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Differentialgeometrie I

Exercise sheet 2

Exercise 1.
Consider the 2-sphere
$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 = 1\}.$$
The straight line through $(u,v,0) \in \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ and the north pole $(0,0,1)$ intersects S^2 in exactly one more point. We denote this point by $x(u,v)$.

(a) Create a figure describing that situation.

(b) Compute $x(u,v)$ and show that $x\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a parameterized piece of a surface.

(c) Let $y\colon \mathbb{R}^2 \to \mathbb{R}^3$ be defined analogously by using the south pole $(0,0,-1)$ instead of the north pole $(0,0,1)$. Determine the map $y^{-1} \circ x$ explicitly and verify that it is differentiable. What is the maximum domain of this map?

(d) Deduce that S^2 is a surface.

(e) Compute the metric tensor (g_{ij}) in that local parametrizations and compute the volume of S^2.

Exercise 2.
Consider a curve in the (r,z)-plane given by $\alpha(t) = (r(t), z(t))$ for $t \in (a,b)$ with $r(t) > 0$. When this is rotated about the z-axis, we obtain a **rotation surface**. We can parameterize this surface as follows. For this purpose it is useful to use the parameters t and ϕ, where t determines the position on the curve and ϕ the angle of rotation. Then we can define
$$x(t,\phi) = \big(r(t)\cos\phi, r(t)\sin\phi, z(t)\big) \;\text{ for }\; t \in (a,b) \;\text{ and }\; \phi \in (0,2\pi).$$
The t-curves are called **meridians** and the ϕ-curves are called **latitudes**.

(a) Show that x is a parametrized piece of a surface if α is regular and injective. Calculate $x_1 = \partial x/\partial_t$, $x_2 = \partial x/\partial_\phi$ and the unit normal vector n.

(b) Consider the rotation surface of $\alpha(t) = (r(t), z(t)) = (2+\cos t, \sin t)$, $t \in (-\pi,\pi)$. Show that the conditions in (a) are satisfied, and describe the mentioned objects from (a) explicitly. What does this surface look like? Describe its meridians and latitude circles in parametrized form and draw a sketch.

(c) Calculate the metric coefficients of a rotation surface.

p.t.o.

Exercise 3.
Consider the parametrized piece of a surface

$$x(r, \phi) = (r \cos \phi, r \sin \phi, r),\ r \in \mathbb{R}^+, \phi \in (0, 2\pi).$$

Determine the components of the metric tensor (g_{ij}). Let $\alpha(t)$ be a curve in this surface piece that is given in the parameter domain by $r(t) = e^{t(\cot \theta)/2}$ and $\phi(t) = t/\sqrt{2}$, with $t \in [0, \pi]$ and a constant θ. Calculate the length of this curve and show that θ is the angle between the curve $\alpha(t)$ and the curves $\phi = \text{const.}$ on the piece of the surface.

Exercise 4.
Describe the behavior of a tangent vector and the metric coefficients under coordinate transformation. Apply your results to the two different parametrizations of the upper hemisphere minus the 0-meridian from the lecture.