) = 0$

L.3
$\Rightarrow$ $f \sim id$

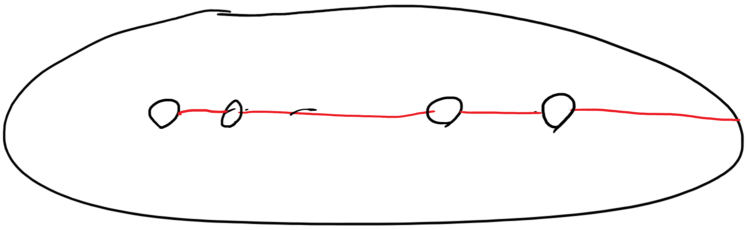
▨

Remark:

- L.3' does <u>NOT</u> follow from the proof of T.2
- $D^3 \cup_f D^3 = S^3$ indep of L.3'
- SMALE: $\mathrm{Diff}^+(S^2) \simeq SO(3)$
- HATCHER: $\mathrm{Diff}^+(S^3) \simeq SO(4)$

Lemma 4 $\Sigma_{0,n+1} =: D^2_n$ = 2-disk with n holes

$MCG(D^2_n)$ is gen. by Dehn twists

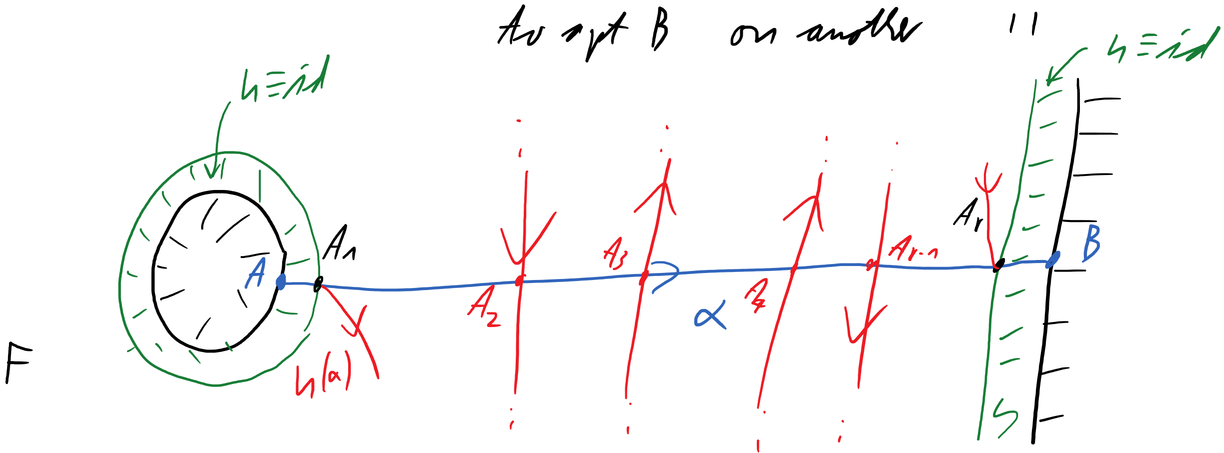


Proof: Induction on n

n=0: in Lemma 3

n→n+1: Let $h : D_{n+1} \xrightarrow[\cong]{C^0} D_{n+1}$, $h|_{\partial D_{n+1}} = id$

$\alpha :=$ oriented path from A on a component of ∂D_{n+1} to a pt B on another ''

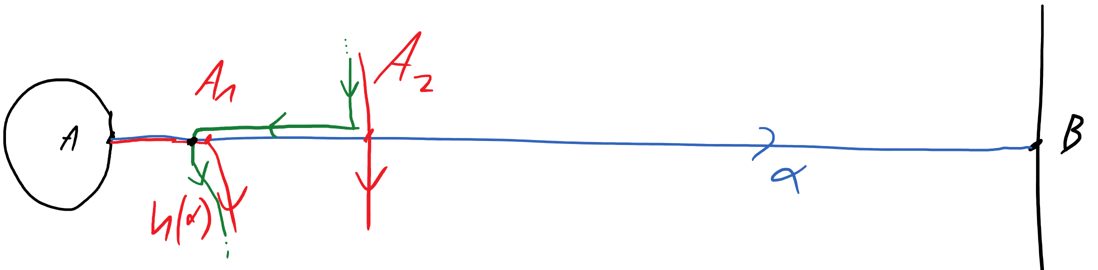


* After isotopy of h we can assume:

$h \equiv id$ on a nbhd of ∂D_{n+1}

$\{A_2, .., A_{r-1}\} := \alpha \pitchfork h(\alpha)$

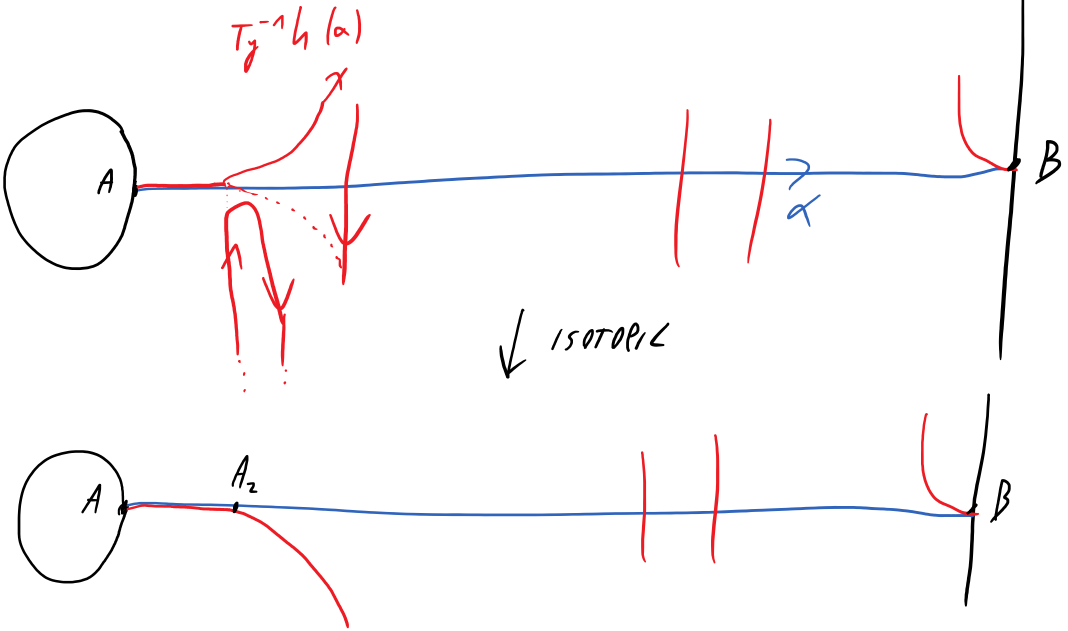
1. CASE: $h(\alpha)$ points at A_1 & A_2 in the same direction



We define γ as in the picture

Remark: $\gamma \pitchfork h(\alpha) = \{pt\}$, but $\gamma \pitchfork \alpha$ = i.g. more pts

* Consider $T_y^{\pm 1} h(\alpha)$:



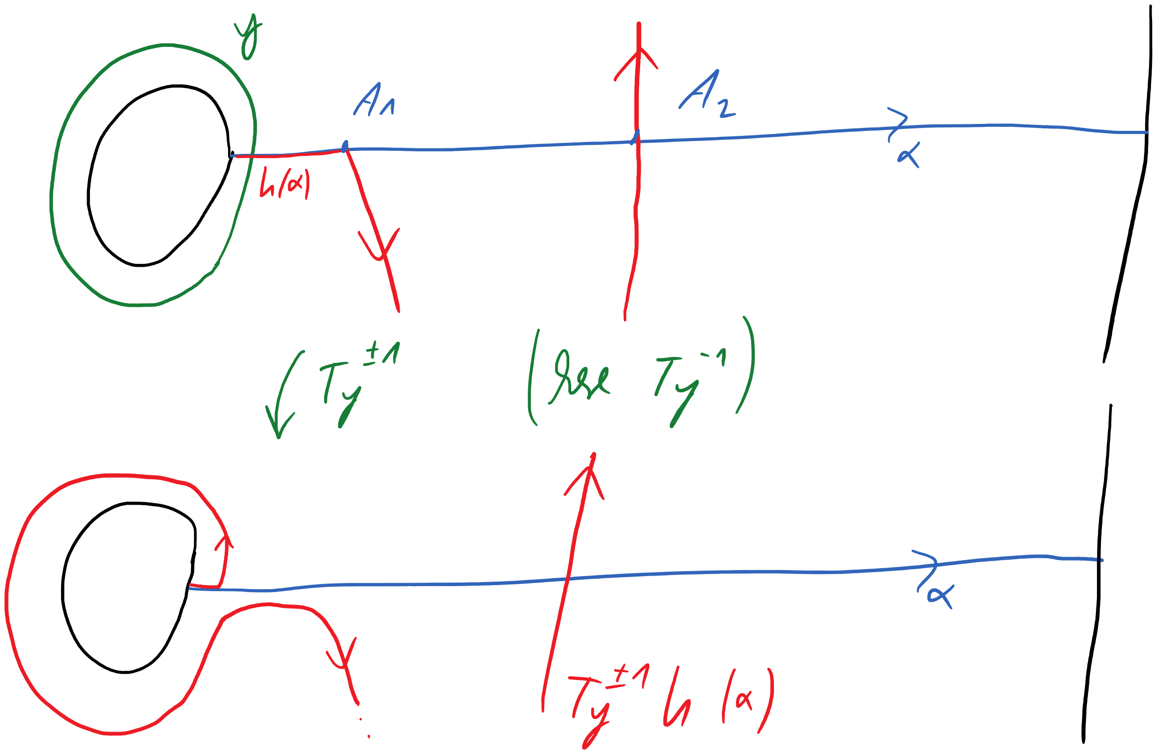
* If $h(\alpha)$ moves to the right of α at A_1 use T_y^{-1}

* " left " T_y^{+1}

$\Rightarrow \exists$ admissible homeo f_1 s.t. $f_1 h \equiv id$ on AA_2 & $A_\gamma B$

$$|f_1 h(\alpha) \pitchfork \alpha| < |h(\alpha) \pitchfork \alpha|$$

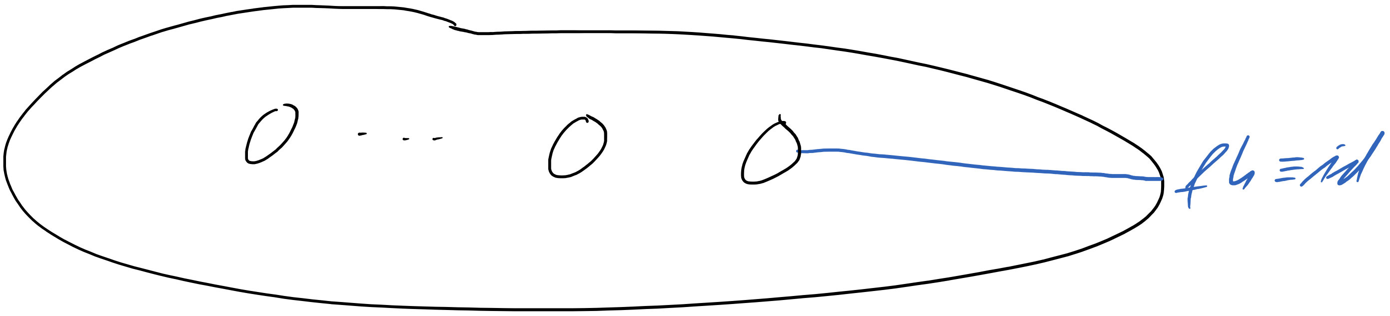
2. CASE: $h(\alpha)$ points at A_1 & A_2 in different directions



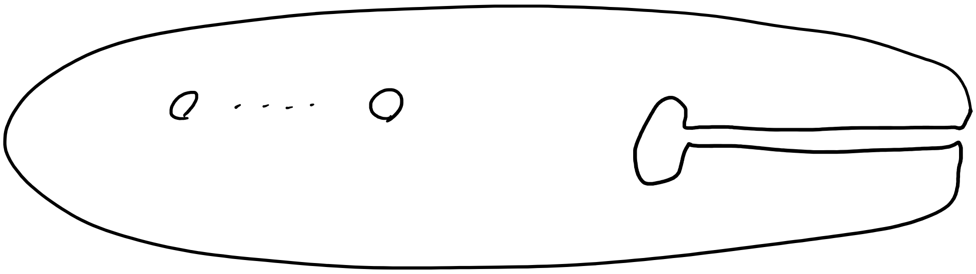
after adm. homeo we get case 1

* After fin many steps we get admissible $f : D_{n+1} \longrightarrow D_{n+1}$

s.t. $fh|_{\partial D_{n+1}} = id$ & $fh|_{\alpha} = id$



* Cut along α:

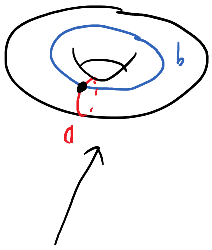


& get $D_n \longrightarrow D_n$ fixing ∂D_n ▨

Lemma 5:

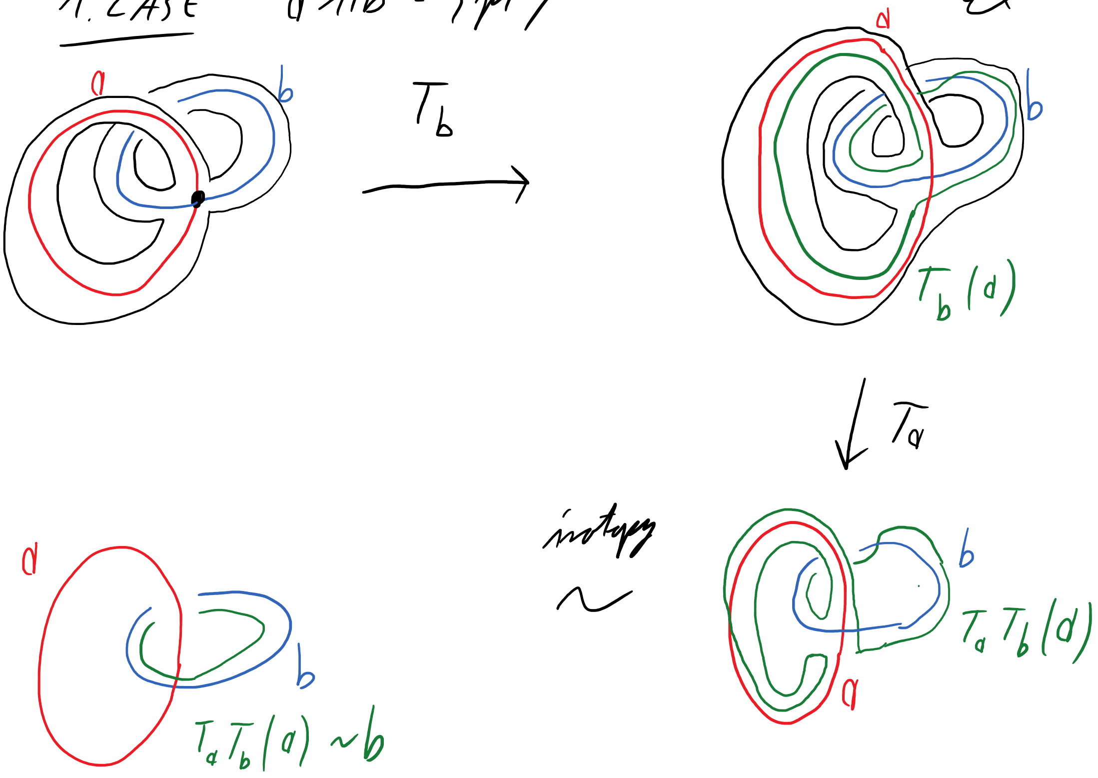
Let $a, b \subset F$ be simple closed curves, NON-SEPARATING (i.e. $F \setminus a$ & $F \setminus b$ are connected)

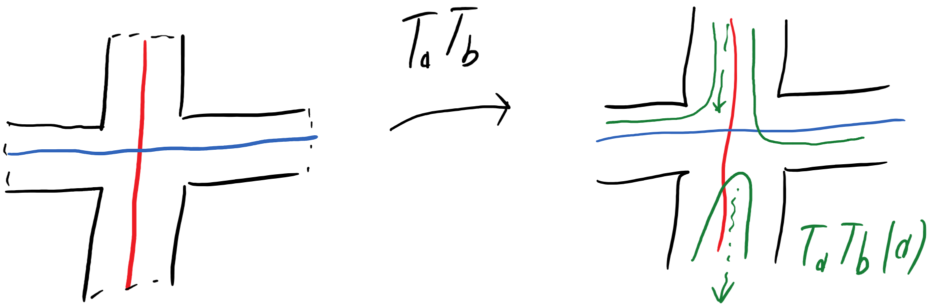
$\Rightarrow \exists\, \varphi \in \mathrm{Homeo}^+(F)$ admissible s.t. $\varphi(a) = b$



Ex

Proof: 1. CASE $a \pitchfork b = \{pt\}$





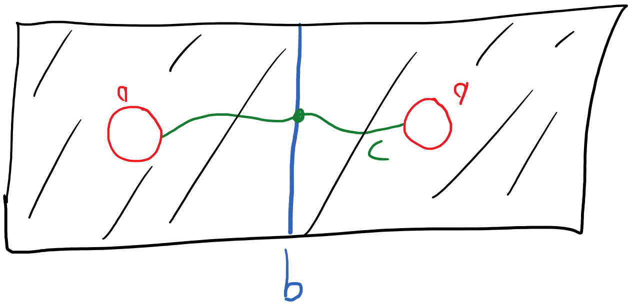
2. CASE $a \cap b = \emptyset$

$*\ \exists$ a simple closed non-sep curve c s.t.

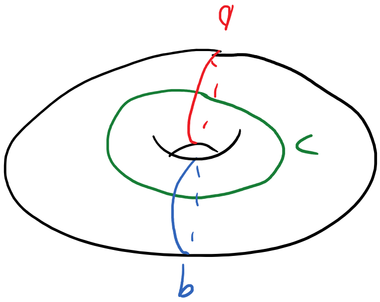
$a \pitchfork c = \{pt\}$ & $b \pitchfork c = \{pt'\}$

(a) If $F \setminus a \cup b$ is disconnected

Cut along a & choose c as:

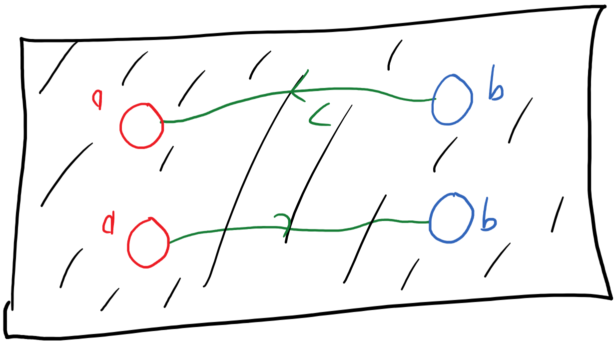


Ex:

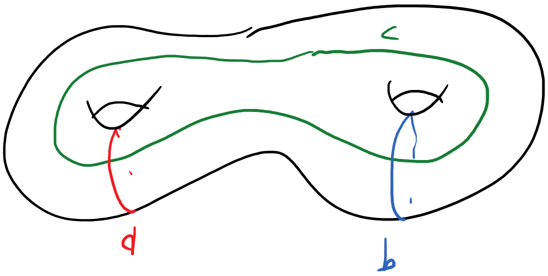


(b) If $F \setminus a \cup b$ is connected:

Cut along a & b & choose c as:



Ex:

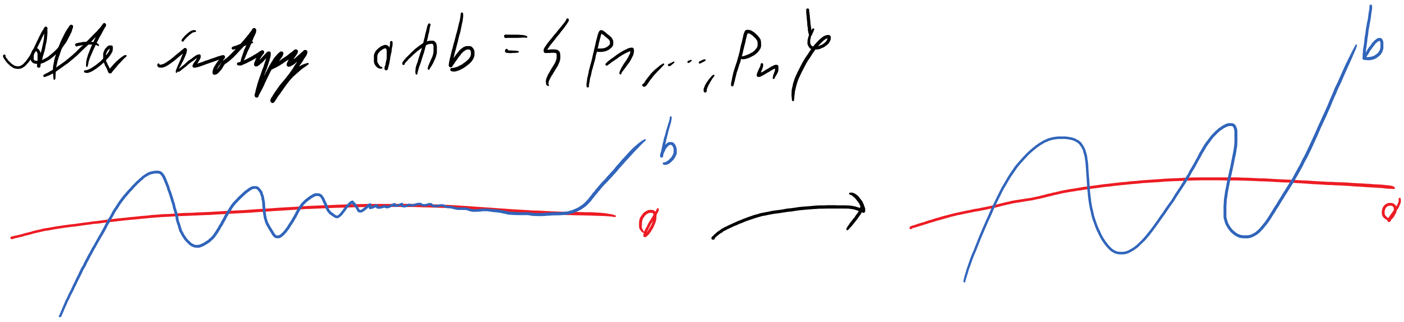


CASE 1

$\Rightarrow$ $\exists$ admissible homeo: $a \longmapsto c \longmapsto b$

3. CASE: a intersects b in more than one pt.

* After isotopy $a \pitchfork b = \{p_1, \dots, p_n\}$



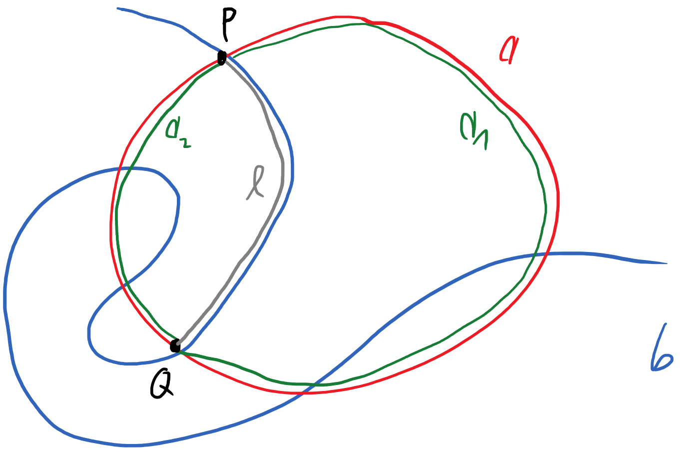
CLAIM: $\exists$ simple closed non-sep curve c s.t.

