SS 2021 Marc Kegel

4-Manifolds and Kirby calculus

Exercise sheet 4

Exercise 1.

(a) Let D_K be a regular diagram of an oriented knot K in S^3. The **blackboard framing** of K with respect to the diagram D_K is given by the parallel knot K_{bb}, that is obtained from D_K by pushing K inside the projection plane away from itself. Show that $\operatorname{lk}(K, K_{bb}) = \operatorname{writhe}(D_K)$, where $\operatorname{writhe}(D_K)$ denotes the number of self-crossings of K in D_K (counted with signs).

(b) Compute the linking numbers of the oriented knots from the Kirby diagrams of $S^1 \times S^2$ shown in Figure 1. To do this, first consider how to compute the homology class of a knot in the standard Kirby diagram of $S^1 \times S^2$ and verify that the knots in Figure 1 are all nullhomologous.

(c) Show by example that the linking number $\operatorname{lk}(K_1, K_2)$ is not well-defined if one (or both) of the knots is not nullhomologous.

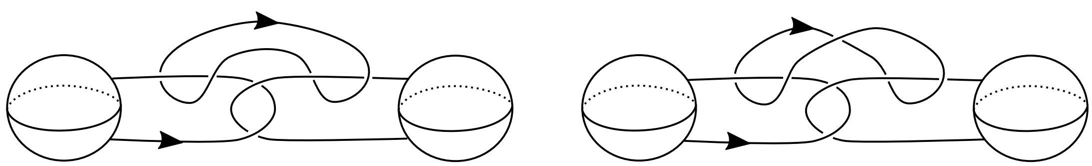

Abbildung 1: Oriented knots in $S^1 \times S^2$.

Exercise 2.

(a) Calculate the intersection form and signature of the 4-torus T^4.

(b) How do you get the intersection form of a connected sum from the intersection forms of the individual summands? How do you get the signature?

(c) The intersection form of a closed oriented 4-manifold is unimodular.

Bonus: The intersection form Q_W of a compact oriented 4-manifold is unimodular if and only if every boundary component of W is a homology sphere.

Exercise 3.

Show that any homology class $a \in H_2(W; \mathbb{Z})$ in a compact, smooth, oriented 4-manifold W can be realized as an embedded surface Σ_a in W.
Hint: Generalize the proof from the lecture for 2-handlebodies.
Bonus: Find a presentation of $H_2(W; \mathbb{Z})$ of a 4-manifold W consisting of a 0-handle and any number of 1- and 2-handles, starting from a description of W as a Kirby diagram. How do you compute the representing matrix of the intersection form of W with respect to this presentation of $H_2(W; \mathbb{Z})$?

Exercise 4.

(a) Show that every symmetric bilinear form $Q\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ is the intersection form of a smooth, simply-connected, compact 4-manifold. Describe Kirby diagrams of these manifolds without 1-handles.

(b) Find Kirby diagrams of simply-connected, closed 4-manifolds W whose intersection form in a basis of H_2 is given by a matrix of the form

$$\begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix}.$$

(c) Any of these matrices transforms after a base change of $H_2(W, \mathbb{Z})$ to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Exercise 5.

Show that the Kirby diagrams from Figure 2 describe the same compact 4-manifolds with boundary.

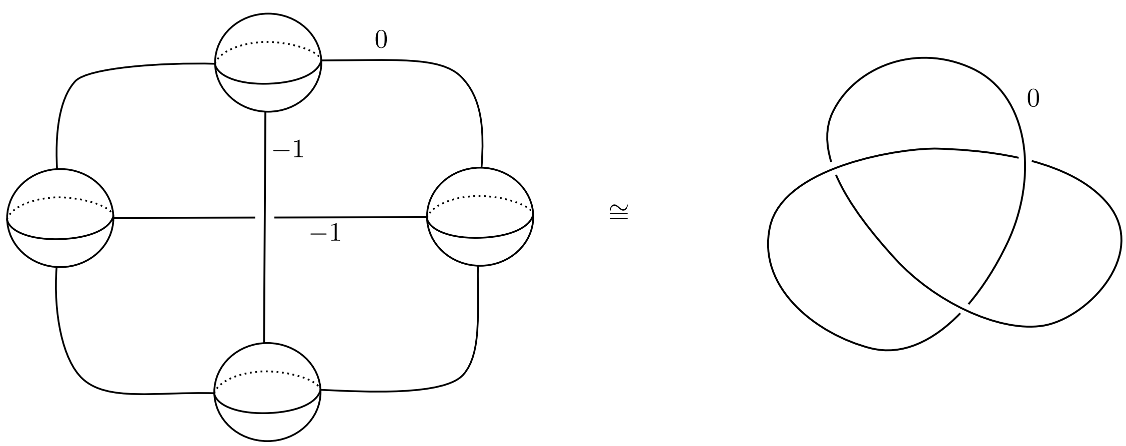

Abbildung 2: Two different Kirby diagrams of the same compact 4 manifold with boundary. The framing coefficients of the 2-handles in the left Kirby diagram are described with respect to the blackboard framing (and thus are not isotopy invariant).

This sheet will be discussed on Friday 4.6. and should be solved by then.