Proving existence of Nash equilibria

Our goal is to prove that every game has a NE.

Def 10 (n-simplex)

$\sigma = \{\theta_0 u_0 + \dots + \theta_n u_n \mid \sum \theta_i = 1 \ \ \theta_i \geq 0 \ \forall i\}$ &

u_i are affinely independent, ie. $\{u_i - u_0\}_{i=1}^n$ lin. independent

Def 11 (standard n-simplex)

$\Delta_n = \{y \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} y_i = 1 \ \ \forall i = 0, \dots, n \ \ y_i \geq 0\}$

(set of all convex combinations of the n+1 unit vectors $e^0, \dots, e^n$)

Brouwer FPT Let $f: \Delta_m \to \Delta_m$ be continuous. Then f has a fixed point (i.e. $\exists$ z s.t. $f(z) = z$)

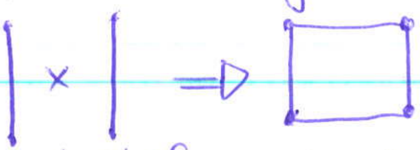
($\Delta_m \underset{\text{homeom.}}{\cong} B^m$) so is clear

Corollary: $K = \prod_{j=1}^{K} \Delta_{m_j}$ (called simpletope) (cart prod. of simplices). Let $f: K \to K$ be continuous. Then f has a fixed point.

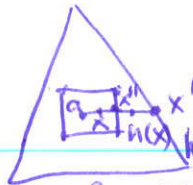
(proof by finding a homeomorphism between K & Δ_m for some m)

Let $h: \Delta_m \to K$ be a homeom. Then $h^{-1} \circ f \circ h: \Delta_m \to \Delta_m$ is cont. $\Rightarrow$ by Brouwer FPT $\exists z'$ s.t. $h^{-1} \circ f \circ h(z') = z'$. Let $z = h(z')$. Then $h^{-1} \circ f(z) = z' = h^{-1}(z)$ since h is injective $\Rightarrow f(z) = z$ so has FP

idea of finding h.



scale the square and put it inside a 2-simplex



x' intersection with bdry of Δ with the ray that pass through a

$a \in$ interior

x'' inters. of bdry of K'' with the same ray

h scales x along the ray by a factor of $\frac{\|x' - a\|}{\|x'' - a\|}$

Product of 1-simp'. — Simpletope

and h can be for example the radial projection

$h(x) = a + \frac{\|x' - a\|}{\|x'' - a\|}(x - a)$ QED

S is a simpletope & each individual's mixed strategy can be understood as a point in a simplex.

Theorem (Nash 1951) Every game with a finite number of players and action profiles has at least one NE.

Proof: Given a strategy profile $s \in S$ $\forall i \in N$ and $a_i \in A_i$

define $\varphi_{i,a_i}(s) = \max\{0, u_i(a_i, s_{-i}) - u_i(s)\}$ (=0 iff s is best response)

Define $f: S \to S$ by $f(s) = s'$, where

$s'_i(a_i) = \frac{s_i(a_i) + \varphi_{i,a_i}(s)}{\sum_{b_i \in A_i} s_i(b_i) + \varphi_{i,b_i}(s)} = \frac{s_i(a_i) + \varphi_{i,a_i}(s)}{1 + \sum_{b_i \in A_i} \varphi_{i,b_i}(s)}$ (*)

Intuitively, this function maps a strategy profiles to a new strategy profile s' in which each agent's actions that are best responses to s recieve increased probability mass.

f is continuous, since each φ_{i,a_i} is continuous. Since S is convex & compact, f must have at least one FP. (u_i cont since linearity is expectation)

Now we need to show that the FP of f are Nash equilibria.

First, if s is a NE then all φs are $0 \Rightarrow$ s is a fixed point.

Conversly: consider arbitrary fp of f, s.

By linearity of expectation, there must exist at east one action in the support of s, say a_i' for which $u_{i a_i'}(s) \leq u_i(s)$

From definition of $\varphi \Rightarrow \varphi_{i a_i'}(s) = 0$. Since s is a fp of f $\Rightarrow$ $s_i'(a_i') = s_i(a_i')$. Consider ⊛, the expression defining $s_i'(a_i')$

$$s_i'(a_i') = \frac{s_i(a_i')}{1 + \sum_{b_i \in A_i} \varphi_{i b_i}(s)} > 0 \text{ (since } a_i' \in \operatorname{supp} s_i) \Rightarrow \text{should be } 1 \Rightarrow$$

$\forall i$ & $b_i \in A_i$ $\varphi_{i,b_i}(s) = 0$. From the definition of φ, this can only occur when no player can improve his expected payoff by moving to a pure strategy $\Longrightarrow$ s is a Nash equilibrium QED.

<u>Examples</u> ① Prisoner's dilemma

A \ B	stay silent (cooperate)	betray
stay silent	1, 1	3, 0
betray	0, 3	2, 2

for each player max reward is 0 jail time and is obtained when their decisions are different. Each player improves their own situation by switching from cooperating to betraying given knowledge that the other player's best decision is to betray $\Rightarrow \exists$ NE, namely when both choose to betray.

