Summer Term 2025

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Topology I

Problem Sheet 7

Exercise 1.

(a) Draw sketches of k-handles and their attachments to the boundary of an n-manifold for all $0 \leq k \leq n \leq 3$.

(b) Describe a handle decomposition of the Klein bottle consisting of exactly one 0-handle.

Exercise 2.

(a) Let $f\colon S^1 \to X$ be a continuous map. As in the lecture, we define

$$X \cup_f D^2 := (X \sqcup D^2)/\sim \quad \text{where } p \sim f(p) \text{ for } p \in \partial D^2 = S^1.$$

If $f, g\colon S^1 \to X$ are homotopic, then

$$X \cup_f D^2 \simeq X \cup_g D^2.$$

(b) In Exercise 1 of Sheet 6, we saw that the dunce hat is simply connected. Show that the dunce hat is even contractible.
Hint: Use part (a).

(c) Show that the dunce hat is triangulable but does not admit a handle decomposition.

Exercise 3.
We call a smooth manifold M **orientable** if M admits an atlas such that all transition functions have positive Jacobian determinants (i.e., for any two charts ψ and ϕ, the Jacobian determinant of $\psi \circ \phi^{-1}$ is positive everywhere).

(a) Show that S^n is orientable by explicitly providing such an atlas.

(b) The Möbius band is not orientable.

(c) A surface is orientable if and only if it does not contain a Möbius band; that is, if and only if there is no closed curve that switches left and right.

Exercise 4.

(a) A manifold is connected if and only if it is path-connected.

(b) A manifold is locally path-connected and semi locally simply-connected.

(c) Every manifold admits an universal cover.

p.t.o.

Bonus Exercise 1.

(a) The connected sum of an n-manifold M with S^n is again homeomorphic to M.

(b) The connected sum of two manifolds is again a manifold.

Bonus Exercise 2.
Consider the surface W of the unit cube

$$W := \big\{(x_1, \dots, x_n) \in \mathbb{R}^n \colon \max_i(|x_i|) = 1\big\}.$$

(a) Show that W is not a smooth submanifold of $\mathbb{R}^n$.

(b) Show that W is a topological manifold and define a smooth structure on W.

These exercises will be discussed in the session on Thursday, June 5.