WS 2022/23

Marc Kegel
Naageswaran Manikandan

Differentialgeometrie I

Exercise sheet 1

Exercise 1.
The 1-sheet hyperboloid

$$\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1\right\}$$

is a doubly ruled surface, i.e. it can be written in two ways as a ruled surface.

Exercise 2.
The tangent line of a regular curve α in $\mathbb{R}^n$ at the point $t=t_0$ is the line

$$\left\{w\in\mathbb{R}^n \,|\, w=\alpha(t_0)+\lambda T(t_0),\ \lambda\in\mathbb{R}\right\}.$$

(a) Let $\alpha(t)=(t,g(t))$ be a planar curve for some given smooth function $g\colon\mathbb{R}\to\mathbb{R}$. Compute the tangent line of α and compare it with the tangent line of a graph from Analysis I.

(b) Show that $\alpha(t)=(\sin 3t\cos t,\sin 3t\sin t)$ is regular and compute the tangent line at $t=\pi/3$.

(c) Compute the tangent line of the Helix $\alpha(t)=(r\cos t,r\sin t,ht)$ (for given $r,h>0$). Show that the angle between $\dot\alpha$ and the vector $(0,0,1)$ is constant.

Exercise 3.
The curve

$$\alpha(s)=\frac{1}{2}\left(s+\sqrt{s^2+1},\left(s+\sqrt{s^2+1}\right)^{-1},\sqrt{2}\log\left(s+\sqrt{s^2+1}\right)\right)$$

is parametrized by arc length.

Exercise 4.

(a) If the plane curve $\alpha(s)=(x(s),y(s))$ is parametrized by arc length then the curvature of α is given by

$$k(s)=|x'(s)y''(s)-x''(s)y'(s)|.$$

(b) The curvature of a regular space curve β is given by

$$k=\frac{|\dot\beta\times\ddot\beta|}{|\dot\beta|^3}$$

p.t.o.

Exercise 5.

(a) A curve with non-vanishing curvature k is a helix if and only if τ/k is constant.

(b) A curve is a circle helix if and only if $\tau = \text{konst} \neq 0$ and $k = \text{konst} > 0$.

(c) If α is parametrized by arc length and $k(s) \neq 0$, $k'(s) \neq 0$, and $\tau(s) \neq 0$ for all s. Then α lies on a sphere if $\rho^2 + (\rho'\sigma)^2 = \text{konst} > 0$, where $\rho = 1/k$ and $\sigma = 1/\tau$.