Smooth manifolds

Recall: (1) A topological space X is called Hausdorff space, if $\forall x,y \in X$ w/ $x \neq y$: $\exists$ nbhs U,V of x,y: $U \cap V = \emptyset$

(2) For a topology $\mathcal{T}$ on X we call $\mathcal{B} \subseteq \mathcal{T}$ basis of $\mathcal{T}$, if $\forall U \in \mathcal{T}: \exists \{U_i\}_{i\in I} \subseteq \mathcal{B}: U = \bigcup_{i\in I} U_i$

Def: A Hausdorff space with countable basis is called n-dimensional topological manifold M^n, if it is locally homeomorphic to $\mathbb{R}^n$, i.e. $\forall p \in M: \exists$ nbh U of p, $U' \subseteq \mathbb{R}^n$ open and homeo $h: U \to U'$ (called "chart").

For charts $h_{\alpha,\beta}: U_{\alpha,\beta} \to U'_{\alpha,\beta}$ we define the transition map by $h_{\alpha\beta} := h_\beta \circ h_\alpha^{-1}: h_\alpha(U_\alpha \cap U_\beta) \to h_\beta(U_\alpha \cap U_\beta)$, i.e. the following diagram commutes

$$\begin{tikzcd}
\mathbb{R}^n \supseteq U'_\alpha \supseteq h_\alpha(U_\alpha \cap U_\beta) \arrow[rr, "h_{\alpha\beta}"] & & h_\beta(U_\alpha \cap U_\beta) \subseteq U'_\beta \subseteq \mathbb{R}^n \\
& U_\alpha \cap U_\beta \arrow[ul, "h_\alpha"] \arrow[ur, "h_\beta"'] &
\end{tikzcd}$$

A set of charts $\{(h_\alpha, U_\alpha)\}_{\alpha\in\Lambda}$ is called atlas, if $\bigcup_{\alpha\in\Lambda} U_\alpha = M$.

An atlas $\{(h_\alpha, U_\alpha)\}_{\alpha\in\Lambda}$ is called smooth, if all its transition maps $h_{\alpha\beta}$ (for $\alpha,\beta\in\Lambda$) are smooth.

For a smooth atlas $\mathcal{A}$ we call

$$\mathcal{D}(\mathcal{A}) := \{\text{all charts, s.t. transition map with chart of } \mathcal{A} \text{ is smooth}\}$$

a differentiable structure (to $\mathcal{A}$) (note that $\mathcal{D}(\mathcal{A})$ is then a smooth maximal atlas containing $\mathcal{A}$).

A topological manifold with a differentiable structure is called smooth.

<u>Example:</u> $\mathbb{R}P^n := \mathbb{R}^{n+1}\setminus\{0\} / _{x \sim \lambda x}$ is a smooth manifold

Hausdorff and countable basis follow from the definition of quotient topology

Consider quotient projection $\pi: \mathbb{R}^{n+1}\setminus\{0\} \longrightarrow \mathbb{R}P^n$
$x \longmapsto [x]$

$U_k := \{ [x_0 : \ldots : x_n] \in \mathbb{R}P^n \,/\, x_k \neq 0 \}$ open

$h_k : U_k \longrightarrow \mathbb{R}^n$ homeo with $h_k^{-1}(y_1, \ldots, y_n) = [y_1 : \ldots : y_k : 1 : y_{k+1} : \ldots : y_n]$

$[x_0 : \ldots : x_n] \longmapsto \left(\frac{x_0}{x_k}, \ldots, \widehat{\frac{x_k}{x_k}}, \ldots, \frac{x_n}{x_k} \right)$

$\Rightarrow$ (wlog $k < l$) transition maps

$$h_l \circ h_k^{-1}(y_1, \ldots, y_n) = \left(\frac{y_1}{y_l}, \ldots, \frac{y_k}{y_l}, \frac{1}{y_l}, \frac{y_{k+1}}{y_l}, \ldots, \widehat{\frac{y_l}{y_l}}, \ldots, \frac{y_n}{y_l} \right)$$

smooth on $h_k(U_k \cap U_l) = \{ y \in \mathbb{R}^n \,/\, y_l \neq 0 \}$

<u>Def:</u> A continuous map $f: M \to N$ is called <u>smooth in $p \in M$</u>,
if $\exists$ charts $h: U \to U'$, $k: V \to V'$ ($p \in U$, $f(p) \in V$) s.t. $k \circ f \circ h^{-1}$ smooth in $h(p) \in U'$.

f is called <u>smooth</u>, if it is smooth in all $p \in M$.

If f is bijective with f and f^{-1} smooth, then f is called <u>diffeomorphism</u>.