(1) $a \pitchfork c = \emptyset$ or $\{pt\}$

(2) $b \pitchfork c$ in less than n pts

1. or 2. CASE $\Rightarrow$ $\exists$ adm. homeom. $a \longmapsto c$ & finish via induction on n

PROOF OF THE CLAIM:



Let P, Q be neighboring intersection points on b

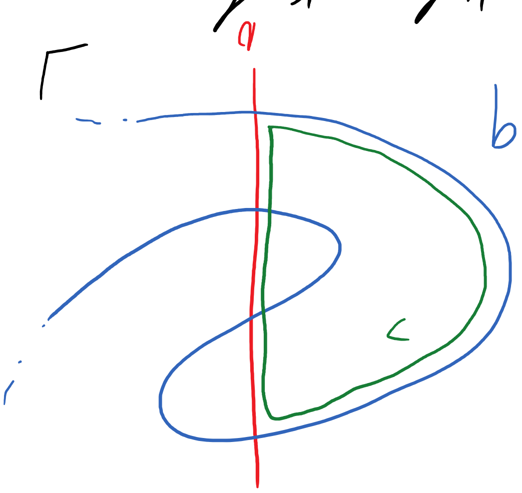
$c_1 := a_1 \cup \ell$

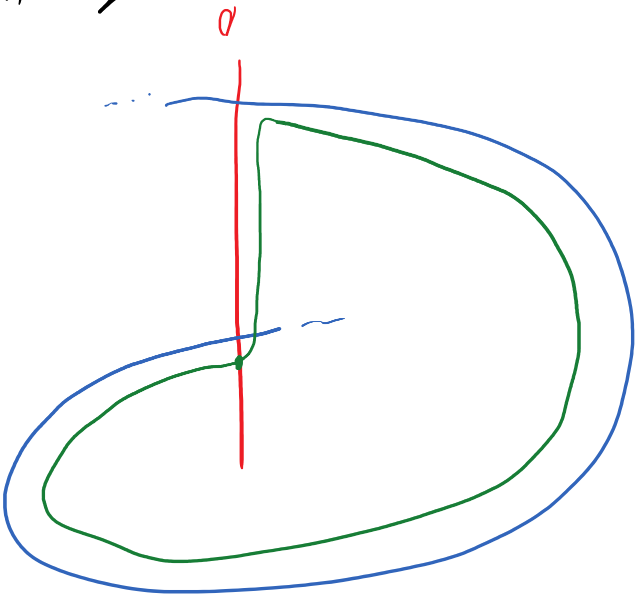
$c_2 := a_2 \cup \ell$

* a is non-separating $\Rightarrow$ c_1 or c_2 is non separating

Let c_1 be non-sep

* Pushing c_1 away from itself yields c

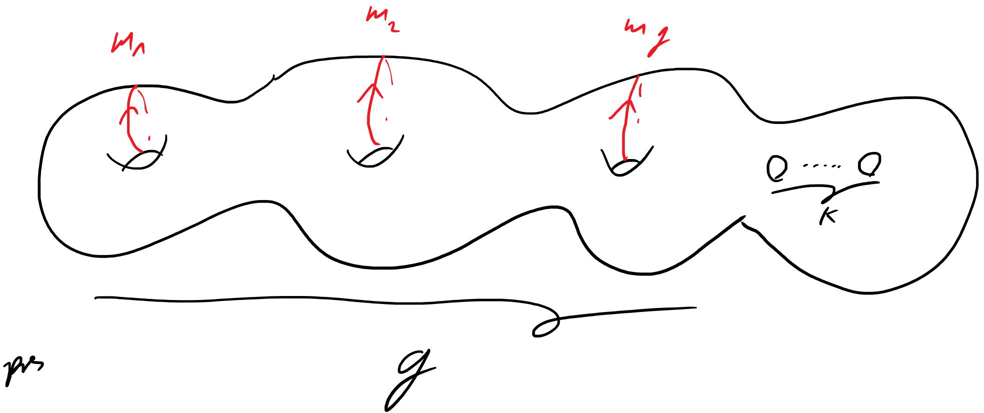




▨

Proof of Thm 2:

$F \cong \Sigma_{g,k} \cong$



Let $h : F \xrightarrow{\overset{C^0,\text{ or pres}}{\cong}} F$ with $h|_{\partial F} \equiv id_{\partial F}$

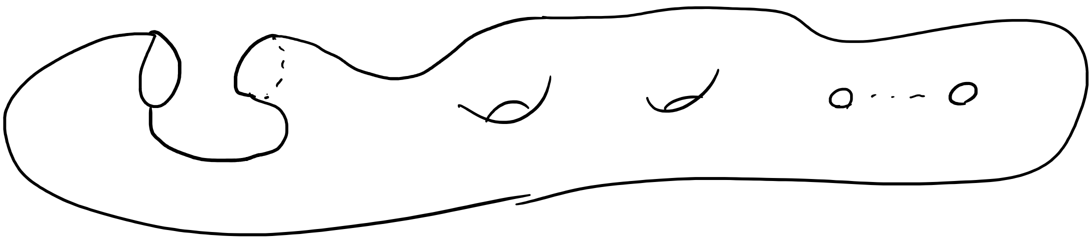
m_1 is non-sep. $\Rightarrow$ $h(m_1)$ is non-sep.

$\overset{L.5}{\Rightarrow}$ $\exists$ adm. homeo f_1 s.t. $f_1 h(m_1) = m_1$

CASE 1: orientations of m_1 & $f_1 h(m_1)$ agree

see SHEET 5
$\Rightarrow$ after isotopy: $f_1 h|_{m_1} = id_{m_1}$

* Cut $\Sigma_{g,k}$ along m_1:



& get $\Sigma_{g-1,k+2} \xrightarrow{\cong} \Sigma_{g-1,k+2}$ fixed ∂

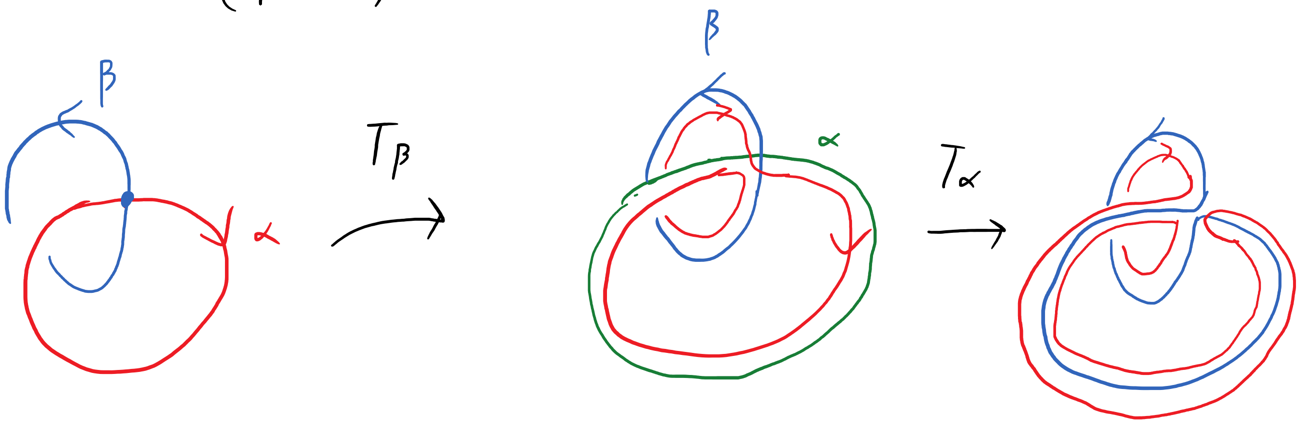
CASE 2 : orientation of m_1 & $f_1 h(m_1)$ are opposite :

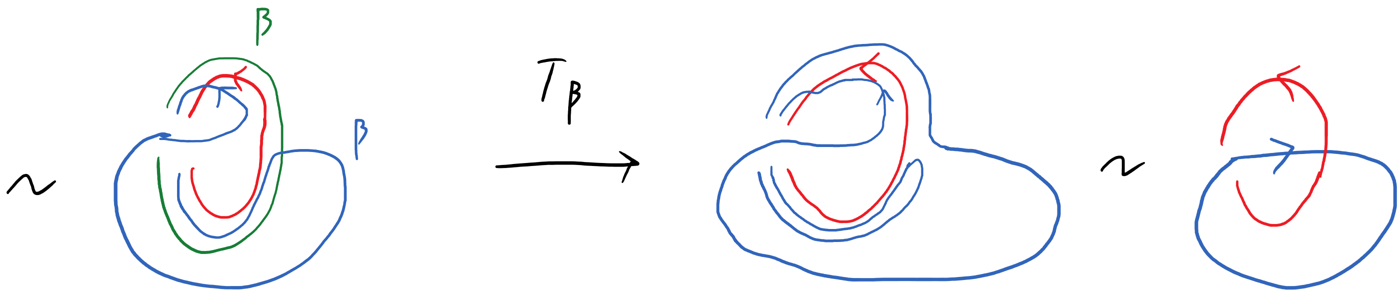
$\alpha := m_1$, $\beta :=$ simple closed curve s.t. $\alpha \cap \beta = \{pt\}$

CLAIM : $(T_\beta T_\alpha T_\beta)^2(\alpha) = -\alpha$

$(T_\beta T_\alpha T_\beta)^2(\beta) = -\beta$

⌜ ⌝





$T_\beta T_\alpha T_\beta :$ $\alpha \longmapsto -\beta$

$\beta \longmapsto \alpha$

⌞ ⌟

$\Rightarrow$ $\exists$ admissible homeo f_1' s.t.

$$f_1' h(m_1) = m_1 \quad (\text{or. preserving})$$

$\rightarrow$ continue in CASE 1

* After g steps: $fh : D^2_{k+2g-1} \xrightarrow{\cong} D^2_{k+2g-1}$

fixing ∂D^2_{k+2g-1} with f admissible

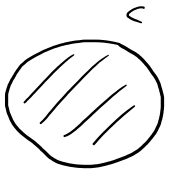
$\overset{L.4.}{\Rightarrow}$ $fh \sim$ comp. of Dehn-twists

$\Rightarrow$ $h \sim$ ''

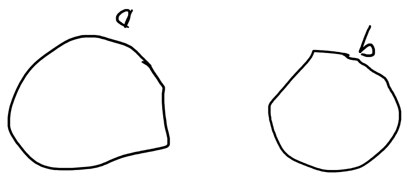
▨

DIGITAL ORAL EXAMS: TUES 28.7.
TUES 27.10

4.2. RELATIONS



Ex: * If $c = \partial D^2 \subset F$ $\overset{L.3}{\Rightarrow}$ $T_c = \mathrm{id}$

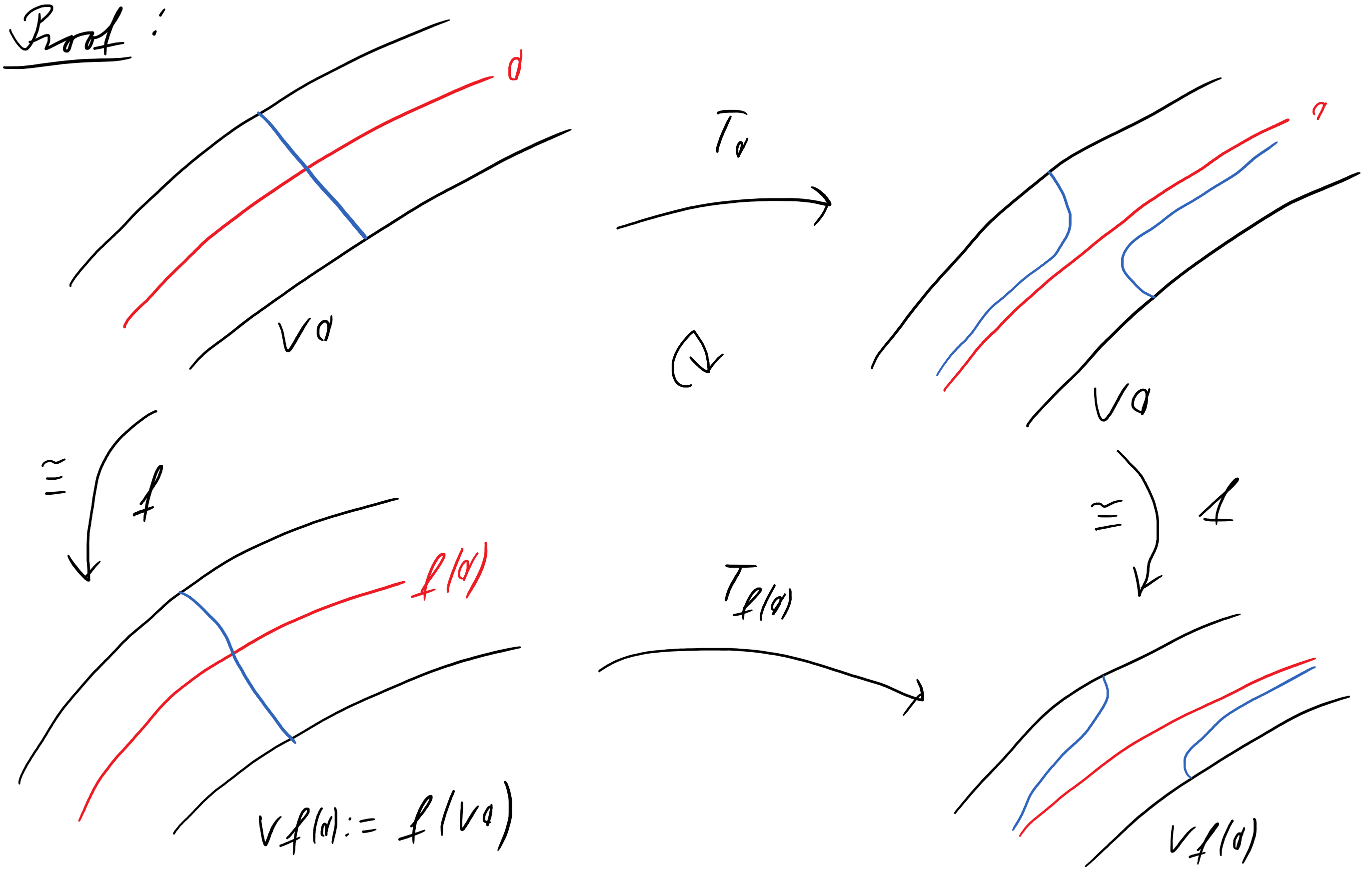


* If $a \cap b = \emptyset$ $\Rightarrow$ $T_a T_b = T_b T_a$

Lemma 6: Let $f \in \mathrm{Homeo}^+(F)$, $a \subset F$ be a s.c.c.

$\Rightarrow$ $\boxed{f T_a f^{-1} = T_{f(a)}}$

Proof:



∎

Lemma 7 (BRAID RELATION)

Let $a, b \subset F$ be s.c.c. s.t. $a \pitchfork b = \{pt\}$

$$\Rightarrow \quad T_a T_b T_a = T_b T_a T_b$$

Proof: $\circledast$ $T_a T_b (a) = b$ (see CASE 1 of L.5)

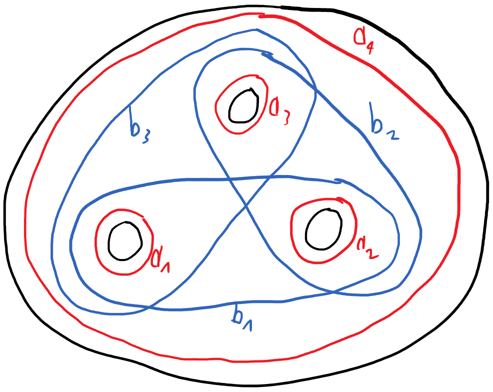
$$\Rightarrow \quad T_a T_b T_a = \underbrace{T_a T_b}_{f} T_a \underbrace{T_b^{-1} T_a^{-1}}_{f^{-1}} T_a T_b$$

$$\overset{L.6}{=} T_{T_a T_b(a)} \quad T_a T_b$$

$$\overset{\circledast}{=} T_b \quad T_a T_b$$

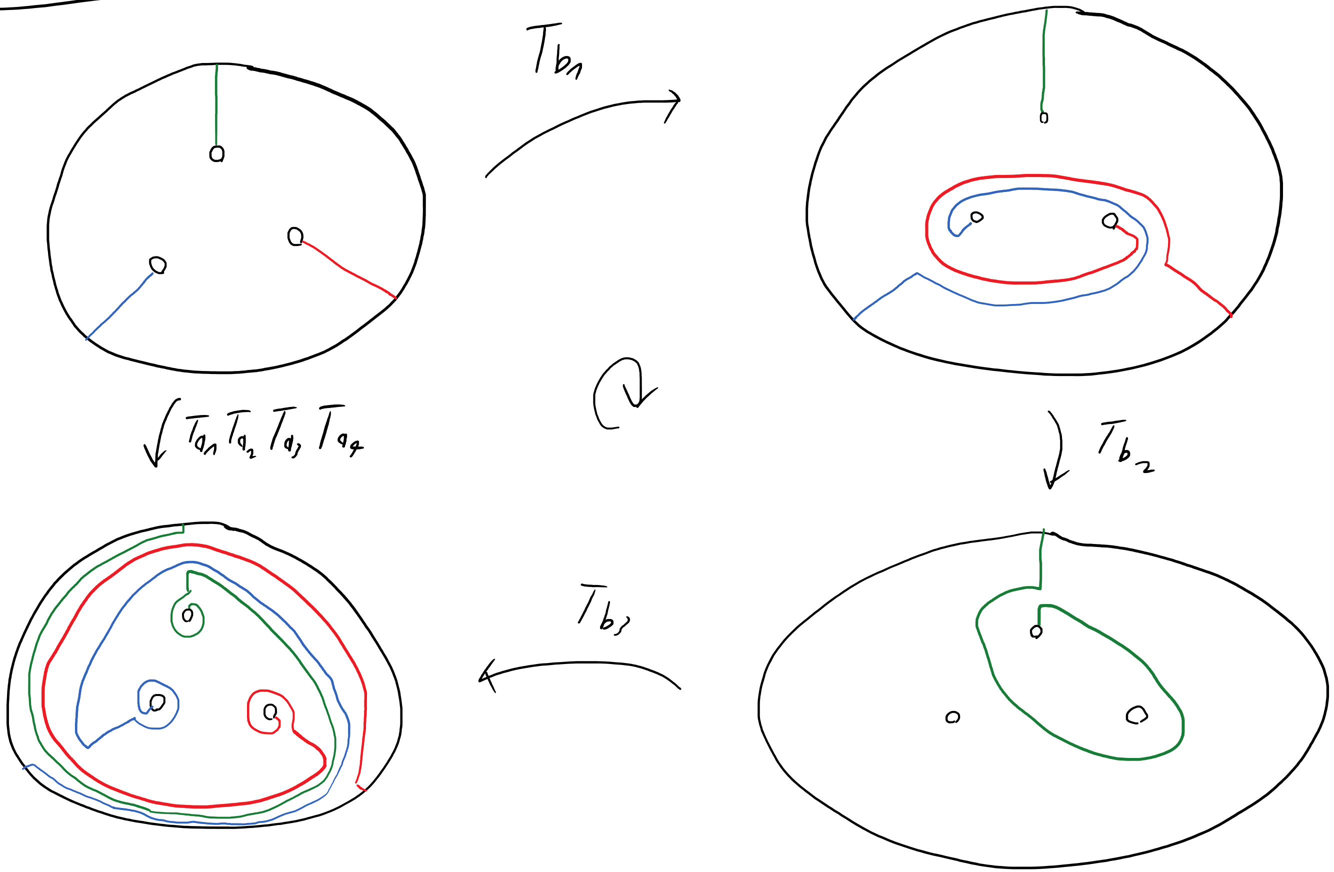
∎

Lemma 8 (LANTERN RELATION)



$$T_{a_1} T_{a_2} T_{a_3} T_{a_4} = T_{b_1} T_{b_2} T_{b_3}$$

Proof sketch:



∎

THM 9 (HATCHER-THURSTON, WAJNRYB, LUO, GERVAIS)

MCG(F) has a presentation with

GENERATORS: $\{ T_c \mid c \subset F \text{ n.s.c.c.} \}$

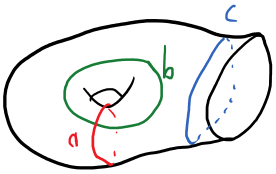
RELATIONS: (I) $T_c = \mathrm{id}$ if $c = \partial D^2 \subset F$

(II) $T_a T_b = T_b T_a$ if $a \cap b = \emptyset$

(III) BRAID RELATION

(IV) LANTERN RELATION

(V) CHAIN RELATION:



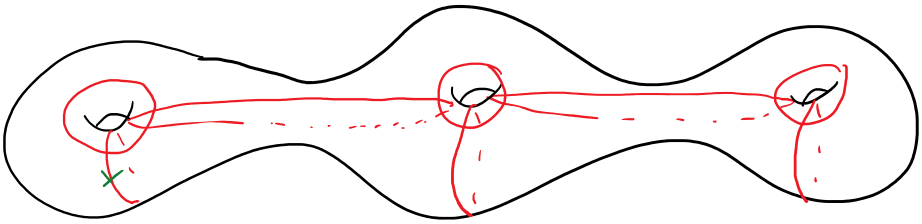
$$(T_a T_b T_a)^4 = T_c$$

c.f. CASE 2 of proof of T.2.

