WS 2020/2021 Marc Kegel

Topology II

Exercise sheet 13

Exercise 1.
Let $Q\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ be a symmetric bilinear form and let e_i be the standard generators of $\mathbb{Z}^n$.

(a) Q is non-singular if and only if its representing matrix $\big(Q(e_i, e_j)\big)_{1\leq i,j\leq n}$ is invertible over $\mathbb{Z}$ (which is equivalent to having determinant ± 1).

(b) Describe the matrices (in a suitable basis) of the intersection forms of $\pm\mathbb{C}P^2$, $S^2 \times S^2$ and compute its ranks, signatures, parities and definiteness.

(c) Show that E_8 represents a non-singular symmetric positive definite bilinear form and compute its rank, signature and parity.

(d) Show that the intersection forms H and $-H$ are isomorphic.

(e) Show that the intersection forms H and $[+1] \oplus [-1]$ are isomorphic over $\mathbb{R}$ but not over $\mathbb{Z}$ and conclude that $S^2 \times S^2$ is not homeomorphic to $-\mathbb{C}P^2 \# \mathbb{C}P^2$, where $-\mathbb{C}P^2$ denotes $\mathbb{C}P^2$ with opposite orientation.

Exercise 2.

(a) Use Corollary 8.3 and the intersection form to compute the ring structure of $\mathbb{C}P^n$.

(b) Show that $\mathbb{C}P^{2m}$ admits no orientation reversing diffeomorphism.

(c) Show that any map $\mathbb{C}P^{2m} \to \mathbb{C}P^{2m}$ has a fixed point.
Hint: Use the Lefschetz fixed point theorem.

Bonus: What can you say about maps $\mathbb{C}P^{2m+1} \to \mathbb{C}P^{2m+1}$?

Exercise 3.
Show that $\mathbb{C}P^2 \# (-\mathbb{C}P^2)$ is homeomorphic to an S^2-bundle over S^2 and use this to deduce results about its homotopy groups.

Exercise 4.
Show that $S^2 \vee S^4$ is not homotopy equivalent to a manifold.

Bonus exercise.

(a) For every integer $k \in \mathbb{Z}$ there exists a map $T^2 \to T^2$ of degree k.
Hint: Remember our construction of maps $S^n \to S^n$ of arbitrary degree.

(b) Now we consider a general genus g surface Σ_g with $g \geq 2$. Construct maps $\Sigma_g \to \Sigma_g$ with degree 0, 1 and -1.

(c) Any map $\Sigma_g \to \Sigma_g$ of non-vanishing degree 0 induces a surjection on fundamental groups.
Hint: Lift the map to a suitable covering of Σ_g, deduce from the non-vanishing of the degree that this covering has to be finite and use the behavior of the Euler characteristic under finite coverings.

(d) Deduce that any map $\Sigma_2 \to \Sigma_2$ has degree 0, 1 or -1.
Hint: Use (c) together with the Hurewicz homomorphism and the cup product structure of Σ_2.

(e) Show the statement from part (d) for arbitrary genus $g \geq 2$.

This sheet will be discussed on Friday 19.2. and should be solved by then.