SS 2020 Marc Kegel

Topology of 3-Manifolds

Exercise sheet 5

Exercise 1.
Let K be a knot in a connected closed oriented 3-manifold M.

(a) There exists a Heegaard splitting of M such that K lies on its Heegaard surface.

(b) Compute the homology class of K in $H_1(M;\mathbb{Z})$ from a Heegaard splitting $(\Sigma_g;\beta_1,\dots,\beta_g)$ of M swith $K\subset\Sigma_g$.

(c) Describe non-nullhomologous knotss in planar Heegaard diagrams of the lens spaces $L(p,1)$ and $S^1\times S^2$. Which homological order have these knots? Show that these knots do **not** admit Seifert surfaces.
Remark: Later we will show, that a knot admits a Seifert surface if and only if it is nullhomologous.

Exercise 2.

(a) Any orientation preserving homeomorphism of S^1 is isotopic to the identity.

(b) Let V be a solid torus. A homeomorphism of ∂V extends to a homeomorphism of V if and only if the meridian μ gets mapped to a curve which is isotopic to $\pm\mu$.

(c) A Dehn twist along a non-separating curve on ∂V is not isotopic to the identity, i.e. represents a non-trivial element in the mapping class group.

Exercise 3.
Determine the isomorphism type of the mapping class group of the annulus $S^1\times I$ and the 2-torus T^2.

Exercise 4.
Fill in the details in the proof sketch of the lantern relation from the lecture and verify that the chain relation holds.

Bonus exercise.
Express the hyperelliptic involution and the order g homeomorphism of the genus g surface from the lecture as a composition of Dehn twists.

Challenge.
Deduce from Theorem 4.9 from the lecture that the mapping class group admits a finite presentation.
Hint: See W. Lickorish: A finite set of generators for the homeotopy group of a 2-manifold, *Proc. Cambridge Philos. Soc.* **60** (1964), 769–778.
Bonus: Describe an explicit finite presentation of the mapping class group.

This sheet will be discussed in the exercise session on 26.6. and should be solved by then.