Proof: see FARB-MARGALIT. ▨

Thm 10 (LICKORISH)

MCG(F) is gen by Dehn-twists along $3g-1$ curves, $g = g(F)$.



4.3. BAER'S THM

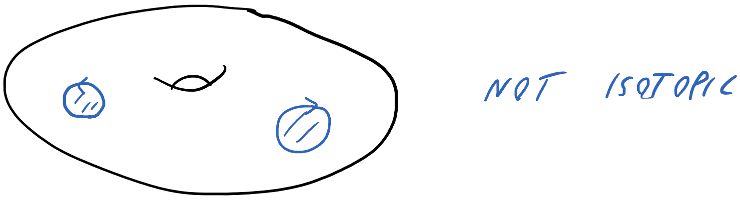
Thm 11 (BAER 1927)

Let $a, b \subset F$ be s.c.c. on a closed or. surface

a is homotopic to b $\Leftrightarrow$ a isotopic to b

Remark: * T.11 is also true for non-trivial oriented curves

Ex:

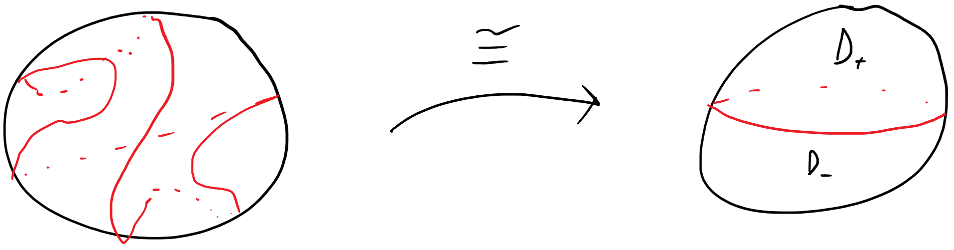


* $F = T^2 \Rightarrow$ L.3.9.

Proof of T.11 for $F = S^2$:

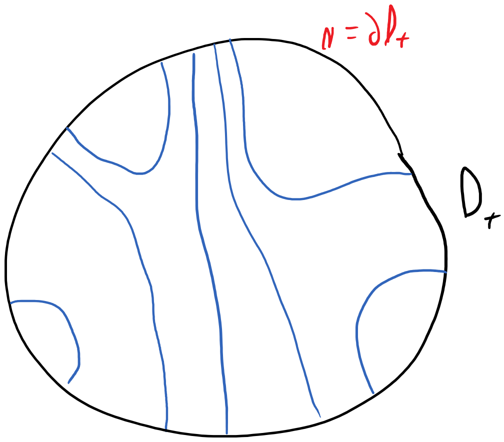
SCHOENFLIES: Let $a \subset S^2$ be a s.c.c.

$\Rightarrow \exists \; (S^2, a) \xrightarrow{\cong} (S^2, \text{equator})$



W.L.O.G. a = equator $\subset S^2$ & $a \pitchfork b$

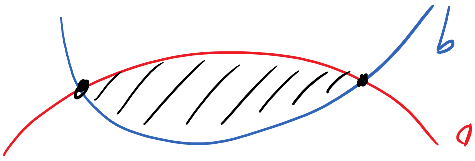
Consider: $b_+ := b \cap D_+$ a compact 1-mfd with ∂b_+ on $\partial D_+ = a$



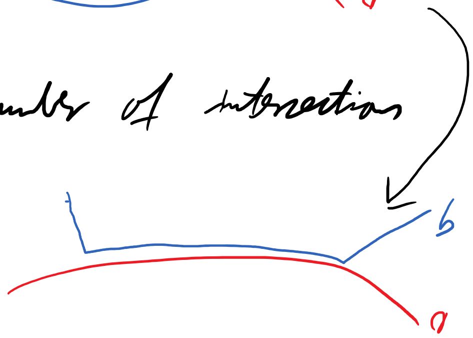
i.e. $\partial b_+ \xleftrightarrow{1:1} a \pitchfork b$

Class. of 1-mfd $\Rightarrow$ b_+ = fin. arcs

$\exists$ a BIGON between a & b



$\Rightarrow$ $\exists$ isotopy of b reducing the number of intersections

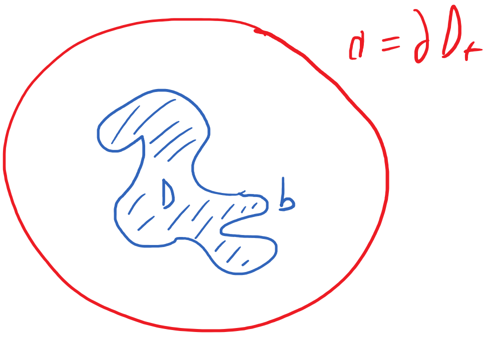


$\Rightarrow$ $\exists$ isotopy s.t. $a \cap b = \emptyset$

$\Rightarrow$ $b \subset \mathring{D}_+$ or $\mathring{D}_-$ let's say $b \subset \mathring{D}_+$

SCHOENFLIES

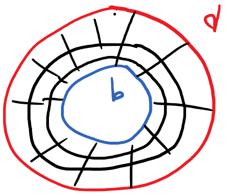
$\Rightarrow$ b bounds a disk D in $D_+ \subset S^2$



Class. of surfaces

$\Rightarrow D_+ \setminus \mathring{D} \cong S^1 \times I$

$\partial(S^1 \times I) = a \sqcup b$



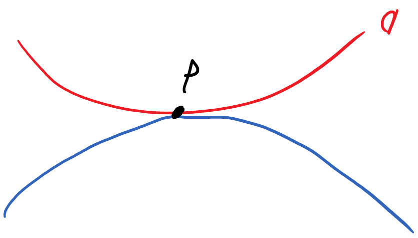
$\Rightarrow$ a is isotopic to b via $S^1 \times \{t\}$

▨

Proof sketch for $F \neq S^2$:

CASE 1 $a \cap b = \emptyset$

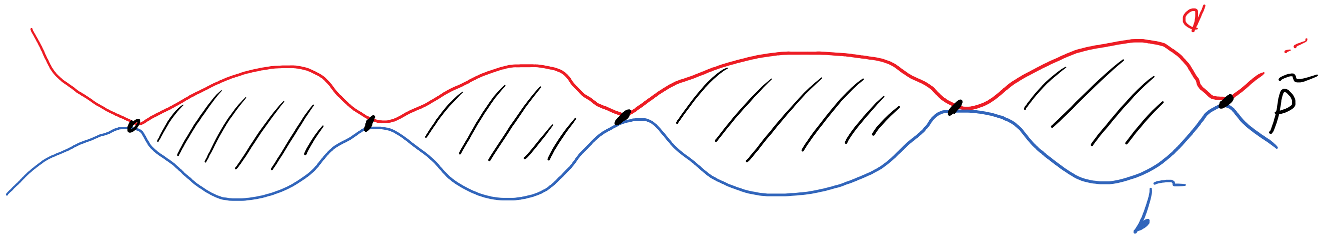
$\Rightarrow$ isotope s.t. $a \cap b = \{p\}$



consider universal cover

$$\begin{tikzcd} \tilde{F} \cong \mathbb{R}^2 \arrow[d, "\pi"'] \\ F \end{tikzcd}$$

$\tilde{a} := \pi^{-1}(a)$
$\tilde{b} := \pi^{-1}(b)$
$\tilde{p} := \pi^{-1}(p)$

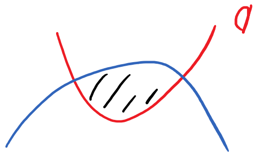


$\Rightarrow \exists$ equivariant isotopy from $\tilde{a}$ to $\tilde{b}$ in $\tilde{F}$

$\Rightarrow \exists$ " a to b in F

GENERAL CASE $a \pitchfork b = \{p_1, \dots, p_k\}$

reduce # of intersections by BIGON CRIT ($\exists$ bigon)



[work in $\tilde{F}$ & similar for S^2]

→ see FARB-MARGALIT for details ∎

5. DEHN SURGERY

5.1. SURGERY & HANDLE BODIES

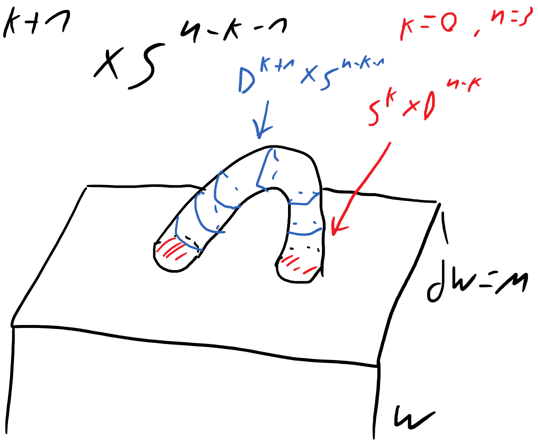
Let W^{n+1} be a compact smooth mfd with $\partial W \supset M^n$

$$W^{n+1} \cup h_{k+1} = W^{n+1} \cup_\varphi D^{k+1}\times D^{n-k}$$

with $\varphi : \partial D^{k+1}\times D^{n-k} \hookrightarrow \partial W = M$

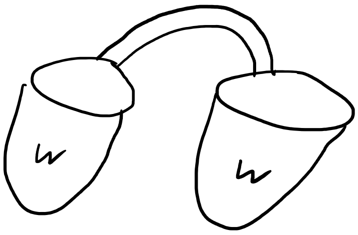
$M = \partial W$ changes to

$$M' := M \setminus \varphi(S^k\times D^{n-k}) \cup D^{k+1}\times S^{n-k-1}$$



M' is obtained from M by SURGERY along $\varphi(S^k\times D^{n-k})$

Ex: $\# \mathrel{\hat{=}}$ 0-SURGERY $\mathrel{\hat{=}}$ ATTACHING 1-HANDLE



„ATTACHING A $(k+1)$-HANDLE TO W

$\mathrel{\hat{=}}$ PERFORMING A k-SURGERY TO ∂W"

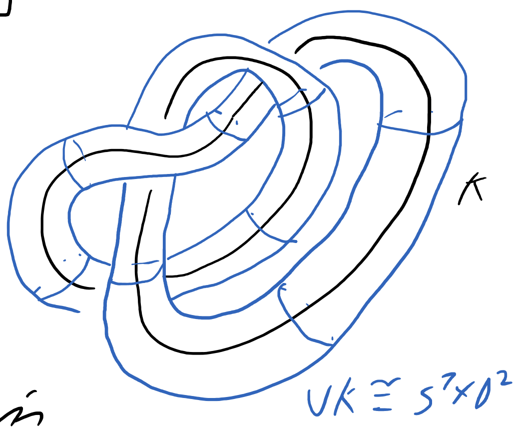
Corollary 1 M' is obtained from M by a fin. sequence of surgeries

$$\Leftrightarrow \partial(\underbrace{I\times M \cup \text{handles}}_{=:W}) = M'$$

$\Leftrightarrow \exists$ COBORDISM W between M & M', i.e.

W compact or mfd s.t. $\partial W = -M \sqcup M'$ ∎

5.2. SURGERY DESCRIPTIONS OF 3-MFDS



Let $K \subset S^3$ be a knot

& $VK :=$ tubular nbhd of K

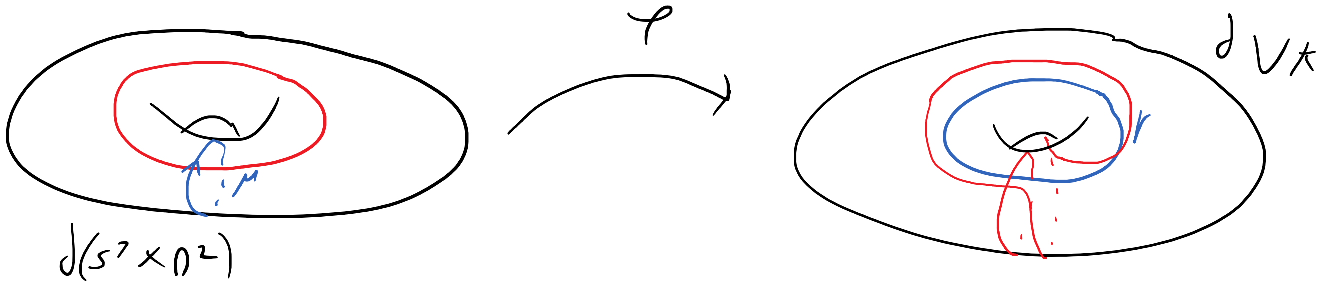
& $r :=$ non-trivial s.c.c. on ∂VK

DEHN SURGERY along K with **SLOPE** r is

$$S^3_K(r) := S^1\times D^2 \cup_\varphi S^3 \setminus \mathring{V}K$$

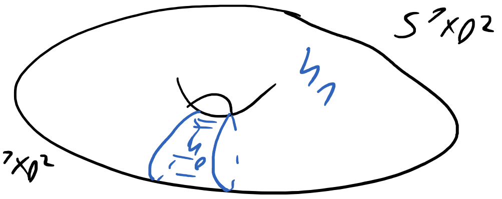
where $\varphi : \partial(S^1\times D^2) \xrightarrow{\cong} \partial VK$ s.t.

$$\{pt\}\times\partial D^2 =: \mu_0 \longmapsto r$$



Lemma 2: $S^3_K(r)$ is indep of the choice of φ

Proof: $S^1\times D^2 = h_0 \cup h_1$



$$S^3_K(r) = S^3\setminus\mathring{V}K \cup \overbrace{h_2 \cup h_3}^{\text{dual hand dec. of } S^1\times D^2}$$

& $\varphi(\mu_0) = r =$ attaching sphere of h_2

Alexander trick $\Rightarrow$ done ▨

Ex: (0) $S^3_K(\mu_K) \cong S^3$

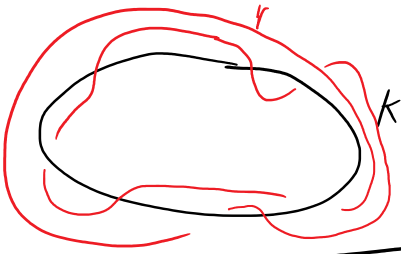
(1) $L(p,q) = \overset{V_0}{\overbrace{S^1\times D^2}} \cup_\varphi \overset{V_1}{\overbrace{S^1\times D^2}}$

$$\mu_0 \xmapsto{\varphi} q\mu_1 - p\lambda_1$$

(where μ_1, λ_1 are mer & long of V_1)

$= S^1\times D^2 \cup_\varphi S^3\setminus \mathring{V}U$

$= S^3_U(q\mu_1 - p\lambda_1)$



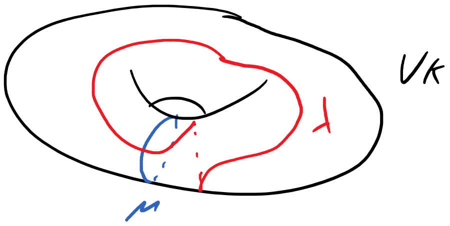
PROBLEM How to describe γ?

5.3. SURGERY COEFFICIENTS & LINKING NUMBERS

Let λ be a longitude on ∂VK

Then we can write any slope γ on ∂VK uniquely as

$\gamma = p\mu + q\lambda$ for p,q coprime (see L.3.9)

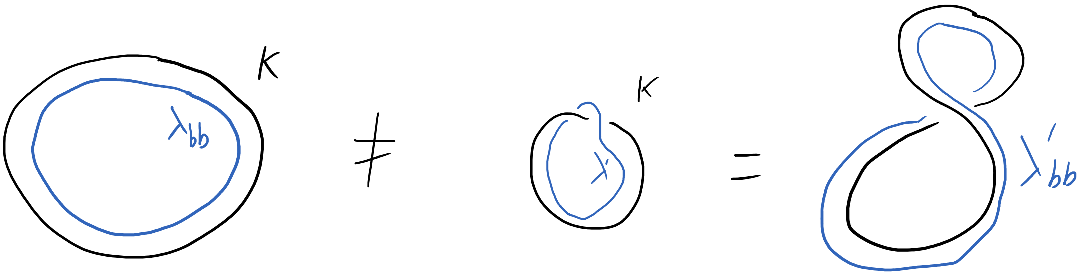


⇒ For a given longitude we can describe γ via the

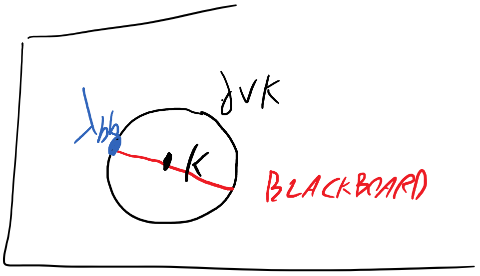
SURGERY COEFFICIENT $\gamma := p/q \in \mathbb{Q} \cup \{\infty\}$

PROBLEM: How to choose λ?

Ex:

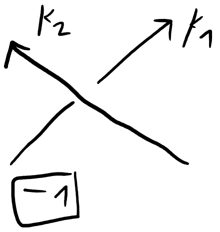


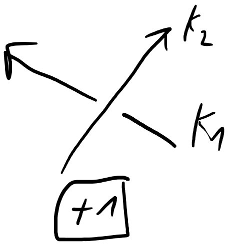
GOAL: Find an isotopy invariant reference λ!



Def: Let K_1, K_2 be oriented knots in S^3. The LINKING NUMBER of K_1 & K_2 is

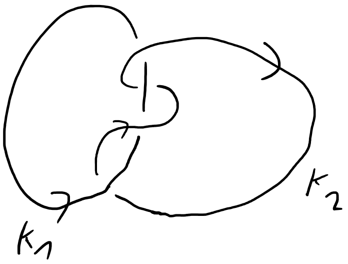
$$lk(K_1,K_2) := \#\text{ crossings of } K_1 \text{ under } K_2 \text{ with signs}$$





Ex:

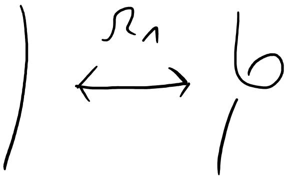
$lk(K_1,K_2) = 1$

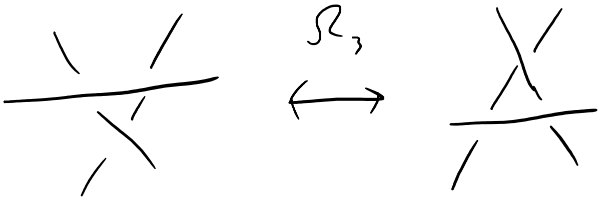


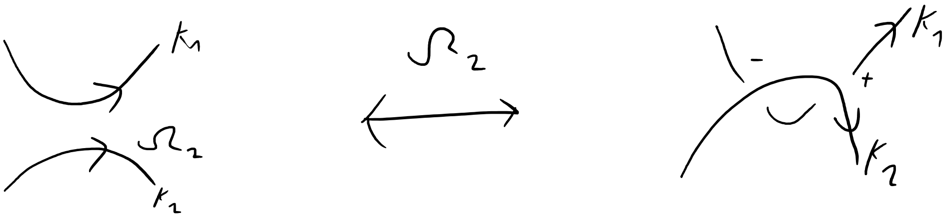
$lk(K_1,K_2) = 2$

Lemma 3: $lk(K_1,K_2)$ is a link invariant.

