Immersion, Embedding and the Whitney Embedding Theorem

1) Immersion:

Def.1: Let M, N be differentiable manifolds. Then an immersion is a differentiable function $f: M \to N$ whose derivative is everywhere injective.

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An immersion is a map that is injective on tangent spaces, that is

$D_p f : T_p M \longrightarrow T_{f(p)} N$ injective for all points $p \in M$.

2) Embedding:

Def.2: A map $f: M \to N$ is called proper if the preimage of every compact set in N is compact in M.

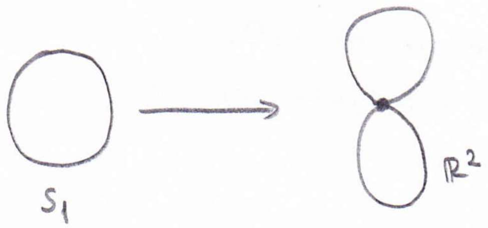
An immersion that is injective and proper is called an embedding.

(That is, an embedding $f: M \to N$ is an immersion, which maps M homeomorphically onto its image. ($f(M) \subset N$))

Proposition 1: If $f: M \to N$ is an embedding, then $f(M)$ is a submanifold of N.

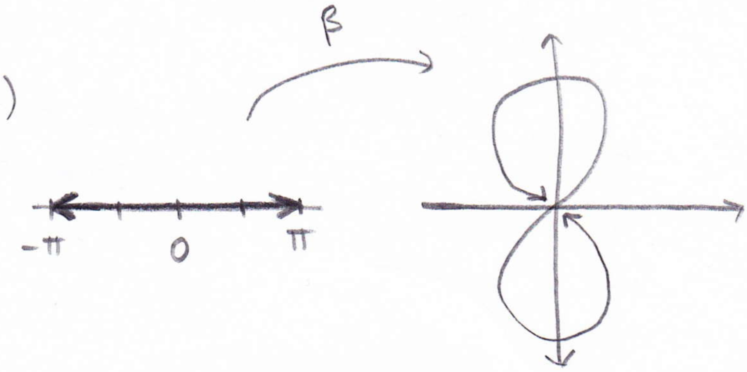
Example

i)



- an immersion of S^1 to $\mathbb{R}^2$
- but the immersion is not injective
- not an embedding.

ii)



- Consider the curve $\beta: (-\pi, \pi) \longrightarrow \mathbb{R}^2$ defined by $\beta(t) = (\sin 2t, \sin t)$
- β is an injective smooth immersion (because $\beta'(t)$ never vanishes)
- It is not an embedding because its image is compact while its domain is not. (open)

(It contains all its limit points, so it is a closed subset of $\mathbb{R}^2$. Since it is bounded as well, by the Heine-Borel Theorem, it is a compact subset of the plane $\mathbb{R}^2$.)

** When M itself is a compact manifold, every map $f: M \to N$ is proper. Thus for compact manifolds, embeddings are just one-to-one immersions.

3) The Whitney Embedding Theorem

Let M be a compact manifold. Then there is an embedding $M \to \mathbb{R}^n$ for sufficiently large n.

The Bump Function

$$C(r) := \{x \in \mathbb{R}^m : \|x\|_\infty < r,\ |x_1|, \dots, |x_m| < r\}$$

There is a C^∞-function φ on $\mathbb{R}^m$ with following properties:

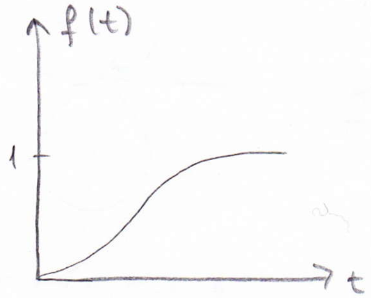
(i) $\varphi \geq 0$

(ii) $\varphi \equiv 1$ on $\overline{C(1)}$

(iii) $\varphi \equiv 0$ on $\mathbb{R}^m \setminus C(2)$

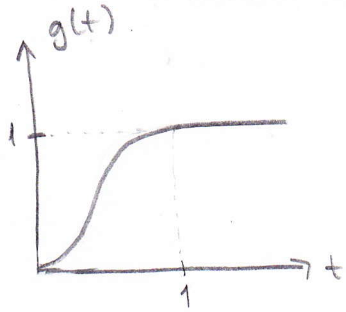
Proof:

(#1) $$f(t) := \begin{cases} e^{-1/t}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$ is C^∞.



