SS 2025

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Topology I

Problem Sheet 9

Exercise 1.

(a) Describe a triangulation of $\mathbb{R}P^n$.

(b) Let K be a simplicial complex. Construct a triangulation of the suspension $\Sigma|K|$ of the underlying topological space $|K|$.

Exercise 2.
Let K and L be simplicial complexes that intersect in a common subcomplex $K \cap L$. Verify the gluing formula for the Euler characteristic:

$$\chi(K \cup L) = \chi(K) + \chi(L) - \chi(K \cap L).$$

In the following two exercises we assume that Theorem 6.4 has already been proven. That is, we assume that the Euler characteristic is a topological invariant of triangulable spaces.

Exercise 3.

(a) Compute the Euler characteristic of the n-sphere.

(b) Let M be a smooth and compact n-manifold with non-empty boundary. Denote by $M \cup h_k$ the result of attaching a k-handle h_k to M. Compute the Euler characteristic of $M \cup h_k$ from that of M.

(c) Use this to determine the Euler characteristic of all surfaces, and show that the Euler characteristic together with orientability is a complete invariant for closed surfaces.

(d) The 2-sphere admits a handle decomposition with any even number of handles, but no handle decomposition with an odd number of handles.

(e) Show that the Euler characteristic of a closed 3-manifold is zero.

(f) How can the Euler characteristic of an n-manifold be computed from its handle decomposition?

Exercise 4.
Let M be a smooth compact n-manifold. Define $\Delta(M)$ as the minimal number of n-simplices in a triangulation of M. Similarly, define $h(M)$ as the minimal number of handles in a handle decomposition of M.

(a) Compute Δ and h for S^2, $\mathbb{R}P^2$, and T^2.

(b) What can you say about Δ and h for other surfaces?

Exercise 5.

(a) A simplicial complex is path-connected if and only if it is connected.

(b) Every simplicial complex is locally simply connected.

(c) The diameter of a simplex equals the length of its longest edge.

Challenge Exercise.
Visualize the first (and second?) barycentric subdivision of a 3-simplex.

These exercises will be discussed in the session on Thursday, June 19.