WS 2020/2021 Marc Kegel

Topology II

Exercise sheet 6

Exercise 1.
Compute the simplicial homology groups of the n-sphere, the Möbius strip and the Klein bottle using directly the definition of simplicial homology.

Exercise 2.
We call a *smooth* manifold M **orientable** if M admits an atlas in which all charts are compatible and the determinant of the Jocobi matrices of all transition maps (i.e. $\psi \circ \phi^{-1}$ for charts ψ and ϕ) are everywhere positive.

(a) Show that S is orientable by explicitly describing such an atlas.

(b) The Möbius strip is not orientable.

(c) A surface is orientable if and only if it contains no Möbius strip, i.e. if and only if there exists no closed path interchanging right and left.

Exercise 3.
An oriented q-simplex $\sigma = (x_0, \dots, x_q)$ **induces** an orientation on any of its $(q-1)$-dimensional faces τ via

$$\tau = (-1)^i (x_0, \dots, \hat{x}_i, \dots, x_q).$$

We call a triangulated n-manifold **orientable**, if there exists orientations on the n-simplices such that any two adjacent n-simplices induce opposite orientations on their common $(n-1)$-dimensional face.

(a) Draw sketches in Dimensions 2 and 3.

(b) Show that this definition of orientability coincides with the definition from Exercise 2 on smooth manifolds.

(c) Let M be a smooth compact n-manifold. Show that

$$H_n(M, \partial M) \cong \begin{cases} \mathbb{Z}; & \text{if } M \text{ is orientable,} \\ 0; & \text{if } M \text{ is not orientable.} \end{cases}$$

Hint: It might be helpful to work with simplicial homology and start with an explicit triangulation of the 2-torus and to identify an explicit 2-cycle generating the second homology. Next, one can consider the Klein bottle. Does there exists a 2-cycle on the Klein bottle? Finally, try to generalize these arguments.

Exercise 4.
The **degree** $\deg(f)$ of a map $f\colon S^n \to S^n$ is defined by

$$\begin{aligned} f_*\colon H_n(S^n) &\longrightarrow H_n(S^n) \\ [S^n] &\longmapsto \deg(f)[S^n]. \end{aligned}$$

(a) Compute the degree of the constant map $\mathrm{c}\colon S^n \to S^n$, the identity $\mathrm{Id}\colon S^n \to S^n$ and the antipodal map $\text{-Id}\colon S^n \to S^n$.

(b) Construct, for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, a map $f\colon S^n \to S^n$ of degree k.
Hint: Construct from a given map $S^{n-1} \to S^{n-1}$ a map $S^n \to S^n$ of the same degree by identifying S^n with the suspension ΣS^{n-1}.

Bonus exercise.

(a) Classify compact 1-manifolds (possibly with boundary).

(b) Workout the details from the proof sketch of the classification theorem of surfaces.
Hint: It might be helpful to have a look at Chapter 5 of:
`https://www.mathematik.hu-berlin.de/~kegemarc/19SSTopologie/Skript.pdf`

Bonus exercise.

(a) Prove a version of the Mayer–Vietoris sequence for simplicial homology groups.

(b) Present a geometric description of the connecting homomorphism in the Mayer–Vietoris sequence.

This sheet will be discussed on Friday 11.12. and should be solved by then.