This function is smooth at $t=0$.
(Each derivative of the function on the right is zero at this point.)

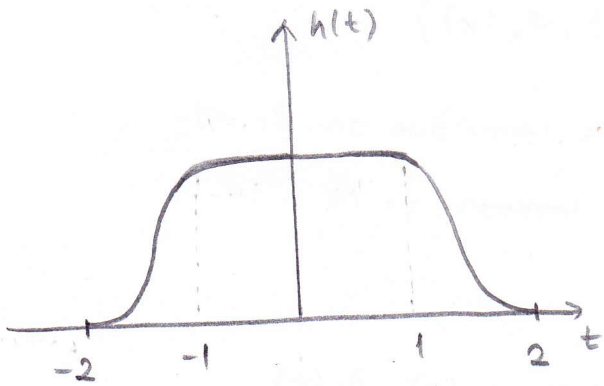
#2 $g(t) := \dfrac{f(t)}{f(t)+f(1-t)}$



For $t \geq 1$, $f(1-t)=0 \Rightarrow g(t)=1$.

For $t \leq 0$, $f(t)=0 \Rightarrow g(t)=0$

#3 $h(t) := g(t+2)\,g(2-t)$



For $t \geq 2$, $g(2-t)=0$, $h(t)=0$

For $t \leq -2$, $g(t+2)=0$, $h(t)=0$

So, $h(t)$ is supported inside $[-2,2]$.

For $-1 \leq t \leq 1$, $h(t)=1$.

$$\boxed{\varphi(x) := h(|x_1|) \cdots h(|x_m|)}$$

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→ The Whitney Embedding Theorem Continued:

Proof: For every point $p \in M$ we can find a chart (U_p, h_p) with $h_p(p)=0$ and $h_p(U_p) = C(3)$

➡ $p \in h_p^{-1}(C(1))$ and $M = \bigcup_{p \in M} h_p^{-1}(C(1))$

➡ M compact $\Rightarrow$ $\exists$ chart $(U_1,h_1), \dots, (U_k,h_k)$ as above,

$$M = \bigcup_{i=1}^{k} h_i^{-1}(C(1))$$

→ $B_i := \varphi_i^{-1}(1)$, where $\begin{cases} \varphi_i := \varphi \circ h_i & \text{on } U_i \\ \varphi_i = 0 & \text{otherwise} \end{cases} \Rightarrow \varphi_i \in C^\infty(M)$

$B_i \subset h_i^{-1}(C(1))$ and $M = \bigcup_{i=1}^{k} B_i$. (finite union of subcovers)

→ Define $f_i \in C^\infty(M, \mathbb{R}^m)$

$$f_i(x) = \begin{cases} \varphi_i(x) h_i(x) & \text{for } x \in U_i \\ 0 & \text{otherwise} \end{cases}$$

→ $f \in C^\infty(M, \mathbb{R}^{k(m+1)})$

$$f(x) = (f_1(x), \varphi_1(x), \ldots\ldots, f_k(x), \varphi_k(x))$$

→ For $x \in B_i \Rightarrow f_i = h_i$, f_i is immersive on $B_i \Rightarrow$ f is immersive on M.

→ For $x \in B_i$ and $y \neq x$:

- $y \in B_i$, then $f_i(y) = h_i(y) \neq h_i(x) = f_i(x)$.
- $y \notin B_i$, then $\varphi_i(y) \neq 1 = \varphi_i(x)$

f is an injective immersion $M \to \mathbb{R}^{k(m+1)}$. Since M is compact, f being an injective immersion is an embedding.

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<u>Lemma</u>:

(a) An immersion $f: M \to W$, which is a homeomorphism of M on $f(M)$, is an embedding.