<u>The dilemma</u>: if both betray they both serve longer sentence than if neither said anything, even though it would be better for them to cooperate, each of them can improve their situation by betraying each other. if strategy sets are cpct & utility fctns cont $\Rightarrow \exists$ NE

② Infinite game with no NE: Player 1 chooses $x > 0$ & player 2 chooses $y > 0$, the payoffs are $x \cdot y$.

for each player $(x+1)y > xy$ is a better response $\Rightarrow$ no best response.

③ Infinite game with NE: two vendors have carts on beach, people come to the closest one. How should they position themselves to get max profit? Assuming for simplicity people are unif. distrib. can think of the beach as $[0,1]$ interval

0 — 1/2 — 1

for each vendor to get maximum they should have more than 1/2 of the business so $v_1, v_2 \geq 1/2$ & $v_1 + v_2 = 1 \Rightarrow v_1, v_2 = 1/2$. So each should get half of the beach. they can be equidistant from the center or in the same location. $v_1 = 0$, $v_2 = 1$ not equil - can move. $v_1 = v_2 = a \neq 1/2$ can move $\Rightarrow$ 1/2 is equilibr.

Game theoretic preliminaries

Def 1: (Normal-form game) A (finite, n-person) n-f game is a tuple (N, A, O, μ, u) where:

- N is a finite set of players, indexed by i
- $A = (A_1, \dots, A_n)$ where A_i is a finite set of pure strategies (actions) available to player i. ~~I~~ each vector $a = (a_1, \dots, a_n) \in A$ is called a pure strategy profile (action profile)
- O is a set of outcomes
- $\mu : A \to O$ determines the outcome as a function of the pure strategy profile
- $u = (u_1, \dots, u_n)$ wher $u_i : O \to \mathbb{R}$ is a real valued utility payoff function for player i

often we do not need the notion of an outcome as distinct from a strategy profile → has simpler form (N, A, u) and for the rest of the talk we'll be using this form.

We've defined the actions available to each player in a game but not yet his set of strategies (or available choices). certainly one kind of strategy is to select a single action and play it, such strategies are called pure, and we call a choice of pure strategy for each player a pure strategy profile.

Players could also follow another, less obvious type of strategy: randomizing over the set of available actions according to some prob. distribution. Such a strategy is called mixed. And is defined as follows:

Def 2: (Mixed strategy): Let $(N, (A_1, \dots, A_n), O, \mu, u)$ be a game, and for any set X let $\Pi(X)$ be set of all probability distributions over X. Then the set of mixed strategies for player i is $S_i = \Pi(A_i)$

a function that provides probabilities of occurence of diff possible outcomes in an experiment

→ Normal, Binomial, Exponential, uniform etc.

Def 3: (Mixed strategy profile) is the cart. prod of ind. mixed str $S_1 \times \dots \times S_n$.

by $s_i(a_i)$ – probability that an action a_i will be played under mixed strategy s_i & $\sum_{j=1}^{|A_i|} s_i(a_j) = 1$

has probability one of occuring

Note: A pure strategy is a special case of a mixed strategy in which the support is a single action.

In order to define payoffs of players given a particular strategy profile we will need to introduce another def.

Def 4. (expected utility of mixed strategy) Given a game (N, A, u), the expected utility u_i for player i of the mixed strategy profile $s = (s_1, \dots, s_n)$ is defined as (is an expectation ⇒ is linear)

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j=1}^{n} s_j(a_j)$$

take average but "weight" the outcomes with probability of that outcome occuring

average of payoffs of the outcomes

probability of each outcome

Now we will look at games from an individual player's point of view, rather than from the viewpoint of an outside observer. This will lead us to the most influential solution concept in game theory - Nash equilibrium.

observe: If a player knew how others were going to play, his strat. problem would become simple. He would be left with the single agent problem of choosing a utility maximizing action. Define $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ ⇒ $s = (s_i, s_{-i})$

If all other players than i were playing s_{-i}, a utility maximizing agent i would face the problem of determining his best response.

Def 5. (Best response) Player i's best response to the strategy profile s_{-i} is a mixed strategy $s_i^* \in S_i$ s.t.

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \forall s_i \in S_i$$

! Best responses are not unique. Except in cases in which there is a unique best response that is a pure strategy, the number of best responses is always infinite.

Of course, in general a player doesn't know what other strategies the other players will adopt, thus the notion of best response is not a solution concept. However we can leverage the idea of best response to define one central notion in non-cooperative game theory - the Nash equilibrium

Def 6 (Nash equilibrium) A strategy profile $s = (s_1, \dots, s_n)$ is a Nash eq. if, for all agents i, s_i is a best response to s_{-i}.

↳ it is a stable strategy profile (no agent would want to change his strategies if he knew what strategies the other players were following.