WS 2021/2022

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Topology II

Exercise sheet 11

Exercise 1.
Let X and Y be connected CW-complexes. We denote by $p_X\colon X\vee Y\to X$ and $p_Y\colon X\vee Y\to Y$ the projection maps.

(a) Show that $p_X^*\oplus p_Y^*\colon H^k(X;R)\oplus H^k(Y;R)\to H^k(X\vee Y;R)$ is an isomorphism for all $k\in\mathbb{N}$.

(b) The cup product $p_X^*(\alpha)\cup p_Y^*(\beta)$ is vanishing for all α and β of non-trivial degree.

(c) Compute the cup product on the cohomology $H^*(\Sigma_2)$ of the genus 2 surface Σ_2.
Hint: Consider maps $\Sigma_2\to T^2$ and $\Sigma_2\to T^2\vee T^2$ and use the calculation of the cup product of T^2 from the lecture.

Bonus: What is the cup product of a general genus-g surface Σ_g?

Exercise 2.
Let M and N be closed oriented n-manifolds and $f\colon M\to N$ a map. Then the induced map on cohomology $f^*\colon H^n(N;G)\to H^n(M;G)$ is the multiplication by $\deg(f)$.

Exercise 3.

(a) Show that $\mathbb{R}P^3$ and $\mathbb{R}P^2\vee S^3$ have isomorphic homology and cohomology groups but different cohomology rings and thus are not homotopy equivalent.

(b) Use the cup product to show that there is no map $\mathbb{R}P^n\to\mathbb{R}P^m$ inducing a nontrivial map on first cohomology with $\mathbb{Z}_2$-coefficients if $n>m$.

(c) Deduce from (b) the Borsuk–Ulam theorem: For every map $f\colon S^n\to\mathbb{R}^n$ there is a point $x\in S$ such that $f(x)=f(-x)$.

Exercise 4.
Show that the following statements are equivalent to the Borsuk–Ulam theorem.

(i) Every antipodal map $S^n\to\mathbb{R}^n$ admits a zero.

(ii) There exists no antipodal map $S^n\to S^{n-1}$.

(iii) There exist no antipodal map $D^n\to S^{n-1}$ that is antipodal on the boundary.

Deduce the Brouwer fixed point theorem from the Borsuk–Ulam theorem.

Bonus exercise 1.
Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of finitely generated abelian groups. We assume that A_1 is free abelian. Show that there exists a connected CW-complex X such that for any $k \in \mathbb{N}$ we have $H^k(X) \cong A_k$.

Remark: In contrast to homology groups, not every sequence of abelian groups can occur as cohomology groups of spaces. In [D. Kan and G. Whitehead, *On the realizability of singular cohomology groups,* Proc. Amer. Math. Soc. **12** (1961), 24–25] it is shown that there is no space X such that $H^k(X) = 0$ and $H^{k+1}(X) \cong \mathbb{Q}$.
It is unknown if $\mathbb{Q}$ can occur as cohomology group of a topological space at all.

Bonus exercise 2.
Let X be a CW-complex. The map

$$H_k(X) \times H_l(X) \xrightarrow{\times} H_{k+l}(X \times X) \xrightarrow{\mathrm{pr}^*} H_{k+l}(X)$$

is trivial, where $\times$ denotes the cross product and pr denotes the projection to one of the X-factors.

This sheet will be discussed on Wednesday 19.1. and should be solved by then.