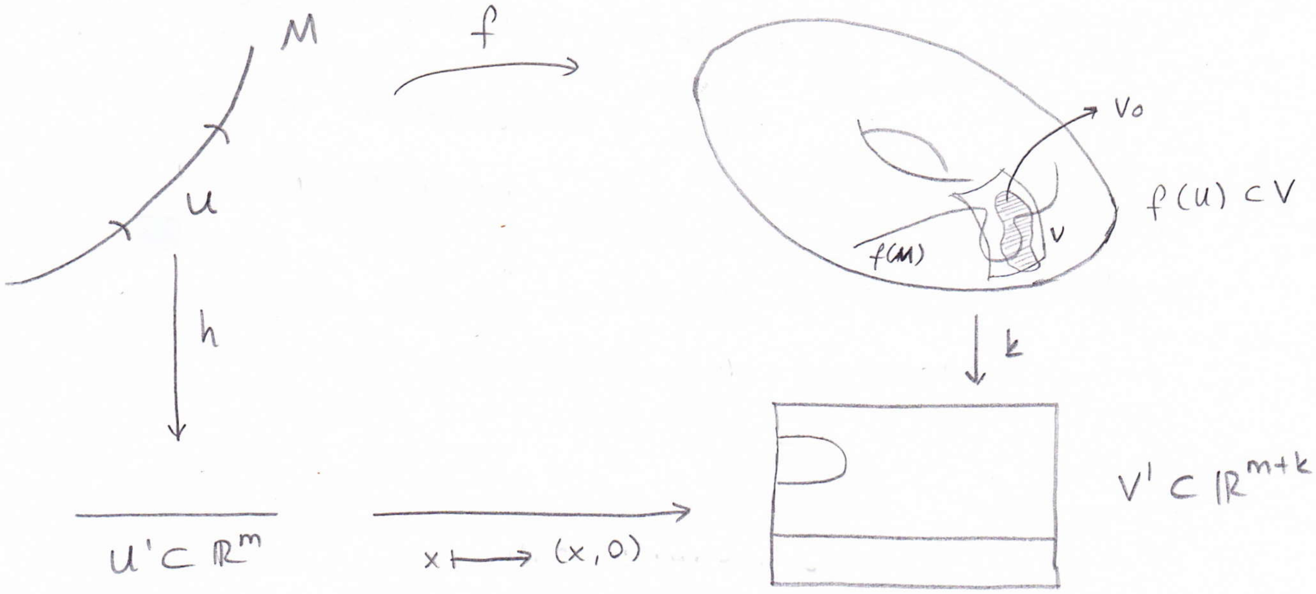
(b) A continuous injective map $f: M \to W$ with compact M is a homeomorphism of M over $f(M)$.

Proof :

(a) For points $p \in M^m$ and $f(p) \in W^{m+k}$, f is in the form of

$$(x_1, \dots, x_m) \longmapsto (x_1, \dots, x_m, 0, \dots, 0) \qquad (*)$$

$(*)$ follows the Local Immersion Theorem. (which states that f is locally equivalent to the canonical immersion near x).



➡ $f(u)$ is open in $f(M) \Rightarrow f(u) = f(M) \cap V_0 \Rightarrow V_0$ open in W

➡ $k(f(M) \cap V \cap V_0) = (\mathbb{R}^m \times \{0\}) \cap k(V \cap V_0)$

➡ $f(M)$ is submanifold of W, f is a homeomorphism from $M \to f(M)$

➡ f is an embedding. (see Proposition 1)

(b)

- $f : M \to f(M)$ is continuous and bijective.
- Let $U \subset M$ be open $\Rightarrow$ $M \setminus U$ is closed and compact since M is comp
- $f(M) \setminus f(U) = f(M \setminus U)$ is compact and continuous.

  $f(M)$ is closed $\Rightarrow$ $f(M)$ Hausdorff
- $f(U)$ is open in $f(M)$
- f^{-1} is continuous.

f is a bijection from a compact space onto a Hausdorff space

$\Downarrow$

f is a homeomorphism onto its image.

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