Proof:





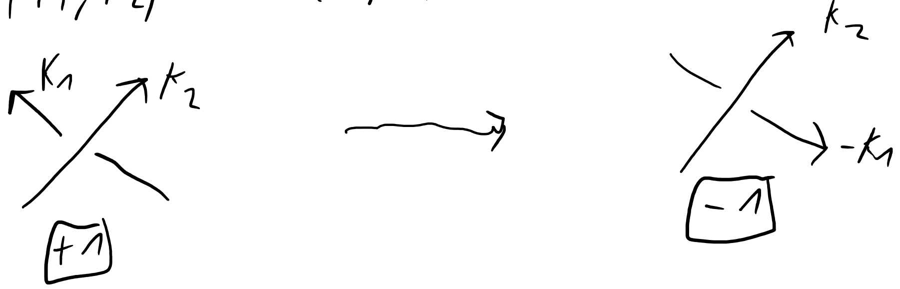


∎

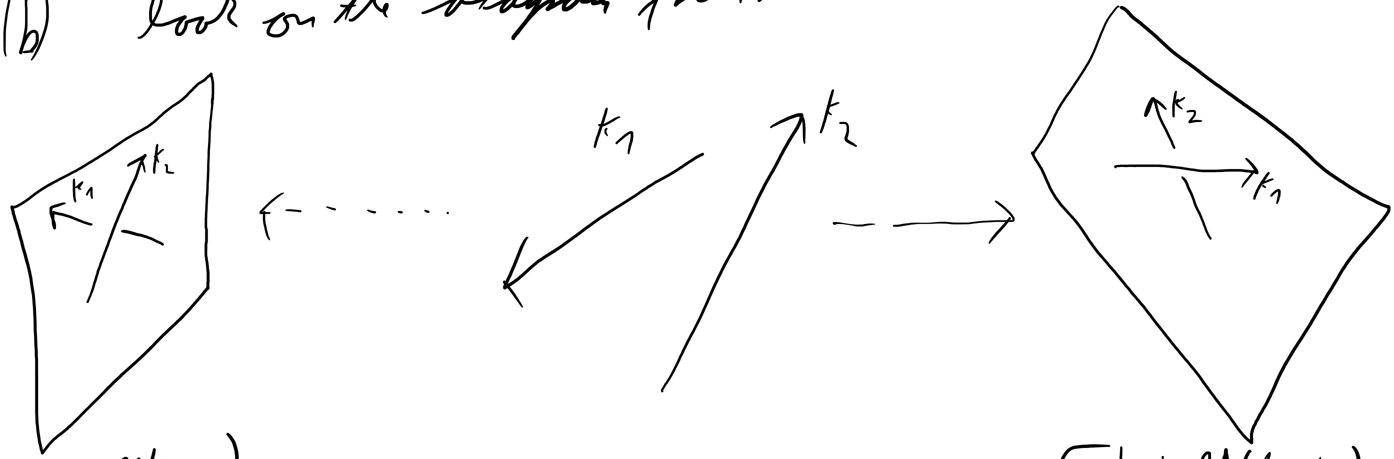
Lemma 4: (a) $lk(-K_1, K_2) = lk(K_1, -K_2) = -lk(K_1, K_2)$

(b) $lk(K_1, K_2) = lk(K_2, K_1)$

Proof: (a)



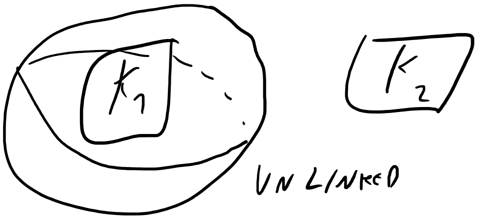
(b) look on the diagram from the other side



$\boxed{+1}$ in $lk(K_1, K_2)$

$\boxed{+1}$ in $lk(K_2, K_1)$ ▨

Lemma 5: $lk(K_1, K_2) \neq 0 \Rightarrow K_1$ & K_2 are <u>LINKED</u>, i.e. K_1 & K_2 cannot be separated by a 2-sphere



Proof:

Let K_1 & K_2 be unlinked

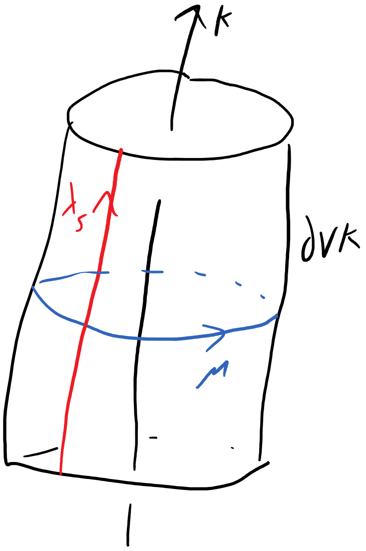
$\Rightarrow K_1 \cup K_2 \sim \boxed{K_1} \sqcup \boxed{K_2}$

$\Rightarrow \exists$ diagr. s.t. there are <u>NO</u> crossings of K_1 & K_2 ▨

$\Rightarrow lk(K_1, K_2) = 0$

Def: Let K be a knot in S^3.

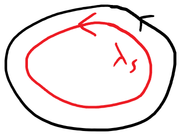
The SEIFERT LONGITUDE $\lambda = \lambda_s \subset \partial \nu K$ of K is the longitude defined by $lk(K,\lambda) = 0$, where we choose or. s.t. K & λ point in same direction.

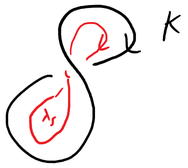


Remark:

$\lambda_s :=$ pushing K into a Seifert surface of K

Ex:





Lemma 6: λ_s is well-def (up to isotopy on $\partial\nu K$) & exists

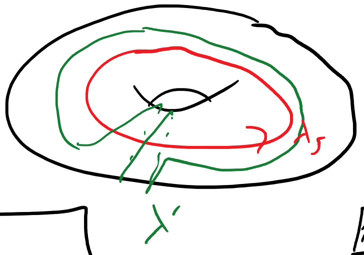
Proof: * λ_s is indep of or. of K by L.4.(a)

* $lk(K,\mu) = 1$

& any other longitude λ' is of the form

$$\lambda' = n\mu + \lambda_s \quad \text{for some } n \in \mathbb{Z} \quad (L.3.9)$$

$$\Rightarrow lk(K,\lambda') = lk(K, n\mu+\lambda_s)$$
$$= n$$



▨

CONVENTION!

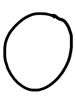
Surgery coefficients are measured w.r.t. λ_s

Ex: (0)

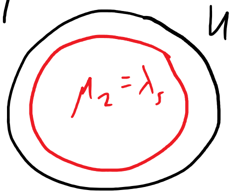
$\infty = \frac{1}{0} \mathrel{\hat{=}} 1\cdot\mu + 0\lambda_s$

$= S^3$

(1) $S^3_u(-p/q) =$



$^{-p/q} = L(p,q)$



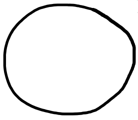
$V_1 = Vu$ with (μ_u, λ_s)

$V_2 = S^3 \setminus \mathring{V}u$ with $(\mu_2, \lambda_2) = (\lambda_s, \mu_u)$

$L(p,q) = V_1 \cup_\varphi V_2$

$\mu_1 \longmapsto q\mu_2 - p\lambda_2 = -p\mu_u + q\lambda_s$

(3)



$^{1/q} = S^3$

5.4. THE POINCARÉ HOMOLOGY SPHERE

Def: A closed n-mfd M^n is called HOMOLOGY SPHERE :⟺

$$H_k(M) \cong H_k(S^n) \cong \begin{cases} \mathbb{Z} & ; k = 0, n \\ 0 & ; \text{else} \end{cases}$$

Remark: M^3 is a hom. sphere ⟺ $\pi_1^{ab}(M) = H_1(M) = 0$

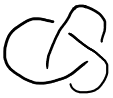
FIRST VERSION OF POINCARÉ'S CONJ (1900)

M^3 a hom. sphere ⟺ $M^3 \overset{C^0}{\cong} S^3$

Thm 7 (POINCARÉ'S COUNTEREXAMPLE, 1904)

∃ 3-mfd P (POINCARÉ HOM. SPHERE) s.t. $\pi_1(P) \neq 1$ but $H_1(P) = 0$

$P =$



$^{-1}$ (DEHN)

Lemma 8: Let $L = L_1 \cup \dots \cup L_n \subset S^3$ be a link & $r_1, \dots, r_n$ surgery coeff of L, $r_i = p_i/q_i$, & $M = S^3_L(r_1, \dots, r_n)$

$$\Rightarrow H_1(M) = \Big\langle \mu_1, \dots, \mu_n \ \Big|\ p_i \mu_i + q_i \sum_{\substack{j=1 \\ j \neq i}} lk(L_i, L_j)\, \mu_j = 0 \Big\rangle_{\mathbb{Z}}$$

Proof: Exercise ▨

Corollary: $H_1(S^3_K(p/q)) = \langle \mu_K \mid p\mu_K \rangle \cong \mathbb{Z}_p$

In particular: $S^3_K(1/q)$ is a homology sphere. ▨

Proof of T.7:

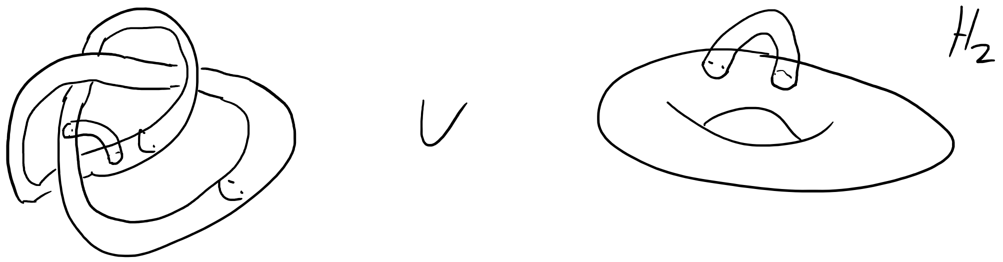
Exercise: $\pi_1(P) \cong$ BINARY ICOSAHEDRAL GROUP I^*

$$=: \langle a, b \mid a^5 = b^3 = (ba)^2 \rangle$$

of order $|I^*| = 120$

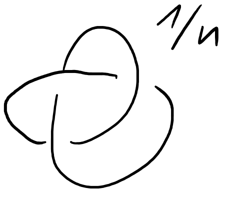
$\Rightarrow \pi_1(P) \neq 1$

See: https://en.wikipedia.org/wiki/Binary_icosahedral_group ▨



$S^3 \setminus H_2$ is a handlebody

Ex:



are non-trivial homology spheres.

5.5. INTEGER SURGERY & THE LICKORISH-WALLACE THM

Thm 8 (LICKORISH-WALLACE)

$\forall M^3$ closed, or., con. $\exists$ LINK $L = L_1 \cup \dots \cup L_k \subset S^3$ s.t.

$$M = S^3_L(n_1, \dots, n_k) \quad \text{for } n_i \in \mathbb{Z}$$

Remark: A surgery with integer surgery coef. is called INTEGER, this is indep of the longitude.

⌜ (coeff w.r.t. λ_S) $= n \in \mathbb{Z}$

$\Leftrightarrow \quad r = n\mu + \lambda_S$

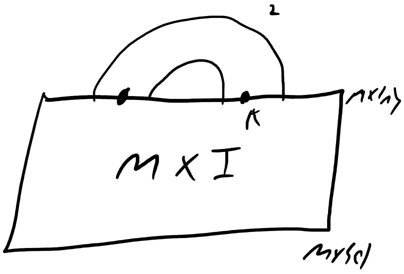
let λ' be a diff. longitude $\Rightarrow \lambda' = k\mu + \lambda_S$ for some $k \in \mathbb{Z}$

$\Rightarrow \quad r = n\mu - k\mu + \lambda'$

$\Rightarrow$ (coeff w.r.t. λ') $= -k + n \in \mathbb{Z}$ ⌟

Lemma 9 Integer surgery on $K \subset M^3$ corresponds to attaching a 4-dim 2-handle to $M \times I$ along $K \subset M \times \{1\}$

Proof: integer surgery: let λ_K be a longitude of K.



$$M_K(n) := S^1 \times D^2 \cup_\varphi M \setminus \mathring{V}_K$$

$$\mu_0 \xmapsto{\varphi} n\mu_K + \lambda_K =: \lambda'$$

$\Rightarrow \quad M_K(n) := M \setminus \Phi(S^1 \times D^2) \cup D^2 \times S^1$ in the obvious way

$\hat{=}$ attaching a 2-handle to $M \times I$ at $M \times \{1\} \equiv M$ see Lect. 5.1. ∎

Corollary 10 (ROHLIN)

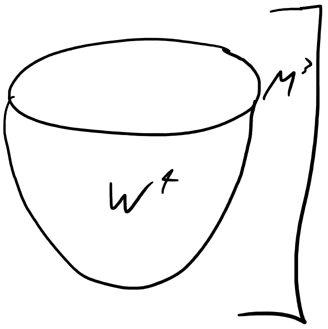
$\forall M^3$ closed, or. $\exists$ compact oriented 4-mfd W^4 with $\pi_1(W) = 1$ s.t.

$\partial W = M$, i.e. $\forall$ 3-mfd is NULL COBORDANT $(\Omega_3 = 0)$

Remark: $\tau : \Omega_4 \xrightarrow{\cong} \mathbb{Z}$

$M^4 \longmapsto \tau(M)$

Ex: $\tau(\mathbb{C}P^2) = 1 \Rightarrow \nexists$ compact or. W^5 s.t. $\partial W^5 = \mathbb{C}P^2$



Proof: * start with 4-dim 0-handle h_0 , $\partial h_0 = \partial D^4 = S^3$

* attach 2-handles to D^4 corresponding to an integer surgery descript of M (T.8.)

$\Rightarrow W^4 = h_0 \cup \{2\text{-handles}\}$ with $\partial W = M$

& $\pi_1(W) = 1$ (no 1-handles)

▨

Remark: Any integer surgery diagram of M det. a compact 4-mfd W with $\pi_1(W) = 1$ & $\partial W = M$.

Proof of T.8:

* write M as a Heegaard splitting (C.3.3)

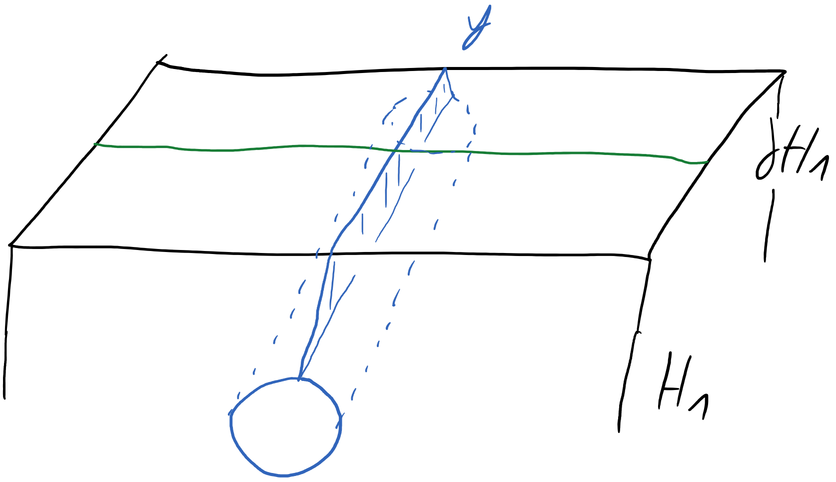
$$M = H_1 \cup_f H_2 \quad \text{for } f : \partial H_1 \xrightarrow{\cong} \partial H_2$$

* $S^3 = H_1 \cup_{f_0} H_2$ for some $f_0 : \partial H_1 \xrightarrow{\cong} \partial H_2$

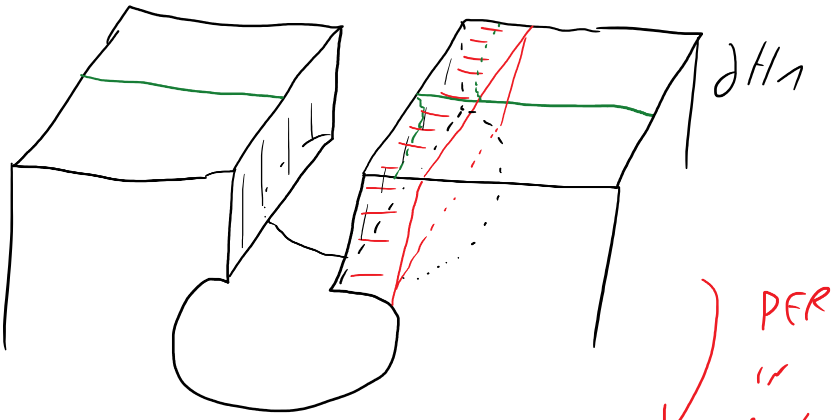
T.4.2. $\Rightarrow$ $f^{-1} f_0 \sim g$ = product of Dehn twists

$$\boxed{\text{„A Dehn-twist } T \text{ along } \gamma \subset \partial H_1 \,\hat{=}\, \text{a surgery along } \gamma \text{ in } H_1 \cup H_2\text{"}}$$

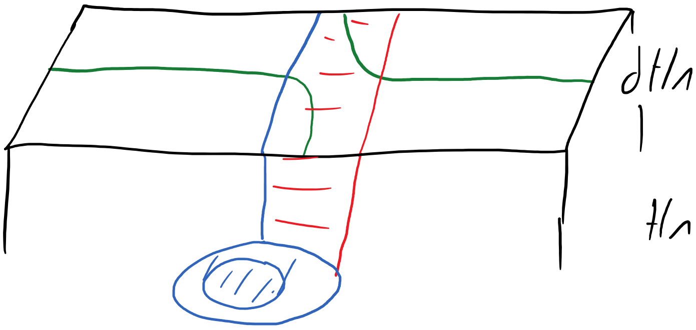
Consider:



↓ CUT ALONG BLUE



PERFORM DEHN-TWIST IN THE RED REGION & GLUE TOGETHER



=) this defines a homeom

$$f_y : H_1 \setminus S^1\times D^2 \xrightarrow{\cong} H_1 \setminus S^1\times D^2$$

s.t. $f_y|_{\partial H_1} = T_y$

* Assume: $f^{-1} f_0 = T_y$

$$\begin{tikzcd}
S^3\setminus S^1\times D^2 \arrow[d, dashed, "\cong"] & = & H_1\setminus S^1\times D^2 \arrow[d, "\cong"', "f_y"] & \cup_{f_0} \arrow[d, phantom, "\circlearrowright"] & H_2 \arrow[d, "\cong\ id"] \\
M\setminus S^1\times D^2 & = & H_1\setminus S^1\times D^2 & \cup_f & H_2
\end{tikzcd}$$

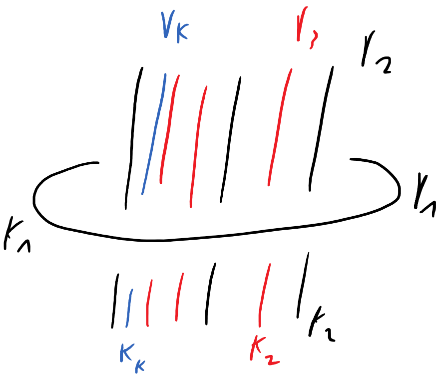
=) $M = S^3_y(\mu - \lambda_H)$ for left-handed Dehn-twist ($\mu + \lambda_H$)

where λ_H is obtained by pushing y onto ∂H_1

* The general case follows by an iteration. ▨

5.6. THE ROLFSEN TWIST & KIRBY'S THEOREM

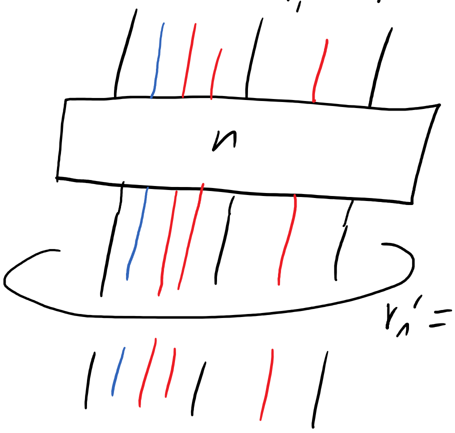
Theorem 11 (ROLFSEN TWIST)



$\forall n \in \mathbb{Z}$

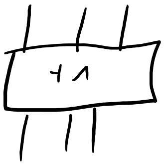
$\overset{C^0}{\cong}$

$r_i' = r_i + n\,lk^2(K_i, K_1)$

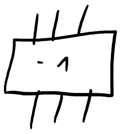


$r_1' = \frac{1}{n + 1/r_1}$

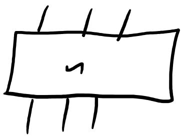
where



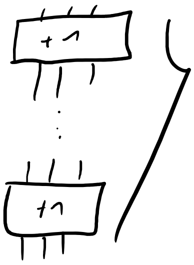
$:=$



= left-handed twist

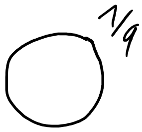


$=$

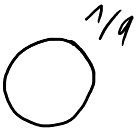


n-times

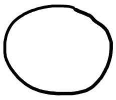
Ex: (1)



$= L(1,9) = S^3$

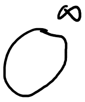


(-9)-fold RT $=$



$\frac{1}{-9+1/1/9}$

$=$

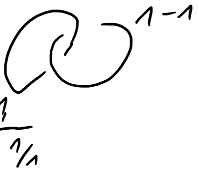


$=$ $\emptyset = S^3$

(2)