Tangent space

For submanifolds of euclidean spaces the tangent space is defined canonically as a vector subspace of the embedding space.
Now for abstract manifolds we need to define it by using solely inner properties of the manifold itself:

Def: $\mathfrak{U}(p) := \{\text{open nbhs of } p\}$

$$C^\infty(\mathfrak{U}(p), N) := \bigcup_{U \in \mathfrak{U}(p)} C^\infty(U, N) = \{f : M \to N \text{ smooth} \,/\, U \in \mathfrak{U}(p)\}$$

On $C^\infty(\mathfrak{U}(p), N)$ define an equivalence relation "$\sim$" via

$$f \sim g \quad :\Leftrightarrow \quad \exists\, U \in \mathfrak{U}(p) : \; f|_U = g|_U \,.$$

$[f] \in C^\infty(\mathfrak{U}(p), N)/_\sim$ is called <u>germ</u> of f around p (write: $[f] : (M,p) \to (N, f(p))$)

$[f] \in C^\infty(\mathfrak{U}(p), \mathbb{R})/_\sim$ is called <u>functional germ</u>

$\mathcal{E}(p) := C^\infty(\mathfrak{U}(p), \mathbb{R})/_\sim$ $\mathbb{R}$-algebra of functional germs.

A germ $(M,p) \xrightarrow{[f]} (N,q)$ gives rise to $\mathbb{R}$-algebra homomorphism

$$f^* : \mathcal{E}(q) \longrightarrow \mathcal{E}(p), \quad [\varphi] \longmapsto [\varphi] \circ [f] := [\varphi \circ f] \,,$$

which is functorial (i.e. $id^* = id$ and $(g \circ f)^* = f^* \circ g^*$).
Thus, for a chart h around $p \in M^n$ the germ $[h] : (M,p) \to (\mathbb{R}^n, 0)$ defines isomorphism $h^* : \mathcal{E}_n := \mathcal{E}(\mathbb{R}^n, 0) \longrightarrow \mathcal{E}(p)$.

Def: A <u>derivation</u> of $\mathcal{E}(p)$ is a linear map $X : \mathcal{E}(p) \to \mathbb{R}$,
s.t. $X([\alpha] \cdot [\beta]) = X([\alpha]) \cdot [\beta](p) + [\alpha](p) \cdot X([\beta])$.

Recall: For $v \in \mathbb{R}^n$ and $x \in U \subseteq \mathbb{R}^n$ there is a linear functional

$$\frac{\partial}{\partial v}(x) : C^\infty(U) \longrightarrow \mathbb{R}$$
$$f \longmapsto \frac{\partial f}{\partial v}(x)$$

Def: $T_pM := \{\text{derivations of } \mathcal{E}(p)\}$ is called <u>tangent space</u> of the manifold M in $p \in M$. This is a $\mathbb{R}$-vector space.

Def: For $f: M \to N$ smooth, $p \in M$, the differential of f in p is defined by
$$df_p : T_pM \to T_{f(p)}N \,. \qquad X \mapsto X \circ f^*$$

Remark: (1) $X(1) = X(1) + X(1)$, i.e. $X(1) = 0 \implies X(c) = 0$

(2)
$$df_p(X)([\varphi]) = X \circ f^*([\varphi]) = X \circ \varphi \circ f$$
$$\implies d(g \circ f)_p = dg_{f(p)} \circ df_p$$

Proposition: the partial derivatives $\frac{\partial}{\partial x_k} : \mathcal{E}_n \to \mathbb{R}$, $[\varphi] \mapsto \frac{\partial \varphi}{\partial x_k}(0)$ from a basis of $T_0\mathbb{R}^n$ (vector space of derivations of $\mathcal{E}_n$).

Proof: (1) Linearly independent:

Suppose $\sum_{k=1}^{n} a_k \frac{\partial}{\partial x_k} = 0$.

$\mathcal{E}_n \ni x_k : \mathbb{R}^n \to \mathbb{R}$, $(v_i) \mapsto v_k$ k-th coordinate function

$$\implies \frac{\partial [x_k]}{\partial x_l} = \frac{\partial x_k}{\partial x_l}(0) = \delta_{kl}$$

$$\implies a_{k_0} = \sum_{k=1}^{n} a_k \frac{\partial}{\partial x_k}[x_{k_0}] = 0 \qquad \forall k_0 \in \{1, \dots, n\}$$

(2) $T_0\mathbb{R}^n = \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}$:

We show:
$$X = \sum_{k=1}^{n} X([x_k]) \frac{\partial}{\partial x_k}$$

It is $Y := X - \sum_{k=1}^{n} X([x_k]) \frac{\partial}{\partial x_k}$ a derivation with $Y([x_k]) = 0 \ \forall k$

