

# Topology II

## Exercise sheet 12

### Exercise 1.

- (a) Let  $M$  be a closed manifold of odd Euler characteristic. Show that there exists no compact manifold  $W$  with boundary  $\partial W$  homeomorphic to  $M$ .
- (b) Find a closed manifold  $M$  in every even dimension that does not occur as the boundary of a compact manifold.
- (c) For every closed manifold  $M$  construct a **non-compact** manifold  $W$  with boundary  $\partial W$  homeomorphic to  $M$ .
- (d) Can the Klein bottle  $\mathbb{R}P^2 \# \mathbb{R}P^2$  appear as the boundary of a compact 3-manifold  $M$ ?
- (e) We call two closed  $n$ -manifolds  $M$  and  $N$  **cobordant** if there exists a compact  $(n+1)$ -manifold  $W$  whose boundary  $\partial W$  is the disjoint union of  $M$  and  $N$ . Show that this defines an equivalence relation.
- (f) The connected sum  $M \# N$  is cobordant to the disjoint union of  $M$  and  $N$  and the disjoint union of  $M$  and  $M$  is **nullcobordant**, i.e. cobordant to the empty set.
- (g) Which closed surfaces are cobordant?
- (h) The set  $\mathfrak{N}_n$  of cobordism classes of closed  $n$ -manifolds is an abelian group with the disjoint union as operation and the cartesian product of manifolds defines a multiplication

$$\begin{aligned} \mathfrak{N}_n \times \mathfrak{N}_m &\longrightarrow \mathfrak{N}_{n+m} \\ ([N^n], [M^m]) &\longmapsto [N^n \times M^m] \end{aligned}$$

which defines a ring structure on

$$\mathfrak{N}_* = \bigoplus_{n \in \mathbb{N}_0} \mathfrak{N}_n.$$

### Exercise 2.

A compact connected manifold  $M$  does not retract onto its connected boundary  $\partial M$ .

*Hint:* Consider the long exact sequence of the pair  $(M, \partial M)$ .

**Bonus exercise 1.**

A map  $f: X \rightarrow Y$  is called **proper** if the preimage of every compact subset of  $Y$  is a compact subset of  $X$ .

- (a) Describe a proper and a non-proper map  $f: \mathbb{R} \rightarrow \mathbb{R}$ .
- (b) Every map from a non-compact space to a compact space is **not** proper.
- (c) Every homeomorphism and every map from a compact space to a Hausdorff space is proper.
- (d) A proper map  $f: X \rightarrow Y$  induces a cochain map

$$f^*: C_c^k(Y; G) \rightarrow C_c^k(X; G)$$

and thus also a well-defined homomorphism

$$f^*: H_c^k(Y; G) \rightarrow H_c^k(X; G).$$

- (e) Homeomorphic topological spaces have isomorphic cohomology groups with compact support.
- (f) Show again that  $\mathbb{R}^n$  is homeomorphic to  $\mathbb{R}^m$  if and only if  $n = m$ .

**Bonus exercise 2.**

Describe an immersion of  $\mathbb{R}P^2$  into  $\mathbb{R}^3$  and describe how to get an embedding into  $\mathbb{R}^4$  from it.

*Hint:* The immersion can be described for example by drawing a detailed figure of its image after watching the Youtube video [J. LEYS: *The Boy surface*] or reading [R. KIRBY: *What is ... Boy's Surface*, Notices of the AMS, **54** (2007), 1306–1307].

Another method is to build an explicit 3-dimensional model, see for example [A. CHÉRITAT: *A model of Boy's surface in constructive solid geometry*, available on his webpage].

If you are not interested in understanding the construction you can alternatively prove that

$$f: D^2 \longrightarrow \mathbb{R}^3 \\ w \longmapsto \frac{1}{g_1^2 + g_2^2 + g_3^2} (g_1, g_2, g_3)$$

induces an immersion  $\mathbb{R}P^2 \rightarrow \mathbb{R}^3$ , where

$$g_1 := -\frac{3}{2} \operatorname{Im} \left[ \frac{w(1-w^4)}{w^6 + \sqrt{5}w^3 - 1} \right] \\ g_2 := -\frac{3}{2} \operatorname{Re} \left[ \frac{w(1+w^4)}{w^6 + \sqrt{5}w^3 - 1} \right] \\ g_3 := \operatorname{Im} \left[ \frac{1+w^6}{w^6 + \sqrt{5}w^3 - 1} \right] - \frac{1}{2}.$$

This sheet will be discussed on Friday 12.2. and should be solved by then.