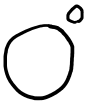
$= S^1 \times S^2$

(-1)-fold RT along K_1 $=$



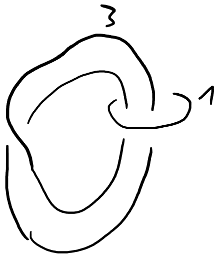
$=$

$=$

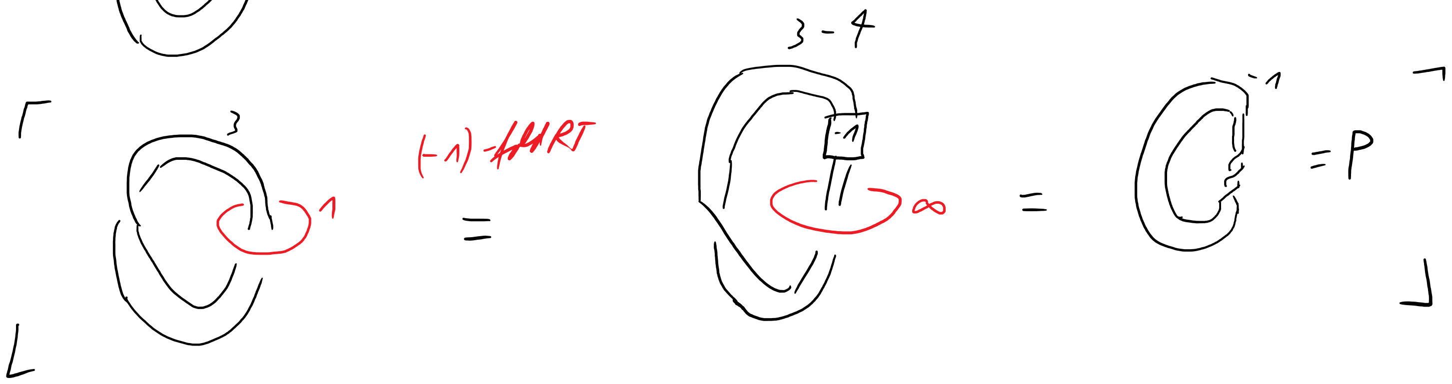


$= S^1 \times S^2$

(3)



$= P$



Theorem 12:

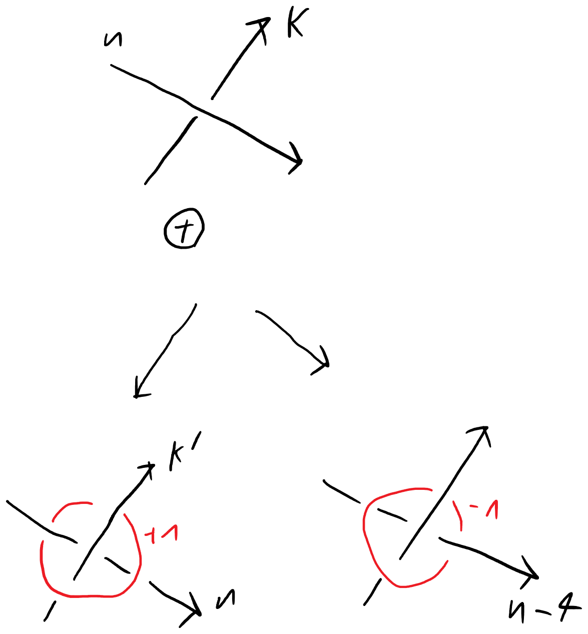
∀ 3-mfd M^3 ∃ link $L = L_1 \cup \dots \cup L_k \subset S^3$ s.t.

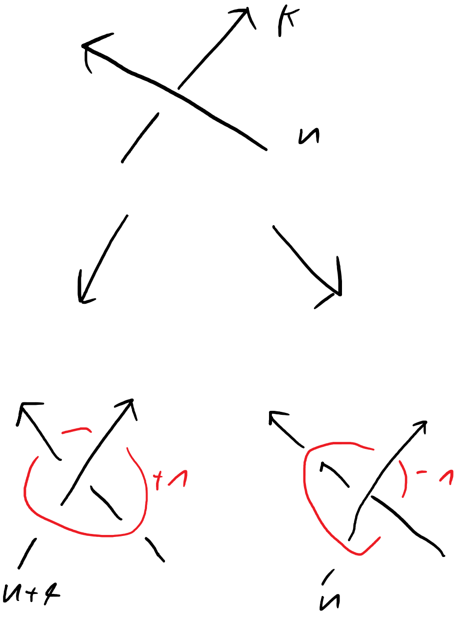
$M = S^3_L(n_1, \dots, n_k)$ for $n_i \in \mathbb{Z}$

& $L_i = O$ ∀ $i = 1, \dots, k$

Proof: TRICK:

we can change a crossing by a Rolfsen twist:

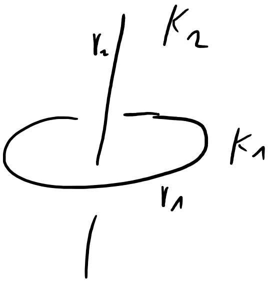




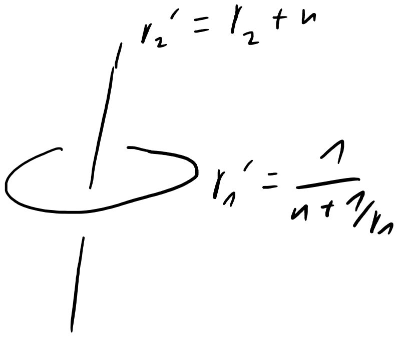
∎

Proof of T.11:

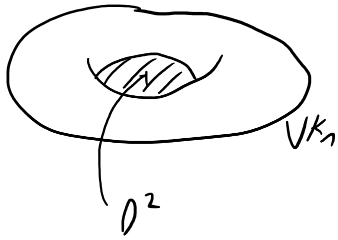
K=1:



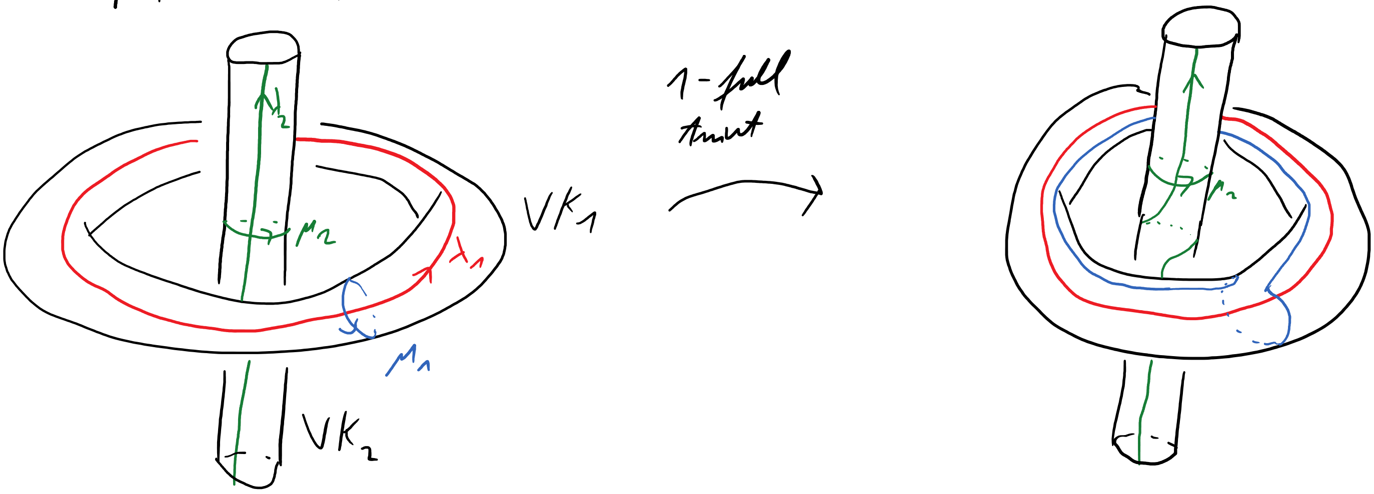
$\overset{!}{\cong}$



Consider: $S^3 \setminus \mathring{V}K_1 \cong S^1 \times D^2$



- Cut $S^3 \setminus \mathring{V}K_1$ along a Seifert disk of K_1
- perform n-full twist & reglue



$$\Rightarrow \quad \mu_1 \longmapsto \mu_1 + n\lambda_1 \qquad \& \qquad \mu_2 \longmapsto \mu_2$$
$$\lambda_1 \longmapsto \lambda_1 \qquad\qquad \lambda_2 \longmapsto \lambda_2 + n\mu_2$$

New surgery coeff.: $r_1 = p_1/q_1$

$$\mu_0 \longmapsto p_1\mu_1 + q_1\lambda_1 \longmapsto p_1(\mu_1 + n\lambda_1) + q_1\lambda_1$$
$$= p_1\mu_1 + (p_1 n + q_1)\lambda_1$$

$$\Rightarrow \quad r_1' = \frac{p_1}{p_1 n + q_1} = \frac{1}{n + 1/r_1}$$

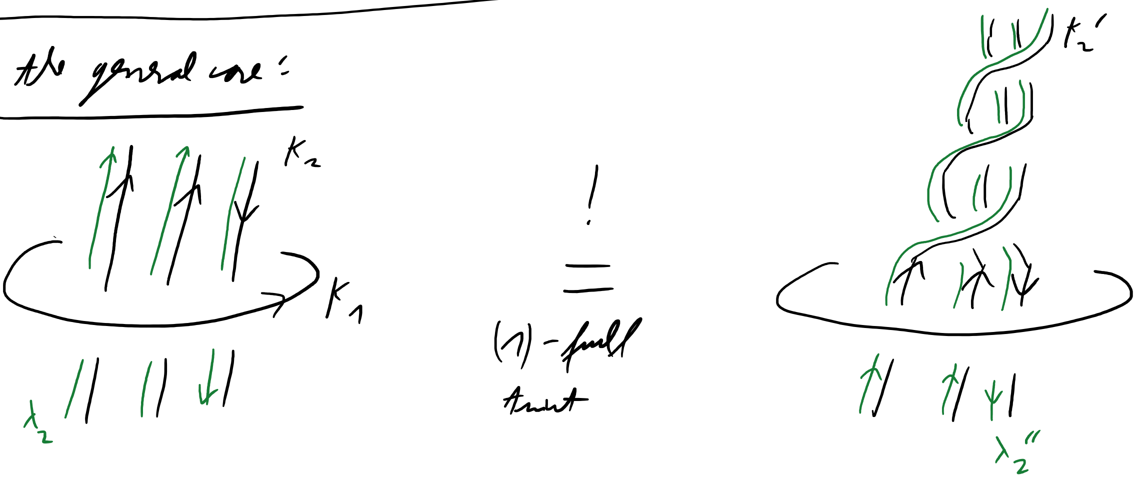
& $\mu_0 \longmapsto p_2\mu_2 + q_2\lambda_2 \longmapsto p_2\mu_2 + q_2(n\mu_2 + \lambda_2)$

$$= (p_2 + n q_2)\mu_2 + q_2\lambda_2$$

$$\Rightarrow \quad r_2' = \frac{p_2 + q_2 n}{q_2} = r_2 + n$$

---

* the general case:



$lk(K_2, \lambda_2) = 0$ $\qquad$ $\lambda_2'' \neq$ Seifert longitude i.g.

$lk(K_1, K_2) = u - d$

u := # strands of K_2 w. up

d := # " down

$$lk(K_2', \lambda_2'') = n\left(u(u-d) + d(d-u)\right)$$

$$= n\,(u-d)^2 = n\, lk^2(K_1, K_2)$$

$\Rightarrow$ The Seifert longitude λ_2' of K_2' is def by $lk(K_2', \lambda_2') = 0$

$$\Rightarrow \quad \lambda_2'' = \lambda_2' + n\, lk^2(K_1, K_2)\mu_2$$

new surgery coeff of K_2':

$\mu_0 \longmapsto p_2\mu_2 + q_2\lambda_2 \longmapsto p_2\mu_2 + q_2\lambda_2'' = p_2\mu_2 + q_2(\lambda_2' + n\, lk^2\mu_2)$

$$= (p_2 + n\, lk^2 q_2)\mu_2 + q_2\lambda_2$$

$$\Rightarrow \quad r_2' = \frac{p_2 + n\, lk^2 q_2}{q_2} = r_2 + n\, lk^2$$

$\boxtimes$

Theorem 13 (KIRBY)

Let $M = S^3_L(r_1,\dots,r_n)$ & $M' = S^3_{L'}(r'_1,\dots,r'_k)$ be surgery prs. of 3-mfds M & M'. Then:

$M \cong M' \iff L(r_1,\dots,r_n)$ can be transformed into $L'(r'_1,\dots,r'_k)$ by fin. many RT & inserting/deleting knots with ∞-coeff.

Proof idea:

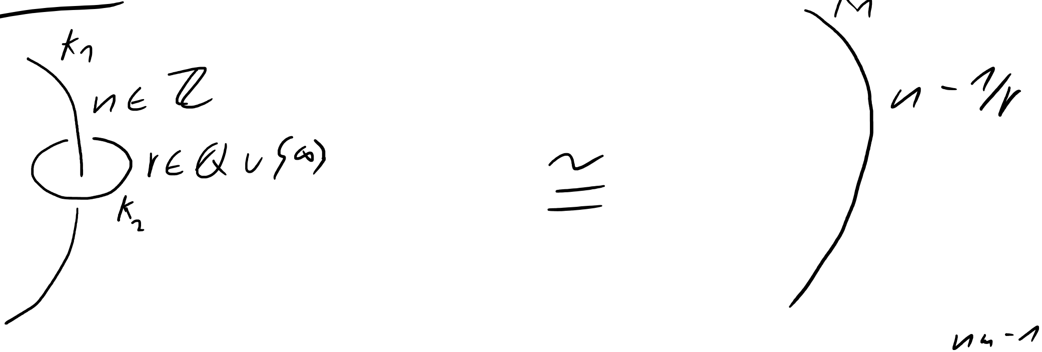
"$\Leftarrow$" in T.11.

"$\Rightarrow$"

$$\begin{tikzcd}
M = L(r_1,\dots,r_n) \arrow[r, squiggly, "\text{see Exercise}"] \arrow[d, "RT"] & \text{Heegaard splitting of } M \arrow[d, "\text{Stab/destab.} \ \& \ \text{MCG relations}", "\text{T.3.11} \ \text{T.4.9.}"'] \\
M' = L'(r'_1,\dots,r'_k) \arrow[r, squiggly, "\text{see Exercise}"] & \text{Heeg. spl. of } M'
\end{tikzcd}$$

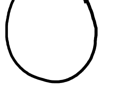
realise Stab & MCG relations by RT $\blacksquare$

Lemma 14 (SLAM DUNK)

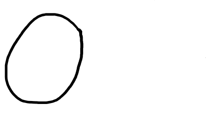


Ex: $*$

$\overset{S.D.}{=}$

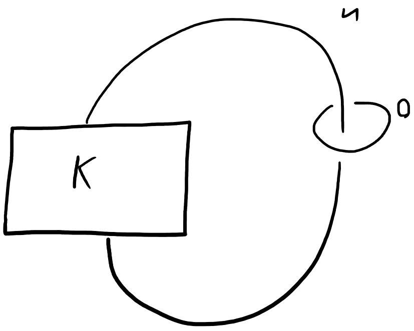


$=$

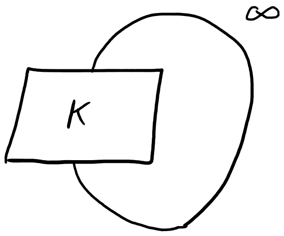


$= L(nm-1, m)$

*

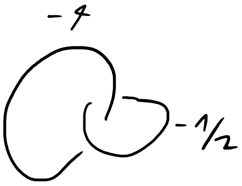


$\overset{SD}{=}$

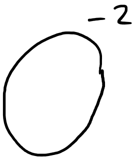


$= S^3$

* WARNING L.13 is WRONG if $n \notin \mathbb{Z}$:

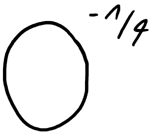


$=$



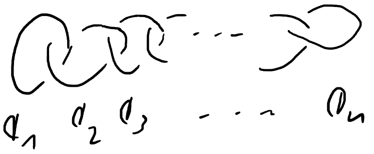
$= L(2,1) = \mathbb{R}P^3$

$\nparallel$



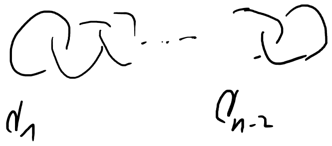
$= S^3$

*

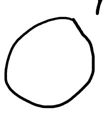


CHAIN OF UNKNOTS, $a_i \in \mathbb{Z}$

|| SD



$a_{n-1} - \frac{1}{a_n} \overset{SD}{=} \cdots \overset{SD}{=}$



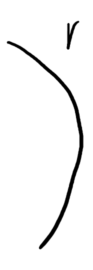
$$p/q = r = a_1 - \cfrac{1}{a_2 - \cdots - \cfrac{1}{a_n}}$$

$$\shortparallel$$

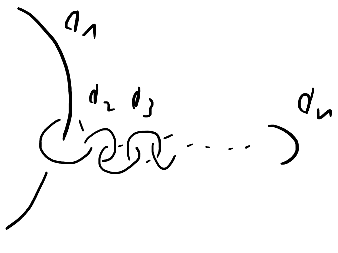
$$[a_1, \dots, a_n]$$

$$= L(p,q)$$

$\Rightarrow$ Conversely, we have an algorithm to convert a surgery diag into an integer surgery diag.



$=$



$r = [a_1, \dots, a_n]$

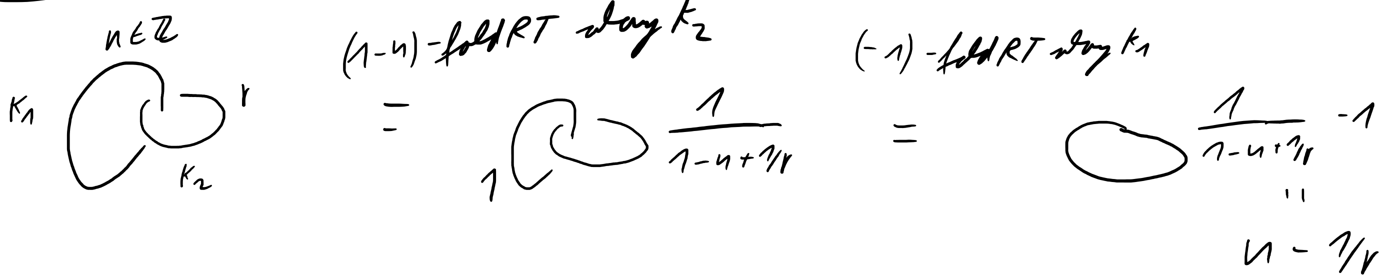
Proof of L.14

easy proof: as in T.11 work in local model & analyse the ugly stuff

→ see GOMPF-STIPSICZ page 163

fun proof: express SD by a sequence of RT.

CASE 1: $K_1 = 0$:



CASE 2: $K_1 \neq 0$

(i) perform RT's to unknot K_1

(ii) perform RT's from case 1

(iii) reverse the RT's from (i) ▨

Further applications of Dehn surgery:
Any 3-manifold admits a codimension 1 foliation, see for example:
B. Albach, Blätterungen von 3-Mannigfaltigkeiten, https://www2.mathematik.hu-berlin.de/~kegemarc/Kirby/Hausarbeit%20B.Albach.pdf
Any 3-manifold admits a contact structure, see for example:
H. Geiges, An Introduction to Contact Topology, Section 4.1.
Any 3-manifold admits a trivial tangent bundle, see for example:
S. Durst, H. Geiges, J. Gonzalo and M. Kegel, Parallelisability of 3-manifolds via surgery, Expo. Math., 38 (2020), 131-137.
Any 3-manifold admits an open book decomposition, see for example:
J. Etnyre, Lectures on open book decompositions and contact structures, in: proceedings of the Floer Homology, Gauge Theory, and Low Dimensional Topology Workshop, (2006), 103-141.

6. BRANCHED COVERINGS

Def: $p: X \longrightarrow Y$ is called COVERING $:\Longleftrightarrow$

$\forall y \in Y$ $\exists$ open nbhd $y \in U_y \subset Y$ s.t.

$$p^{-1}(U_y) = \bigsqcup_i U_i \text{ with } p|_{U_i} : U_i \xrightarrow[\cong]{C^0} U_y$$

Ex: (1) $p: \mathbb{R} \longrightarrow S^1 = \mathbb{R}/\mathbb{Z}$

$t \longmapsto e^{2\pi i t}$

(2) $\mathbb{R}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2 = T^2$

(3) $S^3 \xrightarrow{p\text{-fold}} L(p,q) = S^3/\mathbb{Z}_p$

(4) $\tilde{X} \longrightarrow X = \tilde{X}/\pi_1(X)$

(5) $p: \mathbb{C} \longrightarrow \mathbb{C}$

$z \longmapsto z^k$

$p|_{\mathbb{C}^*} : \mathbb{C}^* \longrightarrow \mathbb{C}^* = \mathbb{C}\setminus\{0\}$ is covering

$p^{-1}(w) = k$ pts

but $p^{-1}(0) = \{0\}$

$\Rightarrow$ p is NOT a covering, BUT the prototype of a br. covering.

