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Differentialgeometrie I

Exercise sheet 9

Exercise 1.

(a) Show that the antipodal map $-\mathrm{id}\colon S^n \to S^n$ is an isometry and use this to construct a Riemannian metric on $\mathbb{R}P^n$ such that the quotient map $\pi\colon S^n \to \mathbb{R}P^n$ is a local isometry.

(b) Construct the *flat* Riemmanian metric on the n-torus as in the lecture.

(c) Show that the quotient of $\mathbb{R}^2$ under the antipodal map is homeomorphic to $\mathbb{R}^2$.

(d) Show that the quotient of $\mathbb{R}^3$ under the antipodal map is not homeomorphic to a manifold.

(e) Can you generalize the statement from (a) and (b) to more general settings?

Exercise 2.
Let M_1 and M_2 be Riemannian manifolds. Construct from this a natural Riemannian metric on $M_1 \times M_2$. Use this to construct a Riemannian metric on the n-torus T^n and show that this metric is isometric to the flat metric on T^n constructed in the lecture (see Exercise 1 (b)).

Exercise 3.

(a) Let $f\colon M \to N$ be an immersion and let g be a Riemannian metric on N. If f would be an embedding then we have defined the induced Riemannian metric f^*g in the lecture. Show that for an immersion the same construction yields a Riemmanian metric on M. Then

$$f\colon (M, f^*g) \to (N, g)$$

is called an **isometric immersion**.

(b) Describe an isometric immersion of the flat n-torus T^n into $\mathbb{R}^{2n}$.

Exercise 4.
We consider the map $f\colon S^2 \to \mathbb{R}^4$ given by

$$(x, y, z) \longmapsto (x^2 - y^2, xy, xz, yz).$$

Show that f induces an embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$. Compute the metric coefficients and the Christoffel symbols of the by f induced metric in a suitable atlas of $\mathbb{R}P^2$.

Exercise 5.
Let (M, g) be a Riemannian manifold. The unit tangent bundle is

$$T^1M := \bigcup_{p\in M} \{X \in T_pM \big| g_p(X,X) = 1\} \subset TM.$$

(a) T^1M is again a smooth manifold. What is its dimension?

(b) If M is compact then T^1M is compact.

(c) Determine the diffeomorphsism type of the unit tangent bundle of S^1 and $\mathbb{R}^n$.

(d) Show that T^1S^2 is diffeomorphic to $SO(3)$.
Hint: Consider the map

$$\begin{aligned} SO(3) \times S^2 &\longrightarrow S^2 \\ (A, x) &\longmapsto A \cdot x. \end{aligned}$$

Exercise 6.

(a) Prove Lemma 6.4: Let Y_1 and Y_2 vector fields that agree on an open neighborhood of $p \in M$ and let X be another vector field on M. Then

$$(\nabla_X Y_1)(p) = (\nabla_X Y_2)(p).$$

(b) Show that the covariant derivative from the first part of the lecture defines a covariant derivative with respect to the definition from Chapter 6. Generalize the definition for surfaces in $\mathbb{R}^3$ from the first part of the lecture to a general Riemmanian submanifold.

Exercise 7.
Let $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_n > 0\}$ denote the upper half space model of the hyperbolic space with its standard Riemannian metric

$$g_x(v_1, v_2) = \frac{\langle v_1, v_2\rangle}{x_n^2},$$

where $v_1, v_2 \in T_x\mathbb{H}^n \cong \mathbb{R}^n$ and $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean inner product. Calculate the metric coefficients and the Christoffel symbols with respect to the canonical global coordinate chart $\phi\colon \mathbb{H}^n \to V \subset \mathbb{R}^n$ given by $\phi(x) = x$.