WS 2021/2022

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Topology II

Exercise sheet 8

Exercise 1. Compute the cellular homology of closed surfaces.
Hint: An easy way to construct a cell decomposition of a manifold, is by starting with a handle decomposition and then following the argument from the lecture to construct a cell decomposition out of the handle decomposition. The cellular boundary map can be computed using the local degree.

Exercise 2.
We denote by F a 2-torus with two open 2-balls removed.
The **surface of infinite genus** Σ_∞ is defined to be the direct limit of the nested sequence of topological spaces $X_0 \subset X_1 \subset X_2 \subset \cdots$ where X_0 is a copy of $S^1 \times I$ and X_{i+1} is obtained from X_i by attaching a copy of F to each boundary component of X_i.
The **Loch Ness monster surface** L_∞ is the non-compact surface obtained as direct limit of $Y_0 \subset Y_1 \subset Y_2 \subset \cdots$ where Y_0 is a 2-disk and Y_{i+1} is obtained from Y_i by attaching a copy of F to its boundary.

(a) Make sketches of Σ_∞ and L_∞. What has L_∞ to do with the Loch Ness monster?

(b) Describe CW-structures on L_∞ and Σ_∞ and compute its cellular homology groups.

(c) Are L_∞ and Σ_∞ homeomorphic?

Exercise 3.
Let $G_1, G_2, \dots$ be a sequence of (not necessarily finitely presented) abelian groups. Construct a CW-complex X with reduced homology groups

$$\widetilde{H}_k(X) \cong \begin{cases} G_k\,; & \text{if } k \in \mathbb{N}, \\ 0\,; & \text{else.} \end{cases}$$

Hint: It may be helpful to start with sequences of finitely presented groups where only finitely many groups are non-trivial.
Bonus: Is such a space unique up to homotopy?

Exercise 4.

(a) Let X be a connected 1-dimensional CW-complex. Show $\pi_n(X) = 0$ for all $n \geq 2$.

(b) Compute all homotopy groups of surfaces Σ_g of genus $g \geq 1$ by applying Hurewitcz's theorem to its universal covering.

(c) Compute the second homotopy groups of $\mathbb{C}P^n$ and $S^1 \vee S^2$.

Bonus exercise.
A connected topological space X with only one non-vanishing homotopy group $\pi_n(X) \cong G$ is called **Eilenberg–MacLane space** $K(G, n)$.

(a) Construct an Eilenberg–MacLane space for arbitrary G and n (assuming G to be abelian if $n \geq 1$).
Hint: It may be helpful to first do Exercise 3.

(b) Let $G_1, G_2, \ldots$ be a sequence of (not necessarily finitely presented) groups (abelian for $n \neq 1$). Construct a connected CW-complex X with homotopy groups

$$\pi_k(X) \cong G_k.$$

(c) When is such a space unique up to homotopy?

Bonus exercise.
Show that the simplicial homology groups of a triangulizable space X are isomorphic to its cellular homology groups.
Hint: Let T be a triangulation of a topological space X. In the lecture we defined a CW-structure on X coming from T. Consider the corresponding simplicial- and cellular chain complexes.

This sheet will be discussed on Wednesday 15.12. and should be solved by then.