Def: Let M^n & M_0^n be n-mfd.

$p : M \longrightarrow M_0$ is called BRANCHED COVERING $:\Leftrightarrow$

$\forall x_0 \in M_0$ $\exists$ nbhd $x_0 \in U_0 \subset M_0$ s.t.

$\forall$ component U of $p^{-1}(U_0)$ $\exists$ comm. diag:

$$\begin{tikzcd}
(z, t_1, \dots, t_{n-2}) \arrow[r, phantom, "\in"] \arrow[d, mapsto] & D^2 \times I^{n-2} \arrow[r, "\cong", "h"'] \arrow[d, "\varphi_k"] \arrow[rd, phantom, "\circlearrowleft"] & U \arrow[d, "p|_U"] \\
(z^k, t_1, \dots, t_{n-2}) \arrow[r, phantom, "\in"] & D^2 \times I^{n-2} \arrow[r, "\cong", "h_0"'] & U_0
\end{tikzcd}$$

with $h_0(0, 1/2, \dots, 1/2) = x_0$

where $k \in \mathbb{Z}$ depends on x_0 and U.

$k :=$ (BRANCHING) INDEX of $x := p^{-1}(x_0) \cap U$

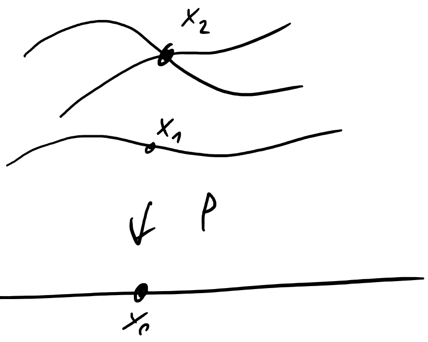
$L := \{ p \in M \mid p$ has index $> 1 \}$ (UPPER) BRANCHING SET

$L_0 := p(L)$ (LOWER) ''

Remark: * $p : F^2 \longrightarrow F_0^2$ is branched cov. $\Leftrightarrow$ ⊛

$\exists$ $\{x_1, \dots, x_\ell\} \subset F_0$ s.t.

$p|_{F \setminus p^{-1}\{x_1, \dots, x_\ell\}}$ is a covering



* i.g. $p^{-1}(L_0) \supsetneq L$

*$F \longrightarrow S^2$ br. covering

$\Rightarrow$ F orient.

$\circledast$ „$\Rightarrow$" ✓

„$\Leftarrow$" S. Stoïlow: Leçons sur les principes topologiques de la théorie des fonctions analytiques, 1938.

Let D^2 be a small closed nbhd around $x_i \in F_0$ s.t. $0 \hat{=} x_i$

$\Rightarrow$ $p|_{p^{-1}(D^2\setminus\{0\})} : p^{-1}(D^2\setminus\{0\}) \longrightarrow D^2\setminus\{0\}$ is a covering

* connected coverings of $D^2\setminus\{0\}$ are classified (up to homeo) by conjugacy classes of subgroups of $\pi_1(D^2\setminus\{0\}) \cong \mathbb{Z}$

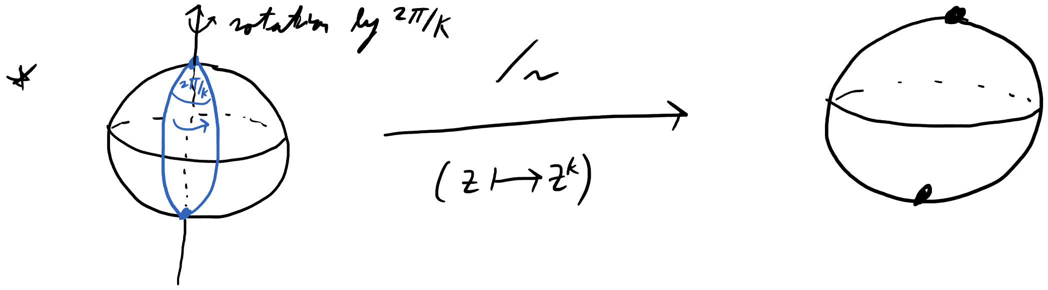
$\Rightarrow$ locally $p: \underset{D^2\setminus\{0\}}{\underset{\cap}{z}} \longmapsto \underset{D^2\setminus\{0\}}{\underset{\cap}{z^n}}$ on every comp. of $p^{-1}(D^2\setminus\{0\})$

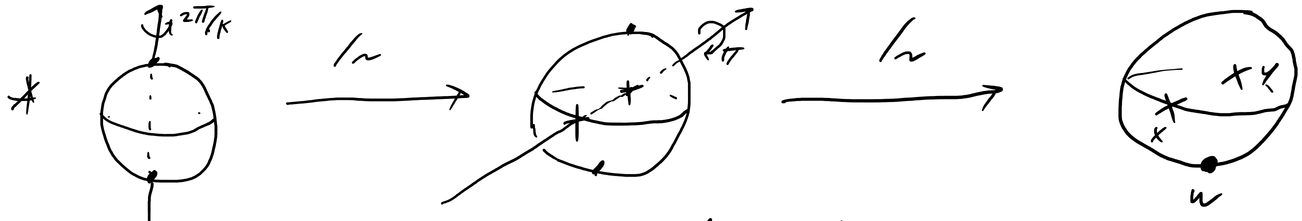
$\Rightarrow$ $\exists!$ cont. extension $D^2 \longrightarrow D^2$
$z \longmapsto z^n$

6.1. BR. COV. OF SURFACES & THE RIEMANN-HURWITZ FORMULA

Thm 1 ∀ surface F^2 ∃ bran. cov. $F \longrightarrow S^2$ with exactly 3 (four) br. pts.

1. Proof: (i) $F = S^2$:



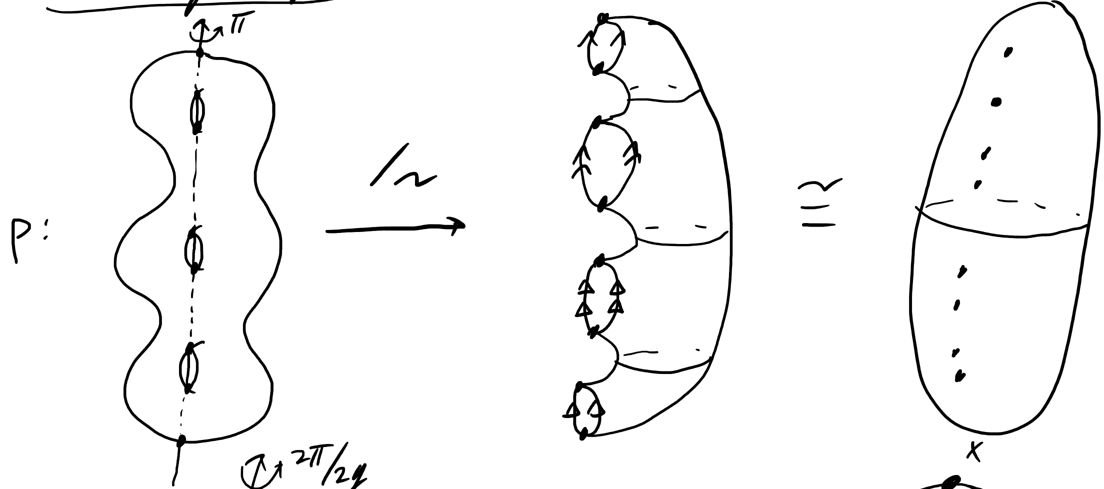


$p^{-1}(w) = 2$ pt of index k

$p^{-1}(x), p^{-1}(y)$ each k pts of index 2

away from br. set it is a $2k$-fold cov.

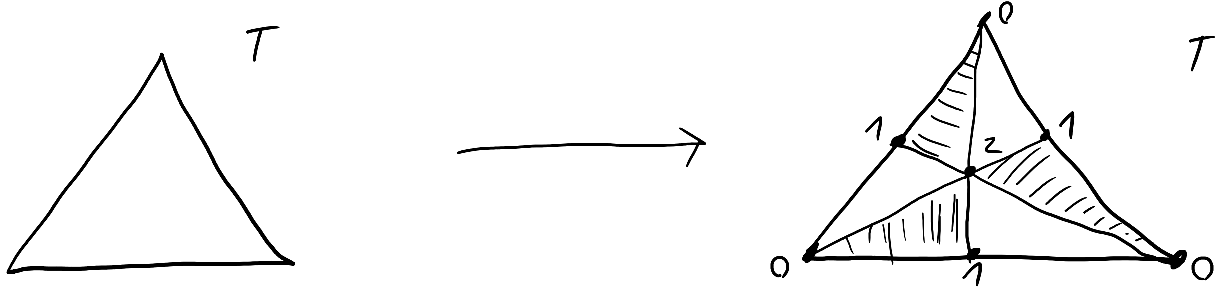
(ii) $F = \Sigma_g$, $g \geq 1$:



▨

2. Proof: * Choose an or. of F

* Choose a triangulation T of F and take its barycentric subdivision T':



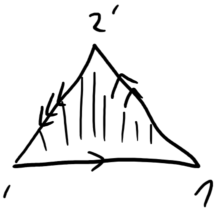
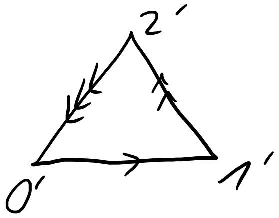
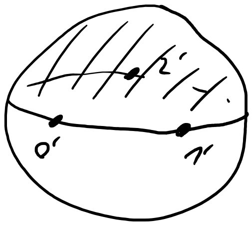
* label the vertices of T' by 0 if vertex ≙ vertex of T

1 " ≙ edge of T

2 " ≙ Δ of T

* label a triangle 012 in T' black if it is pos. or.

" white " neg "

* $S^2 =$  $\cup$  $\cong$ 

* def $p: F \longrightarrow S^2$ by

$$\Delta^{\text{black/white}}_{012} \longmapsto \Delta^{\text{black/white}}_{0'1'2'}$$

p is a covering away from vertices. ∎

The second proof gen. to arb. dim:

Theorem 2 (ALEXANDER)

∀ closed, or. connected PL mfd M^n ∃ (gen) br. covering $M^n \longrightarrow S^n$ branched along the $(n-2)$-skeleton of an n-simplex. ∎

Theorem 3 (RIEMANN-HURWITZ)

Let $p: F^2 \longrightarrow F_0^2$ be an n-fold br. cov.

$y_0, \ldots, y_k \in F_0$ the (lower) br. pts

$\{x_1, \ldots, x_\ell\} = p^{-1}(\{y_0, \ldots, y_k\})$

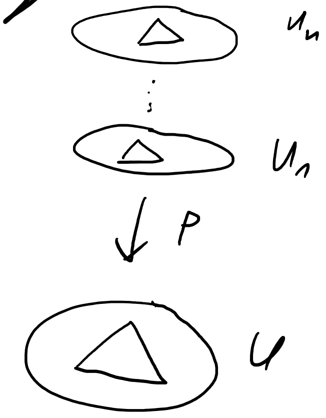
$d_i :=$ index of x_i & $b_j := |p^{-1}(y_j)|$

$$\Rightarrow \boxed{\chi(F) + \sum_{i=1}^{\ell} (d_i - 1) = n\,\chi(F_0)}$$

$$\boxed{\chi(F) = n(\chi(F_0) - k) + \sum_{j=1}^{k} b_j}$$

Proof: (1) p is an unbranched covering.

* Choose a triangulation T of F_0 with suff. small triangles



$\Rightarrow p^{-1}(T)$ induces a triangulation T' of F s.t.

p^{-1}(vertex of T) = n vertices of T'

p^{-1}(edges ") = n edges "

p^{-1}(Δ ") = n Δ "

✓

$\Rightarrow \chi(F) = n\,\chi(F_0)$

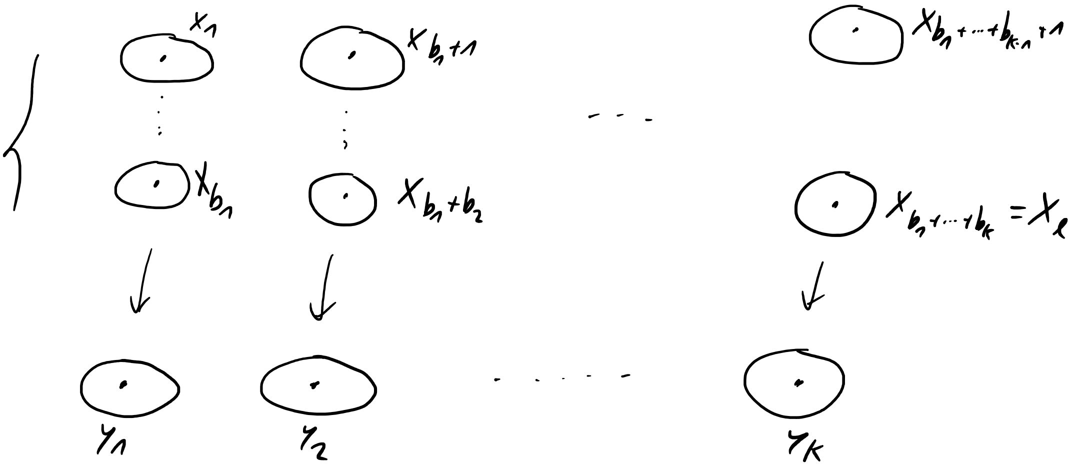
(2) general case:

$$A_0 := \bigcup_{i=1}^{k} D^2_\varepsilon(y_i) \subset F_0 \qquad A := p^{-1}(A_0) \subset F$$

for ε suff. small s.t. $A = \bigsqcup$ of 2-disks

$$B_0 = \overline{F_0 \setminus A_0} \qquad B = \overline{F \setminus A}$$

$$\begin{tikzcd} F \arrow[d, "p"] & \supset & A \\ F_0 & \supset & A_0 \end{tikzcd}$$



$$\Rightarrow * \quad \chi(F) = \chi(B) + \chi(A) - \underbrace{\underbrace{\chi(A \cap B)}_{= \sqcup S^1}}_{=0} = \chi(B) + \chi(A)$$

$$\& \quad \chi(F_0) = \chi(B_0) + \chi(A_0)$$

$*$ $p|_B : B \longrightarrow B_0$ is an n-fold unbr. cover

$$\Rightarrow \chi(B) = n\,\chi(B_0)$$

$$* \quad \chi(A_0) = \chi\Big(\bigsqcup_{i=1}^{k} D^2_\varepsilon\Big) = k$$

$$\& \quad \chi(A) = \chi\Big(\bigsqcup_{i=1}^{l} D^2\Big) = l$$

$$\Rightarrow \chi(F) = \chi(B) + \chi(A) = n\chi(B_0) + l = n\big(\chi(B_0) + \chi(A_0) - k\big) + \sum_{j=1}^{k} b_j$$

$$= n\big(\chi(F_0) - k\big) + \sum_{j=1}^{k} b_j$$

* The first formula follows from:

$$d_1 + \dots + d_{b_1} = n = d_{b_1+1} + \dots + d_{b_1+b_2} = \dots$$

$$\Rightarrow \sum_{i=1}^{\ell}(d_i - 1) = (n-b_1) + (n-b_2) + \dots = kn - b_1 - \dots - b_k$$

▨

Corollary 4

$\exists$ bran. cov. $F^2 \longrightarrow S^2$ with less than 3 pts $\Leftrightarrow F \cong S^2$

Proof: "$\Leftarrow$" $id : S^2 \to S^2$

"$\Rightarrow$" $\chi(F) \overset{T.3}{=} n(\chi(S^2) - k) + \sum_{j=1}^{k} b_j$

$$= \underbrace{n(2-k)}_{>0} + \underbrace{\sum_{j=1}^{k} b_j}_{>0}$$

$\boxed{\text{for } k \leq 2}$

> 0

$\Rightarrow F \cong S^2$

▨

NEXT WEEK LEC 9:15

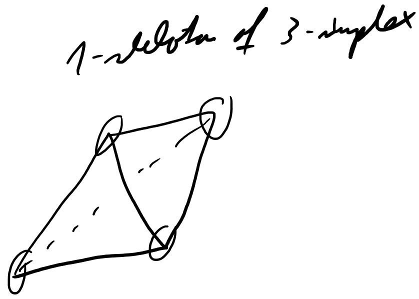
OPTIONAL LECTURE 13:00

WEDNESDAY 22.7. 9:15 OFFICE HOUR

6.2. BRANCHED COVERS OF 3-MFDS

Thm 5: (HILDEN-HIRSCH-MONTESINOS, 1976)

∀ 3-mfd M ∃ 3-fold branched covering $p: M^3 \longrightarrow S^3$ branched along a knot.



Remark:
- * 3-fold
- * br set = submanifold
- * " = connected

Thm 6 (THURSTON?)

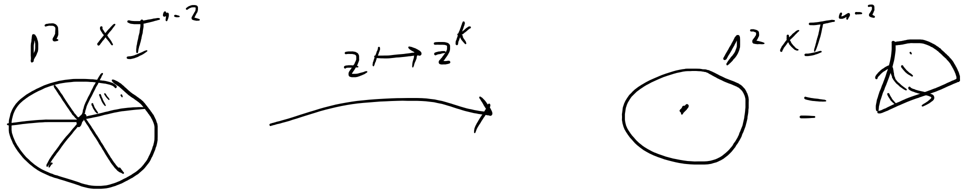
∀ 3-mfd M ∃ branched covering $p: M^3 \longrightarrow S^3$ branched along the figure 8-knot

Remark:
- * A knot with this property is called UNIVERSAL.
- * The Borromean rings are universal (→ see PS)
- * The trefoil is NOT universal.

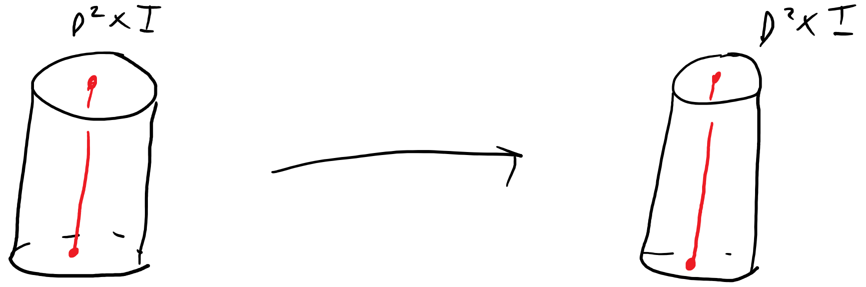
$p: M^n \longrightarrow M_0^n$ br cov.

locally

$$p: \underset{\underset{D^2 \times I^{n-2}}{\rotatebox{90}{$\in$}}}{(z, t_1, \dots, t_{n-2})} \longmapsto \underset{\underset{D^2 \times I^{n-2}}{\rotatebox{90}{$\in$}}}{(z^k, t_1, \dots, t_{n-2})}$$



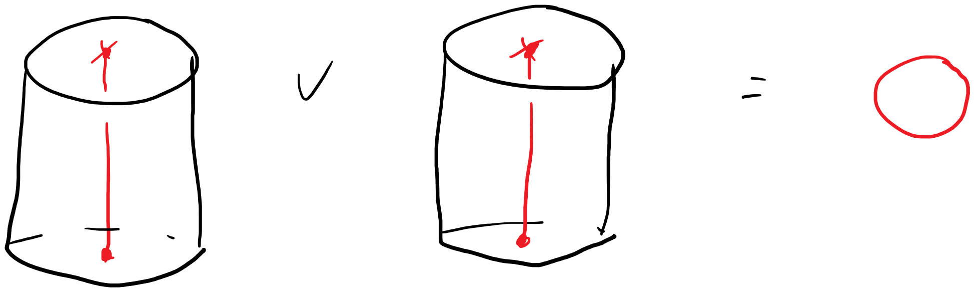
$n=3$



Lemma 7: $\forall n \geq 2 \ \exists$ n-fold br. cov $S^3 \longrightarrow S^3$ br. along $\bigcirc$

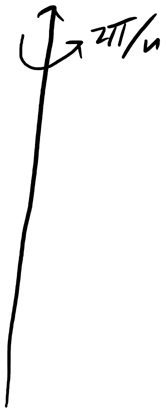
Proof:

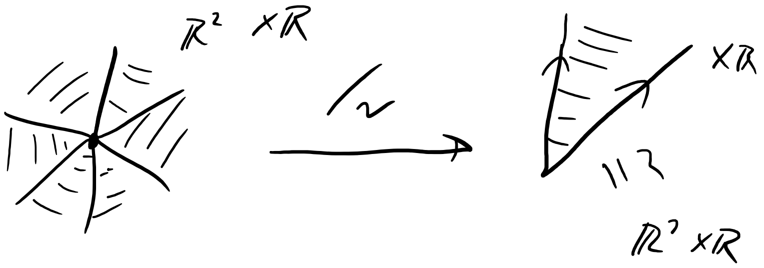
$$\begin{tikzcd}
S^3 \arrow[dddd, dashed, "p"] & = & D^3 & \cup_{id} & D^3 \\
 & = & D^2\times I & \cup_{id} & D^2\times I \\
 & & (z,t) \arrow[u, phantom, sloped, "\in"] \arrow[d, mapsto] & \circlearrowright & (z,t) \arrow[u, phantom, sloped, "\in"] \arrow[d, mapsto] \\
 & & (z^n,t) & & (z^n,t) \\
S^3 & = & D^2\times I & \cup_{id} & D^2\times I
\end{tikzcd}$$

br. set 

$\subset S^3$

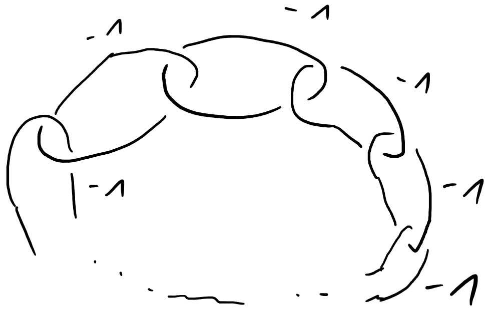
$S^3 = \mathbb{R}^3 \cup \{\infty\}$





▨

$M_n :=$



RING OF n-UNKNOTS

$\leftarrow \#(\text{UNKNOTS}) = n$

Theorem 8:

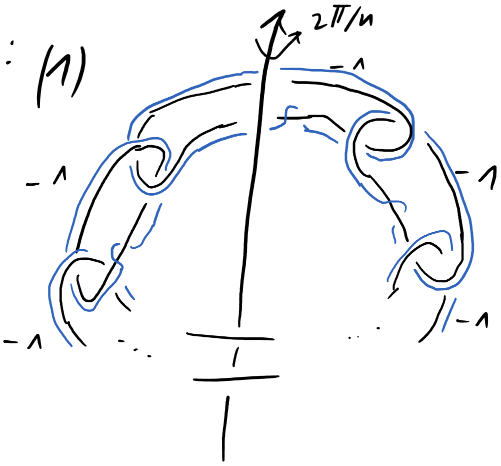
(1) $\exists$ n-fold br. cover $M_n \longrightarrow S^3$ br. along the right-handed trefoil

(2) $\exists$ 5-fold br. cover $\underset{\text{POINCARÉ HOM. SP.}}{\overset{}{P}} \longrightarrow S^3$ "

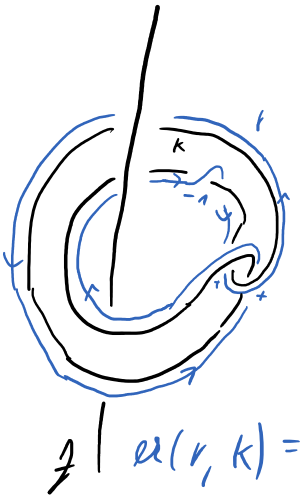
(3) $\exists$ 2-fold " $-L(3,1) \longrightarrow S^3$ "

Proof: (1)

$M_n =$



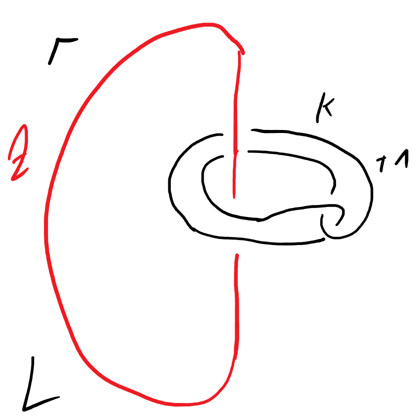
n-fold br. cover $\longrightarrow$



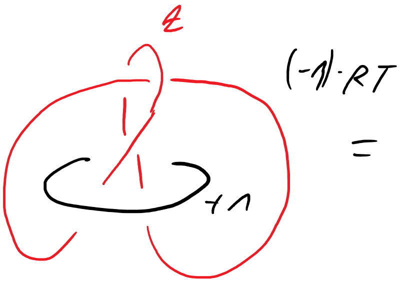
$lk(r, K) = -1+2=1$

$P: M_n \xrightarrow{n\text{-fold}} \bigcirc^{+1} \overset{RT}{=} S^3$ br. along Z

* Z is the trefoil:



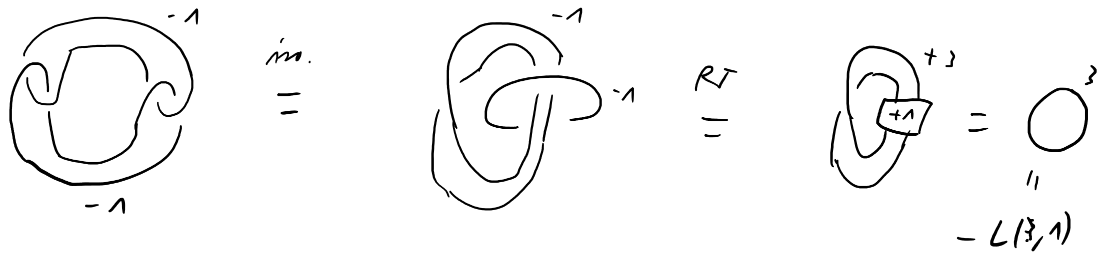
isotopy $=$



$(-1)\cdot RT$ $=$

" right-handed trefoil

(3) $M_2 =$



(2) $M_5 = P$ (see SHEET 6 Ex 2(b))

(c.f. KLO) ▨

Proof idea of T.5:

(1) CONSTRUCT A „GOOD" 3-fold br. covering $p: S^3 \longrightarrow S^3$

(2) Start with a surgery descr. of M & make it symmetric w.r.t. p via RT.

(3) conclude as in T.8. ▨

(details today in afternoon)

Ex: $\forall$ 3-mfd M is parallelisable i.e. $TM = M \times \mathbb{R}^3$

⌜ ⌝

$$\begin{tikzcd} M \arrow[d] \\ S^3 \end{tikzcd}$$

Assume br. set $\subset D^3 \subset M^3$

'' $\subset D^3 \subset M^3$

* S^3 is parallel. (X_1, X_2, X_3) 3-lin. ind VF
* lift par. to $M^3 \setminus D^3$

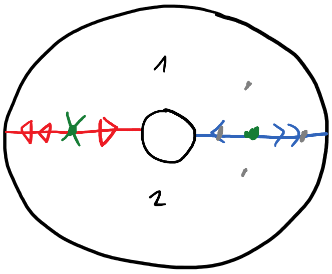
$$S^2 = \partial D^3 \xrightarrow{f} SO(3) = \mathbb{RP}^3$$

⌞ extend to D^3 $\Leftrightarrow$ $0 = \pi_2(\mathbb{RP}^3) \ni [f] = 0$ ⌟

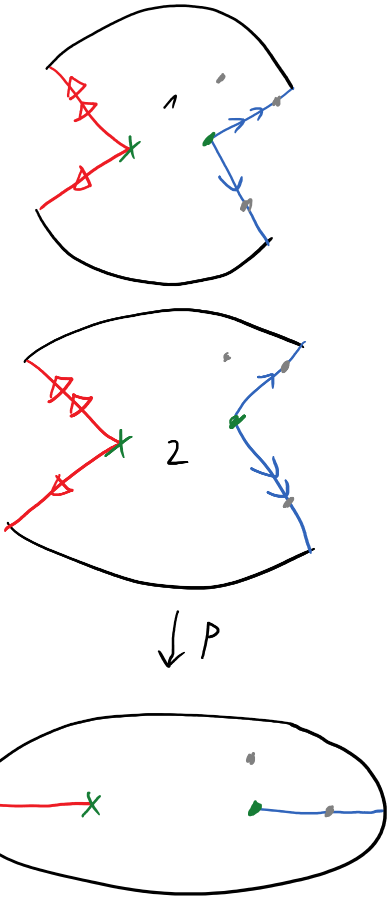
DETAILS FOR THE PROOF OF T.5:

BUILDING BLOCK 1:

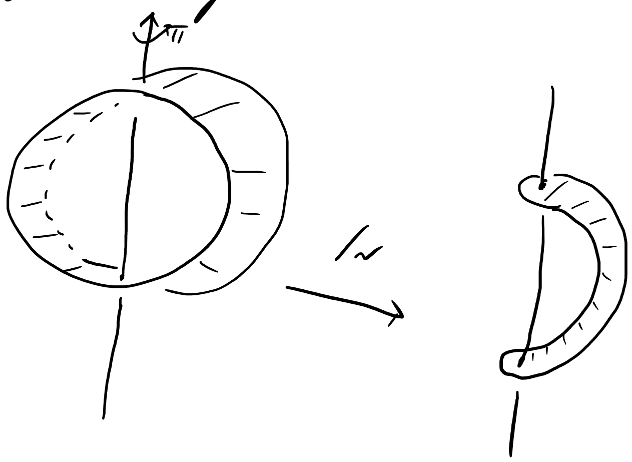
2-fold br. cov $p: S^1 \times I \longrightarrow D^2$



$\cong$



alternatively

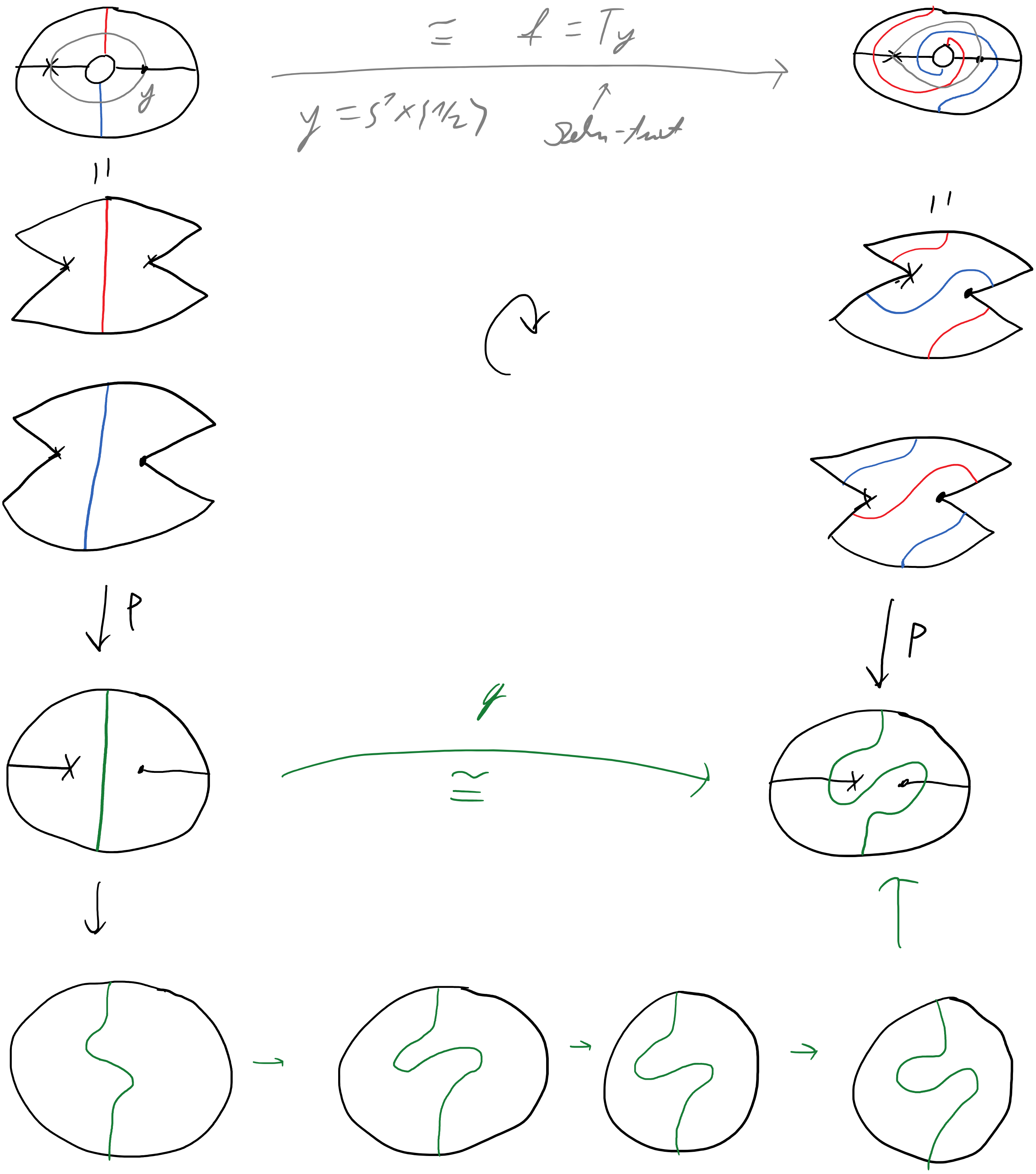


Let $g: D^2 \xrightarrow{\cong} D^2$ interchanging the branching pt by π-rotation

& $g|_{\partial D^2} \equiv id$

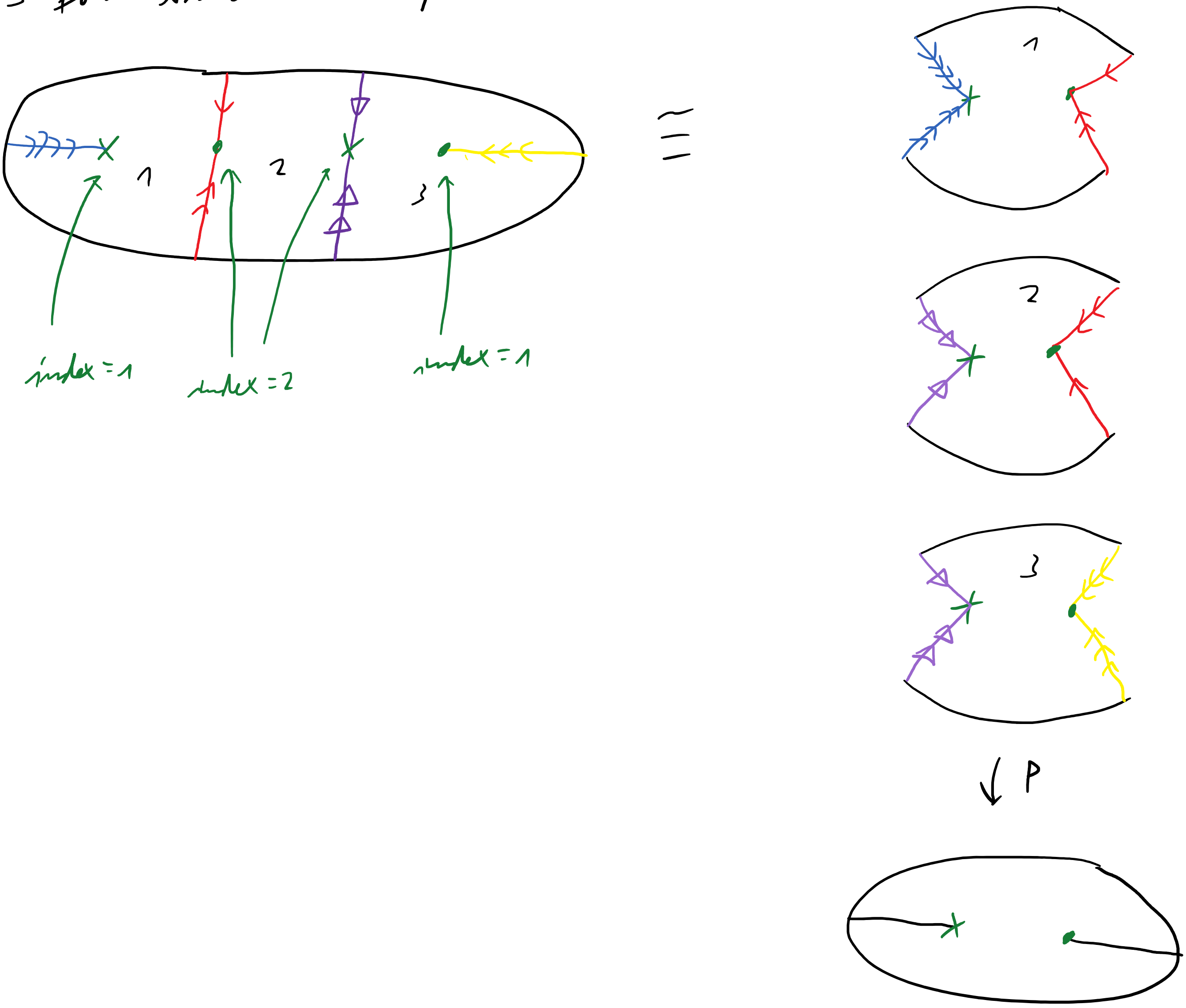
$\Rightarrow$ g lifts to $f: S^1 \times I \xrightarrow{i.e.} S^1 \times I$, i.e.

$g \circ p = p \circ f$ with $f|_{\partial(S^1 \times I)} \equiv id$



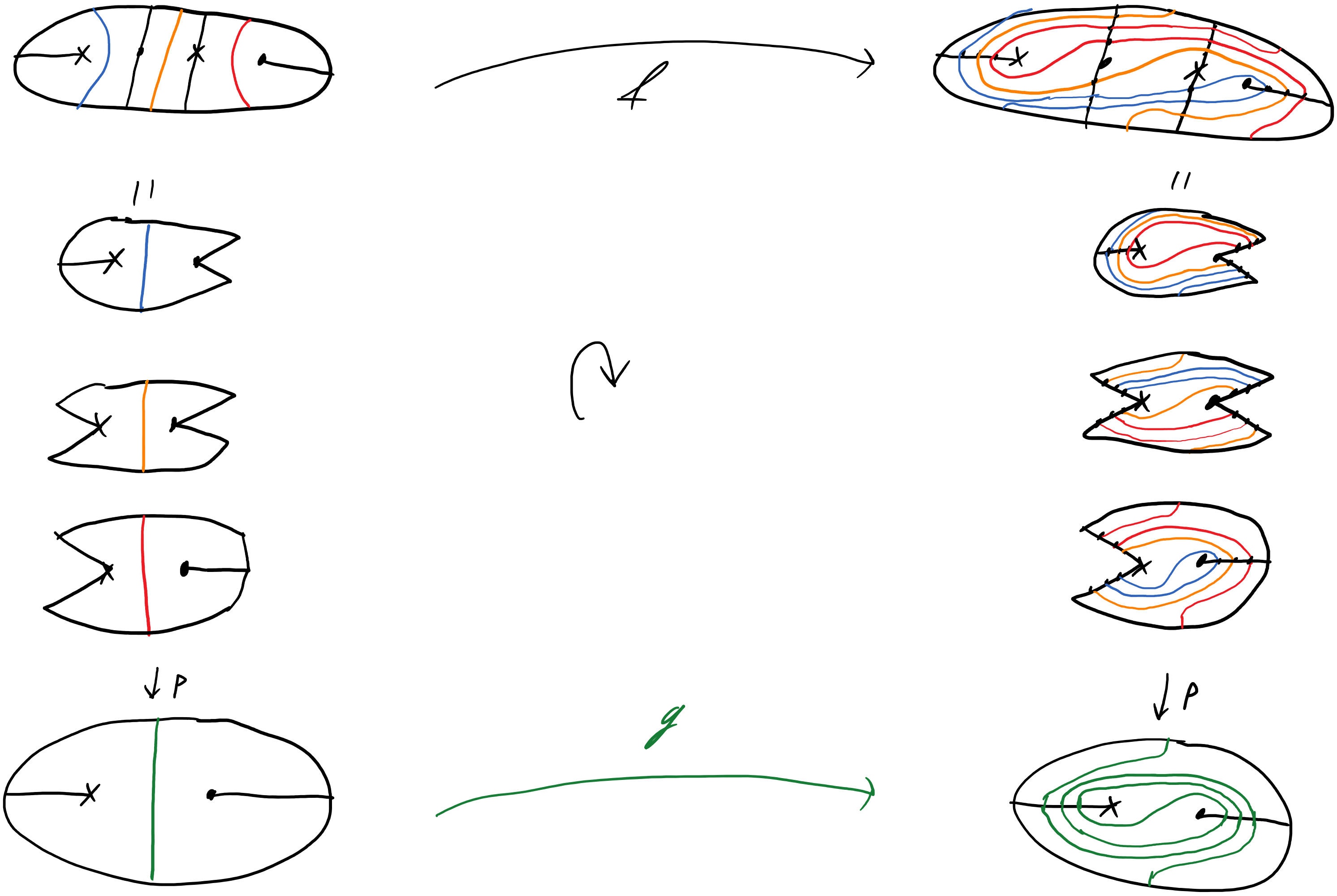
BUILDING BLOCK 2:

3-fold branched cover $p: D^2 \longrightarrow D^2$



Let $g : D^2 \xrightarrow{\cong} D^2$ interchanging the br. pts (downstairs) by a 3π-rotation & $g|_{\partial D^2} = id$

$\Rightarrow$ g lifts to $f : D^2 \xrightarrow{\cong} D^2$ (upstairs) with $f|_{\partial D^2} = id$



