SS 2020 Marc Kegel

Topology of 3-Manifolds

Exercise sheet 3

Exercise 1.

(a) Describe an explicit Morse function of $\mathbb{R}P^2$ inducing a handle decomposition of $\mathbb{R}P^2$ with exactly one 0-handle, one 1-handle and one 2-handle.

(b) Sketch an embedding of the surface Σ_2 of genus 2 into $\mathbb{R}^3$, such that the height function is a Morse function on Σ_2 inducing a handle decomposition of Σ_2 with exactly one 0-handle and exactly one 2-handle.

(c) Draw sketches of all handle cancellations and handle slides in dimensions 1, 2 and 3. Indicate in your sketches also the attaching spheres, the belt spheres, the cores, the cocores and the attaching regions.

Exercise 2.

(a) Use the Alexander trick to show that any manifold obtained by gluing two n-disks is homeomorphic to the n-sphere.

(b) In the lecture we constructed Milnor's exotic 7-sphere E^7 by gluing to copies of $S^3 \times D^4$ via a diffeomorphism of their boundaries. Verify that this construction defines a natural smooth structure on E^7.

(c) Describe an explicit Morse function on E^7 with exactly two critical points and conclude that E^7 is homeomorphic to S^7.
Hint: Consider the suitable scaled real part of the S^3-factor in the first copy of $S^3 \times D^4$ and try to extend that map over the second copy of $S^3 \times D^4$ (where we see S^3 again as in the lecture as the unit sphere in the quaternions). Of course one could also look into Milnor's original paper and just copy the formula and compute that it is a Morse function with the desired properties, but then you will not learn much from this exercise.

Exercise 3.
We consider the 3-torus $T^3 := S^1 \times S^1 \times S^1$.

(a) Show that we can obtain T^3 from the cube $I \times I \times I$ by identifying opposite sides.

(b) Describe a handle decomposition of T^3 (as simple as possible).

(c) Draw a planar Heegaard diagram of T^3.

Exercise 4.

(a) Describe a way to compute the fundamental group of a manifold with a given handle decomposition.

(b) The fundamental group of a compact smooth manifold is finitely presented. Conversely, we can get for any $n \geq 5$ any finitely presented group as the fundamental group of a closed oriented n-manifold.

Challenge: Can you show the same for $n = 4$?

(c) On the other hand, not every finitely presented group occurs as the fundamental group of a closed orientable 3-manifold. Groups arising as the fundamental group of a closed orientable 3-manifolds are called **3-manifold groups**.
Hint: Let $\langle g_1, \dots g_n | r_1, \dots r_k \rangle$ be a finite presentation of a group G. We call $n - k$ the deficiency of this presentation. The **deficiency** of a finitely presented group G is the maximum deficiency of a finite presentation for G. Then you need to show that any 3-manifold group has non-negative deficiency and find a group with negative deficiency.

Bonus exercise.

(a) Use Cerf's theorem to show that the number of handles modulo two in a handle decomposition of a manifold M is an invariant of M and deduce that the 2-sphere admits a handle decomposition with an arbitrary even number of handles, but no handle decomposition with an odd number of handles.

(b) Find a formula for computing the Euler characteristic from a handle decomposition and deduce that a closed oriented 3-manifold has vanishing Euler characteristic. Show that your formula for the Euler characteristic is independent of the chosen handle decomposition and thus an invariant of the manifold by using Cerf's theorem.

(c) Describe a way to compute the homology groups of a compact manifold M with a given handle decomposition.

Challenge: Use Cerf's theorem to verify that the homology groups are independent of the handle decomposition and only depend on M without recourse to the invariance of singular or simplicial homology.
Hint: See S. Durst, H. Geiges and M. Kegel, Handle homology of manifolds, *Topology Appl.*, **256** (2019), 113–127.

Bonus exercise.
Use handle decompositions (or more precisely Kirby diagrams of 1- and 2-manifolds) to classify closed (orientable) 1- and 2-manifolds. Can you distinguish all 1- and 2-manifolds by just using handle decompositions (without using invariants from algebraic topology like homology, fundamental group...)?

This sheet will be discussed in the exercise session on 29.5. and should be solved by then.