Contact Geometry

Exercise sheet 1

Exercise 1.
Prove (as stated in the lecture) that the standard contact structure on $S^{2n-1} \subset \mathbb{R}^{2n}$ is given by the kernel of

$$\alpha_{st} = \sum_{j=1}^{n} x_j \, dy_j - y_j \, dx_j,$$

and compute the Reeb flow of α_{st}. Which orbits of the Reeb flow are closed?

Exercise 2.
Describe an explicit contact form on $S^1 \times S^2$, draw its contact planes and compute the Reeb flow.

Exercise 3.
Consider $S^1 \times \mathbb{R}^2$ with an angular coordinate θ on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and cartesian coordinates (x, y) on $\mathbb{R}^2$. Let $n \in \mathbb{Z} \setminus \{0\}$. Show that

$$\alpha_n = \cos(n\theta) \, dx - \sin(n\theta) \, dy$$

is a contact form and sketch the contact structures. Show that $(S^1 \times \mathbb{R}^2, \xi_n = \ker \alpha_n)$ is contactomorphic to $(S^1 \times \mathbb{R}^2, \xi_m = \ker \alpha_m)$ for all n and m.
Challenge: Are these contact structures also isotopic?

Bonus exercise 1.

(a) Draw sketches of the three contact structures on $\mathbb{R}^3$ introduced in the lecture.

(b) Describe a 2-plane field on a 3-manifold that is not contact but to which no surface is tangent.

(c) Describe a hyperplane field $\xi = \ker \alpha$ that is not contact but for which $\alpha \wedge d\alpha$ is non-vanishing.

(d) Describe a hyperplane field that is nowhere contact and nowhere a foliation.

(e) Describe a non-coorientable hyperplane field on an orientable manifold.

(f) Describe a non-coorientable hyperplane field on a non-orientable manifold.

(g) Describe a coorientable hyperplane field on a non-orientable manifold.

Bonus exercise 2.

(a) Describe a 1-dimensional foliation on T^2 that admits only closed leaves.

(b) Describe a 1-dimensional foliation on T^2 that admits only non-closed leaves.

(c) Describe a 1-dimensional foliation on T^2 that admits closed and non-closed leaves.

(d) Describe a 2-dimensional foliation of $S^1 \times D^2$ that admits exactly one closed leave.

(e) Describe a 2-dimensional foliation of S^3 that admits exactly one closed leave.
Hint: Consider the boundary of $D^2 \times D^2$ and use (d).

(f) Show that the 2-plane field on $\mathbb{R}^3$ defined as the kernel of the 1-form

$$dz - z\,dy$$

is induced by a foliation and describe its leaves.

This sheet will be discussed on Thursday 26.10. and should be solved by then.