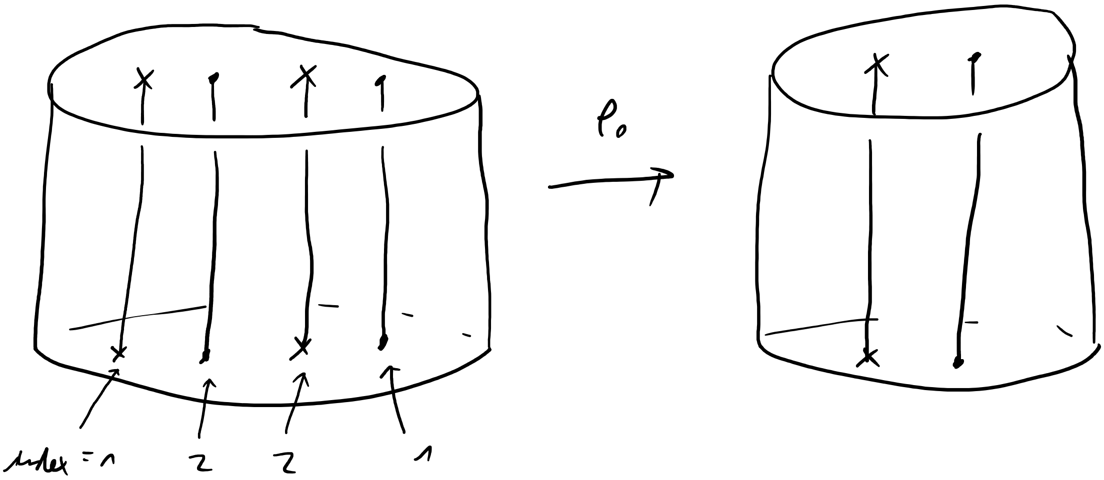
<u>Remark</u>: Rot by π & 2π do <u>NOT</u> lift to homeo fixing ∂D^2

Proof of T.5:

(i) Construct a br. covering $p_0 : S^3_u \longrightarrow S^3_d$

Building block 2 × I

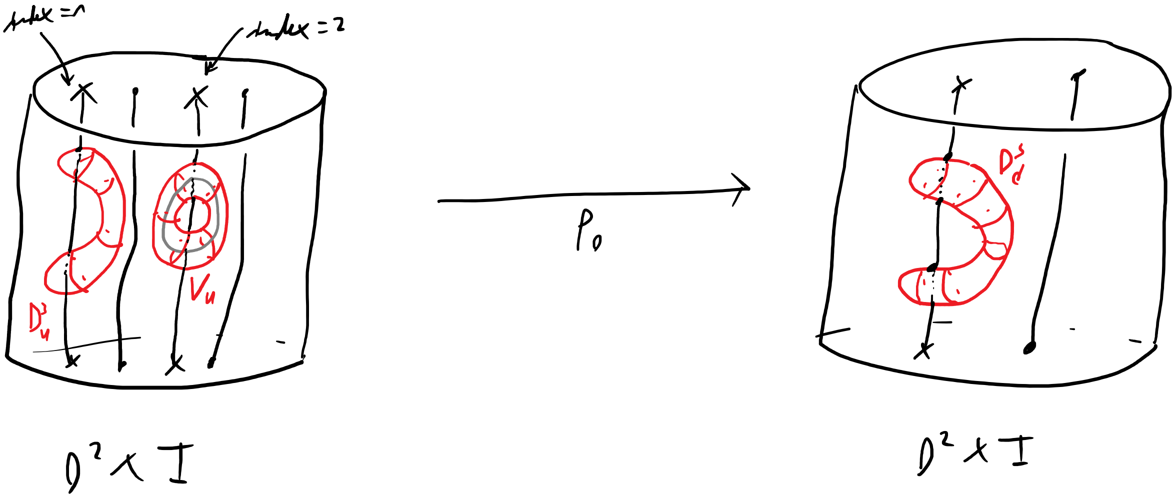
$$D^3 \cong D^2 \times I \longrightarrow D^2 \times I \cong D^3$$



$$P_0 : S^3_u = D^2 \times I \cup_{id} D^2 \times I \longrightarrow D^2 \times I \cup_{id} D^2 \times I = S^3_d$$

(ii) A (trivial) surgery along $D^3_d \subset S^3_d$ downstairs „lifts" to a Dehn-surgery upstairs:

* Choose $D^3_d \subset S^3_d$ as follows



$p_0^{-1}(D^3_d) = D^3_u \sqcup V_u$, $V_u \cong S^2\times D^2$

* Cut D^3_d out & reglue via g from building block 1:

$$S^3 = D^3 \cup S^3\setminus D^3_d$$

$$= D^2\times I \cup_{g_1} S^3\setminus D^2\times I$$

where $g_1 : \partial(D^2\times I) \xrightarrow{\cong} \partial(D^2\times I)$

$(x,1) \longmapsto (g(x),1)$; $x\in\partial D^2$

$z \longmapsto z$; else

g_1 lifts to $f_1 : \partial(S^2\times I\times I) \longrightarrow \partial(S^2\times I\times I)$

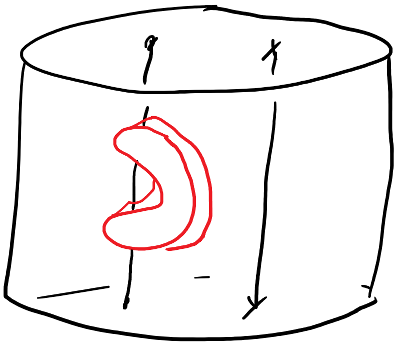
$(x,1) \longmapsto (f(x),1)$

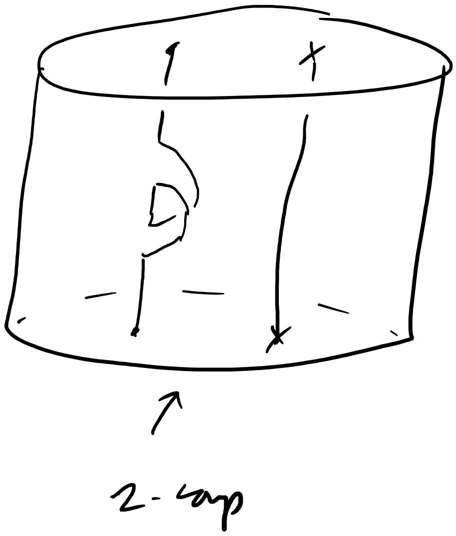
$z \longmapsto z$; else

f from building block 1

$$\begin{tikzcd}
S^3_y(\pm1) = M \arrow[d, "p_1\ (3\text{-fold})"] & = & S^3\setminus(D^3_u \sqcup V_u) \arrow[d, "p_0|_{\dots}"] & \cup_{q_1} & D^3 \arrow[d, "2d"] & \cup_{f_1} & S^2\times D^2 \arrow[dll, "{[illegible]} \times I"] \\
S^3 & = & S^3\setminus D^3_d & \cup_{q_1} & D^3 & &
\end{tikzcd}$$

* The branching set downstairs changes as:



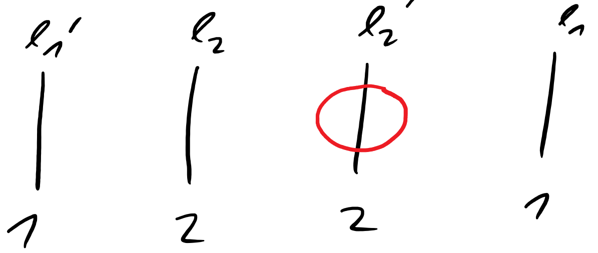


(iii) Every 3-mfd M^3 admits a surgery diagr $L(r_1,\dots,r_n)$ s.t.

(a) $L_i = O$ $\forall\, i=1,\dots,n$

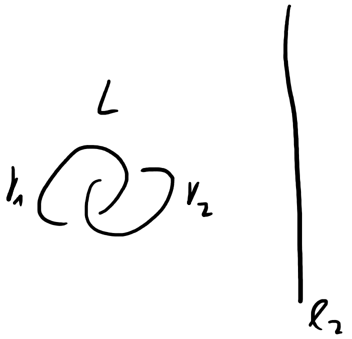
(b) $r_i = \pm 1$ $\forall\, i=1,\dots,n$

(c) $\forall\, L_i$ intersects exactly one branching knot upstairs of index $=2$ in exactly 2 pts & is unlinked from the other branching knots

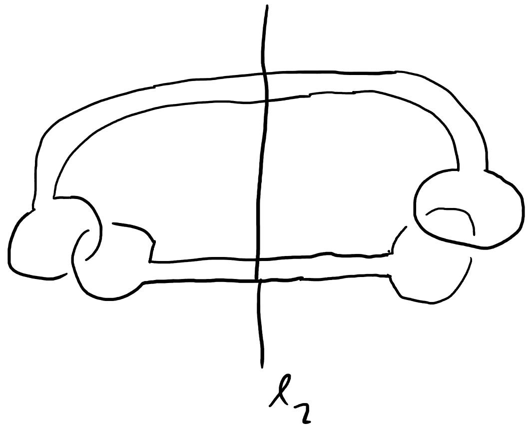


⌐ * Start with integer surgery descr. diagr. upstairs $L(r_1,\dots,r_n)$ (T.5.12)

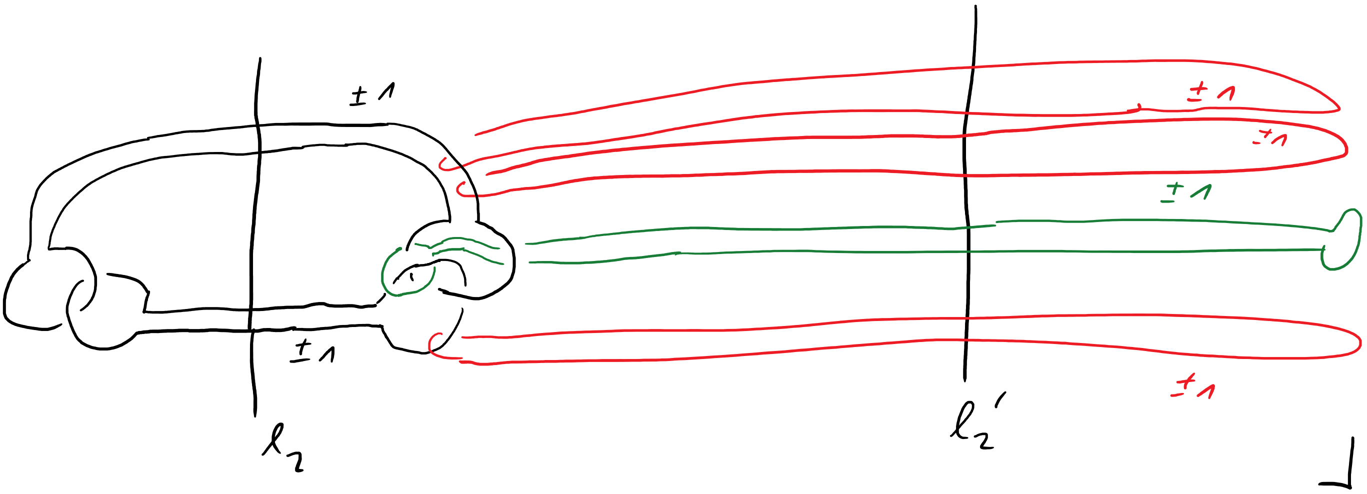
* Move L in a small 3-ball near l_2:



* Isotope L s.t. the <u>diagram</u> of L is symmetric w.r.t. l_2:

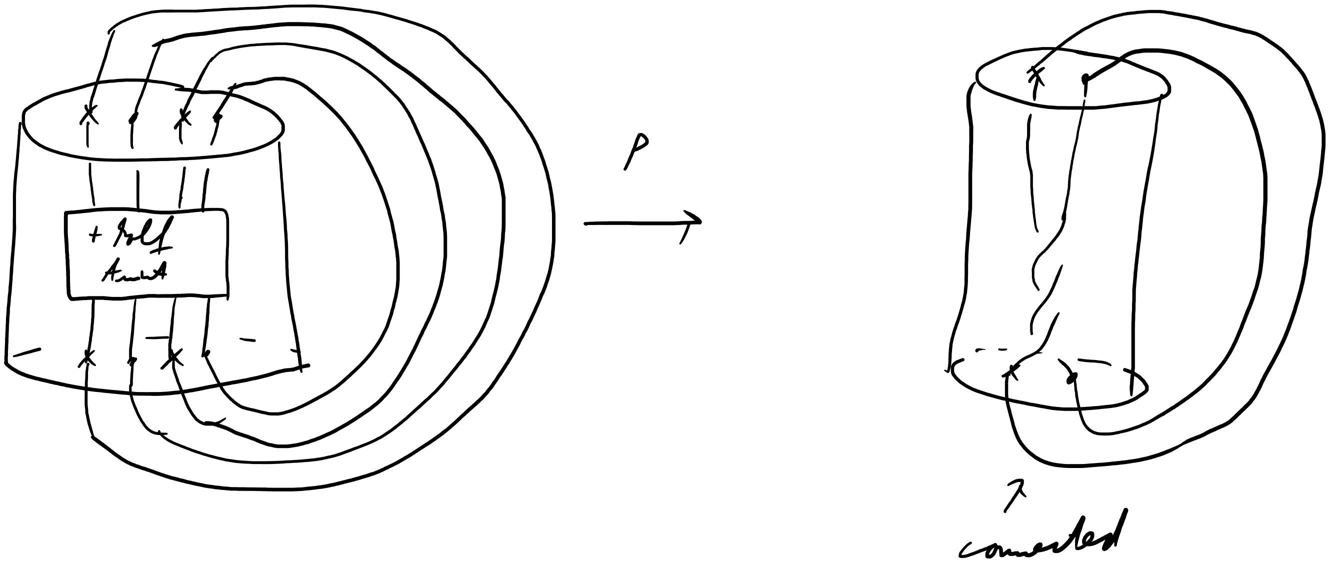


* Change crossings by RT to get a symmetric link:
& add further RT to change all coeff to ±1



(iv) The br. set downstairs can be chosen to be connected:

Replace $D^2 \times I$ via map p from building block 2 downstairs
" " f " upstairs

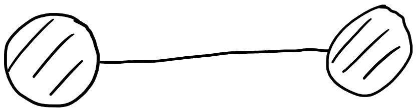


∎

7. OUTLOOK : 4-MFDS – KIRBY CALCULUS & TRISECTION

Recall: $M^3 = h_0 \cup h_1 \cup \dots \cup h_1 \cup h_2 \cup \dots \cup h_2 \cup h_3$

$\partial h_0 = \partial D^3 = S^2 = \mathbb{R}^2 \cup \{\infty\}$



HEEG. DIAG

KIRBY DIAGRAMS:

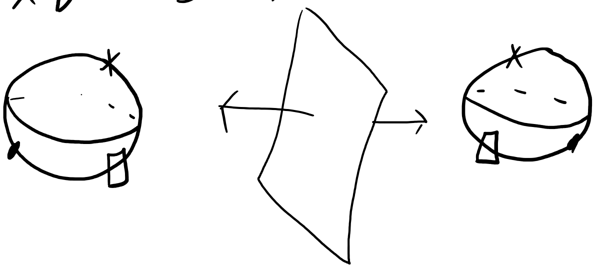
Let M^4 be a SMOOTH closed, or., con. 4-mfd with a handle decomp.

$M^4 = h_0 \cup h_1 \cup \dots \cup h_1 \cup h_2 \cup \dots \cup h_2 \cup h_3 \cup \dots \cup h_3 \cup h_4$

$\partial h_0 = \partial D^4 = S^3 = \mathbb{R}^3 \cup \{\infty\}$

* Attaching region of 1-handle

$\partial D^1 \times D^3 = S^0 \times D^3 = D^3 \sqcup D^3$

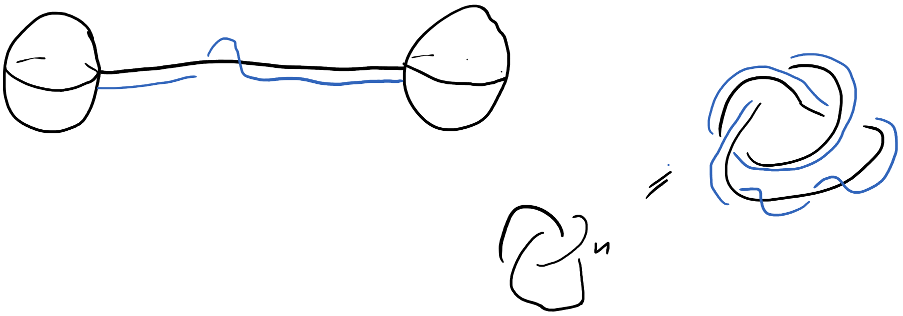


$\subset \mathbb{R}^3 \subset \partial h_0$

* Attaching region of 2-handle:

$\partial D^2 \times D^2 = S^1 \times D^2$

Draw the attaching sphere $S^1 \times \{0\}$ together with a framing $\hat{=}$ longitude



* LAUDENBACH-POENARU:

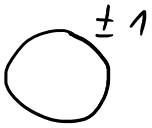
$\forall \ f: \#_k S^1 \times S^2 \xrightarrow{\overset{C^\infty}{\cong}} \#_k S^1 \times S^2$ extends to

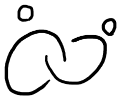
$F: \natural_k S^1 \times D^3 \xrightarrow{\overset{C^\infty}{\cong}} \natural_k S^1 \times D^3$

* $h_3 \cup \dots \cup h_3 \cup h_4 \overset{DUAL}{\cong} h_0 \cup h_1 \cup \dots \cup h_1 \overset{L3.1}{=} \natural_k S^1 \times D^3$

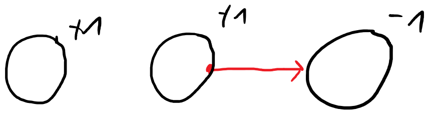
$$\begin{tikzcd}
M \arrow[d, squiggly, "C^\infty \cong"] & = & M_2 \arrow[d, "\cong"', "id"] & \cup_f \arrow[d, phantom, "\circlearrowleft"] & \natural_k S^1 \times D^3 \arrow[d, "\text{extension of } f \circ f'"] \\
M' & = & M_2 & \cup_{f'} & \natural_k S^1 \times D^3
\end{tikzcd}$$

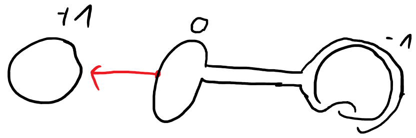
→ There is a unique way to glue in 3- & 4-handles

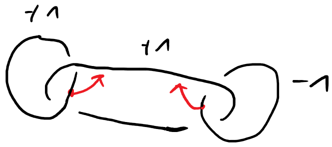
Ex:  $= \pm \mathbb{C}P^2$

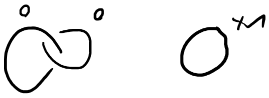
 $= S^2 \times S^2$

Ex (Handle slide) $S^2 \times S^2 \# \mathbb{C}P^2 \overset{C^\infty}{\cong} \mathbb{C}P^2 \# \mathbb{C}P^2 \# (-\mathbb{C}P^2)$

$\mathbb{C}P^2 \# \mathbb{C}P^2 \# -\mathbb{C}P^2 =$ 

$=$ 

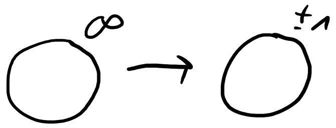
$=$ 

$=$  $= S^2 \times S^2 \# \mathbb{C}P^2$

Ex: M^4 simply con $\Rightarrow M^4 \#_k \mathbb{CP}^2 \#_l -\mathbb{CP}^2 \overset{C^\infty}{\cong} \#_n \mathbb{CP}^2 \#_m -\mathbb{CP}^2$

Proof: via KIRBY'S THM & RT

RT $\hat{=}$ $\#\pm\mathbb{CP}^2$



→ LECTURE SS 21

Recall: $\forall$ 3-mfd M^3 $\exists$ $M^3 = \natural_k S^1\times D^2 \cup_f \natural_k S^1\times D^2$

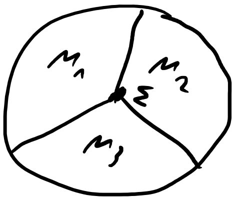
→ THIS IS NOT POS FOR 4-MFDS!

$$\natural_k S^1\times D^3 \cup_f \natural_k S^1\times D^3 \underset{\substack{\uparrow \\ \text{L.-P.}}}{\cong} \#_k S^1\times S^3$$

TRISECTION:

$\forall M^4$ $\exists$ TRISECTION, i.e.

$M^4 = M_1 \cup M_2 \cup M_3$ s.t.



- $M_i = \natural_k S^1\times D^3$
- $M_i \cap M_j \cong \natural_g S^1\times D^2$
- $M_1 \cap M_2 \cap M_3 =$ surface Σ

→ TRISECTION DIAG $(\Sigma, \alpha, \beta, \gamma)$

TRISECTION GENUS $g(M^4) := \min\{g(\Sigma) \mid \Sigma$ comes from TRIS.$\}$

EASY: $g(M_1\#M_2) \leq g(M_1)+g(M_2)$

* If $g(M_1\#M_2) = g(M_1)+g(M_2) \Rightarrow$ S4PC is true