Lemma: $\forall\, \mathbb{R}^n \xrightarrow{f} \mathbb{R}$ diff. : $\exists\, \mathbb{R}^n \xrightarrow{f_i} \mathbb{R}$ diff. $(i = 1, \dots, n)$:
$$f(x) = f(0) + \sum_{k=1}^{n} x_k \cdot f_k(x)$$

Proof: Put $f_k(x) := \int_0^1 \frac{\partial f}{\partial x_k}(tx)\, dt$. $\square$

Now $[f] \in \mathcal{E}_n \underset{\text{Lemma}}{\implies} [f] = [f](0) + \sum_{k=1}^{n} [x_k] \cdot [f_k]$

$$\implies Y([f]) = Y(f(0)) + \sum_{k=1}^{n} Y([x_k]) \cdot f_k(0) = 0$$

$\square$

Note: $\dim(T_pM^n) = n$

Proposition: Suppose we have local coordinates
$(x_1, \dots, x_n)$ around $p \in N^n$
$(y_1, \dots, y_m)$ around $q := f(p) \in M^m$

$$\begin{tikzcd}(N,p) \arrow[r, "{[f]}"] \arrow[d, "\varphi"'] & (M,q) \arrow[d, "\psi"] \\ (\mathbb{R}^n, 0) \arrow[r, "\psi \circ f \circ \varphi^{-1} \hat{=} f"'] & (\mathbb{R}^m, 0)\end{tikzcd}$$

Then the differential of a germ $[f] : (N,p) \to (M,q)$ (concerning the bases of derivations of T_pN resp. T_qM) is:

$$Df_0 : \mathbb{R}^n \to \mathbb{R}^m \quad \text{with} \quad Df_0 := \left(\frac{\partial f_i}{\partial x_j}(0)\right) \text{ Jacobi matrix}$$

Proof: $[\varphi] \in \mathcal{E}_m \Rightarrow$
$$df_0\left(\frac{\partial}{\partial x_i}\right)([\varphi]) = \frac{\partial}{\partial x_i}([\varphi] \circ [f])$$
$$= \sum_{j=1}^m \frac{\partial \varphi}{\partial y_j}(0) \cdot \frac{\partial f_j}{\partial x_i}(0)$$
$$\Rightarrow df_0\left(\frac{\partial}{\partial x_i}\right) = \sum_{j=1}^m \frac{\partial f_j}{\partial x_i}(0) \frac{\partial}{\partial y_j}$$

$\square$

Alternative definition of tangent space (geometrical approach):

Def: $W_p := \{\text{diff. germs } \bar{w} : (\mathbb{R}, 0) \to (N,p)\}$
$\bar{v} \sim \bar{w} \;:\Leftrightarrow\; \forall \bar{f} \in \mathcal{E}(p): \frac{d}{dt} \bar{f} \circ \bar{w}(0) = \frac{d}{dt} \bar{f} \circ \bar{v}(0)$
Then the (geometrical) tangent space of N in $p \in N$ is $(T_pN)_{geom} := W_p/_\sim$.

We define the derivation $X_w(\bar{f}) := \frac{d}{dt} \bar{f} \circ \bar{w}(0)$ and the mapping

$$\tau : (T_pN)_{geom} \longrightarrow T_pN$$
$$[\bar{w}] \longmapsto X_w$$

Then τ is an isomorphism:

- τ injective: $X_w(\bar{f}) = X_v(\bar{f}) \Rightarrow \frac{d}{dt} \bar{f} \circ \bar{w}(0) = \frac{d}{dt} \bar{f} \circ \bar{v}(0)$
  $\Leftrightarrow \bar{v} \sim \bar{w}$, i.e. $[\bar{v}] = [\bar{w}]$
- τ surjective: $w(t) := t\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \Rightarrow X_w = \sum_{k=1}^n a_k \frac{\partial}{\partial x_k}$.

The differential can be defined analogously

$$\begin{aligned} df_p : (T_pN)_{geom} &\longrightarrow (T_pM)_{geom} \\ [\bar{w}] &\longmapsto [\bar{f} \circ \bar{w}] \end{aligned}$$

Then the definitions are equivalent:

$$X_{f \circ w}(\bar{\varphi}) = \frac{d}{dt} \bar{\varphi} \bar{f} \bar{w}(0) = X_w(\bar{\varphi} f) = df_p(X_w)(\bar{\varphi})$$

$$\Rightarrow \quad \begin{tikzcd} (T_pN)_{geom} \arrow[r, "df_p"] \arrow[d, "\tau"'] & (T_qM)_{geom} \arrow[d, "\tau"] \\ T_pN \arrow[r, "df_p"'] & T_qM \end{tikzcd} \quad \text{commutes}$$