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Marc Kegel
Naageswaran Manikandan

Differentialgeometrie I

Exercise sheet 8

Exercise 1.

(a) The product $M \times N$ of two smooth manifolds M and N is again a smooth manifold.

(b) The topology on the tangent bundle as defined in the lecture is Hausdorff and has countable basis.

(c) $T(M \times N)$ is diffeomorphic to $TM \times TN$.

Exercise 2.

(a) The composition and the product of embeddings are again embeddings.

(b) $S^n \times \mathbb{R}$ can be embedded in R^{n+1}.

(c) $S^{n_1} \times \cdots \times S^{n_k}$ can be embedded in $\mathbb{R}^{n_1+\cdots+n_k+1}$.

Exercise 3.
Verify the following properties of the Lie bracket:

(i) $[X, Y] = -[Y, X]$.

(ii) $[X + Y, Z] = [X, Z] + [Y, Z]$.

(iii) The Jacobi identity: $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

Let (U, h) be a chart of M and let $\left(\frac{\partial}{\partial x^1}\right)_p, \ldots, \left(\frac{\partial}{\partial x^m}\right)_p$ the induced basis of T_pM for a point $p \in M$. Then we have:

(iv) $\left[\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^j}\right] = 0.$

(v) For $X = X^i \dfrac{\partial}{\partial x^i}$ and $Y = Y^j \dfrac{\partial}{\partial x^j}$ we have $[X, Y] = \left(X^i \dfrac{\partial Y^j}{\partial x^i} - Y^i \dfrac{\partial X^j}{\partial x^i}\right) \dfrac{\partial}{\partial x^j}$.

Use the last equation to compute the Lie bracket of $X = x^1 \dfrac{\partial}{\partial x^2}$ and $Y = x^1 x^2 \dfrac{\partial}{\partial x^1}$.

Exercise 4.
The tangent bundle TM of an m-manifold M is called **trivial**, if there is a diffeomorphism $TM \to M \times \mathbb{R}^m$ which linearly maps T_pM to $\{p\} \times \mathbb{R}^m$ for every $p \in M$.

(a) TM is trivial if and only if there exist m vector fields $X_1, \dots, X_m$ on M such that $X_1(p), \dots, X_m(p)$ are linearly independent for every $p \in M$.

(b) $T\mathbb{R}^m$ is trivial.

(c) TS^1 and TS^3 are trivial.

(d) TS^2 is not trivial. *Hint:* Use the Poincaré–Hopf theorem.

Bonus exercise 1.

(a) Any submanifold of $\mathbb{R}^n$ is automatically Hausdorff and has countable basis.

(b) Where have we used the Hausdorff property and a countable basis in the proof of the Whitney embedding theorem?

(c) Any Riemannian manifold has automatically a countable basis. Is it also Hausdorff?

Bonus exercise 2.
Describe an immersion of the Klein bottle into $\mathbb{R}^3$ and describe how to get an embedding into $\mathbb{R}^4$ from it.
Challenge: Describe an immersion of $\mathbb{R}P^2$ into $\mathbb{R}^3$ and describe how to get an embedding into $\mathbb{R}^4$ from it.
Hint for the challenge: The immersion can be described for example by drawing a detailed figure of its image after watching the Youtube video [J. Leys: *The Boy surface*] or reading [R. Kirby: *What is … Boy's Surface*, Notices of the AMS, **54** (2007), 1306–1307].
Another method is to build an explicit 3-dimensional model, see for example [A. Chéritat: *A model of Boy's surface in constructive solid geometry*, available on his webpage].
If you are not interested in understanding the construction you can alternatively prove that

$$\begin{aligned} f\colon D^2 &\longrightarrow \mathbb{R}^3 \\ w &\longmapsto \frac{1}{g_1^2 + g_2^2 + g_3^2}(g_1, g_2, g_3) \end{aligned}$$

induces an immersion $\mathbb{R}P^2 \to \mathbb{R}^3$, where

$$\begin{aligned} g_1 &:= -\frac{3}{2} \operatorname{Im}\left[\frac{w\left(1 - w^4\right)}{w^6 + \sqrt{5}w^3 - 1}\right] \\ g_2 &:= -\frac{3}{2} \operatorname{Re}\left[\frac{w\left(1 + w^4\right)}{w^6 + \sqrt{5}w^3 - 1}\right] \\ g_3 &:= \operatorname{Im}\left[\frac{1 + w^6}{w^6 + \sqrt{5}w^3 - 1}\right] - \frac{1}{2}. \end{aligned}$$