SS 2021 Marc Kegel

4-Manifolds and Kirby calculus

Exercise sheet 1

The goal of this exercise sheet is to give a proof of the following theorem via Kirby calculus of surfaces.

Theorem 0.
For any connected, closed, orientable surface F there exists exactly one $k \in \mathbb{N}_0$ such that F is homeomorphic to $\#_k T^2$, the k-fold connected sum of 2-tori. (Here we define $\#_0 T^2$ to be S^2.)

Exercise 1.

(a) Describe handle decompositions of S^2 and T^2 with exactly one 0-handle and exactly one 2-handle.

(b) Let F_1 and F_2 be surfaces with handle decompositions with exactly one 0-handle and exactly one 2-handle. Describe a handle decomposition of $F_1 \# F_2$.
Hint: Choose disks along which to perform the connected sum that intersect the 0- and the 2-handles in 'half-disks'.

Exercise 2.

(a) Any closed, connected 1-manifold is homeomorphic to S^1.

(b) Any connected, closed surface admits a handle decomposition with exactly one 0-handle and exactly one 2-handle. In the following we will consider only such handle decompositions.

Hint: Here we assume that any closed surface (and 1-manifold) admits a triangulation. Then we can get a handle decomposition by taking regular neighborhoods of the simplicies. An acessible proof of the statement that any surface admits a triangulation can be found in:
A. Hatcher, *The Kirby torus trick for surfaces*, `arXiv:1312.3518`.

Exercise 3.

(a) Let F be a compact surface and $\phi_1, \phi_2 \colon \partial D^1 \times D^1 \to \partial F$ be embeddings and $h \colon F \to F$ be a homeomorphism of F such that $h \circ \phi_1 = \phi_2$. Show that the results of attaching a 1-handle h_1 to F via ϕ_1 or ϕ_2 are homeomorphic.

(b) Any homeomorphism $S^1 \to S^1$ extends to a homeomorphism $D^2 \to D^2$.

(c) Let F be a compact surface and $\phi_1, \phi_2 \colon \partial D^2 \times \{0\} \to \partial F$ be embeddings with equal images $\phi_1(\partial D^2) = \phi_2(\partial D^2)$ then the surfaces obtained from F by attaching a 2-handle via ϕ_1 or ϕ_2 are homeomorphic.
Hint: Use Exercise (b).

p.t.o.

Exercise 4.

(a) Embeddings of $[-1,1] \to \mathbb{R}$ are exactly the strictly monotonic functions $[-1,1] \to \mathbb{R}$.

(b) Let $\phi_1, \phi_2 \colon [-1,1] \to \mathbb{R}$ be strictly increasing functions. Show that there exists a homeomorphism $h \colon \mathbb{R}^2_- \to \mathbb{R}^2_-$ such that $h = \mathrm{Id}$ away from a compactum and $h \circ \phi_1 = \phi_2$. Here we see $\mathbb{R}$ as the boundary of $\mathbb{R}^2_- := \{(x,y) \mid y < 0\}$.
Hint: Show first the existence of such a homeomorphism $\mathbb{R} \to \mathbb{R}$.

(c) Let F be a handle decomposition of a surface F. Show that we can isotope the handle decomposition such that all 1-handles are attached to ∂h_0. In the following we will consider only such handle decompositions.
Hint: Use Exercises $3(a)$, $4(a)$ and $4(b)$.

Exercise 5.
Let F be a surface with a handle decomposition. We choose a point ∞ on $\partial h_0 = S^1$ that lies in the complement of the attaching regions of the 1-handles and idetify $\partial h_0 \setminus \{\infty\}$ with $\mathbb{R}$. We draw the attaching spheres $\partial D^1 \times \{0\}$ of the 1-handles on $\mathbb{R}$ an mark pairs of points that belong to the same 1-handle h_1 with the same letter. We call the resulting 1-dimensional diagram a **Kirby diagram** of F.

(a) Draw Kirby diagrams of S^2, T^2 and explain how the connected sum operation looks in Kirby diagrams.

(b) A Kirby diagram of F uniquely determines a handle decomposition of F and thus the homeomorphism type of F.

(c) Discuss possible changes on a Kirby diagram that do not change the homeomorphism type of a surface.
Hint: It is possible to 'slide' a 1-handle over another 1-handle and one can move a 1-handle through ∞. How do these operation look in Kirby diagrams? To show that these moves do not change the homeomorphism type of the surface use exercises 3 and 4.

(d) Prove Theorem 0 by transforming a Kirby diagram of an arbitrary handle decomposition of a surface F into a Kirby diagram of $\#_k T^2$.
Bonus: Can you show that the integer k in Theorem 0 is unique without using invariants from algebraic topology?

(e) Deduce the Poincaré conjecture in dimension 2, i.e. show that any closed surface that is homotopy equivalent to S^2 is homeomorphic to S^2.

Bonus exercise.

(a) Draw Kirby diagrams of $\mathbb{R}P^2$ and the Klein bottle.

(b) Show that the Klein bottle is homeomorphic to $\mathbb{R}P^2 \# \mathbb{R}P^2$.

(c) Show that $T^2 \# \mathbb{R}P^2$ is homeomorphic to $\#_3 \mathbb{R}P^2$.

(d) Classify connected closed non-orientable surfaces.

This sheet will be discussed on Friday 23.4. and should be solved by then.