The Pontryagin construction

Open Question: How many homotopy classes are there for mappings $S^m \longrightarrow S^p$ where $m > p$?

To answer that question Pontryagin developed the theory of framed cobordism that is introduced in the following. This presentation will be centered around three main theorems that will enable the reader to classify all mappings $S^m \longrightarrow S^p$ $(m>p)$ up to homotopy.

Def: (Cobordism, Codimension)

Let N, N' be cpct., n-dim. submanifolds of M, where M is m-dim, cpct & $\partial M = \emptyset$. Let $\partial N = \partial N' = \emptyset$ too.

The codimension of N & N' is defined as $\mathcal{E} := m-n$.

N and N' are said to be cobordant within M if $N \times [0,\varepsilon) \cup N' \times (1-\varepsilon, 1]$ can be extended to a cpct. mf. $X \subset M \times [0,1]$ s.t. ⓐ $\partial X = N \times \{0\} \cup N' \times \{1\}$ and

ⓑ $X \cap \left(M \times \{0\} \cup M \times \{1\}\right) = \partial X$.

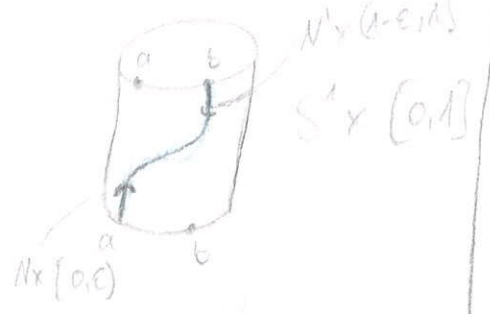
The triple $(X; N, N')$ is called a cobordism.

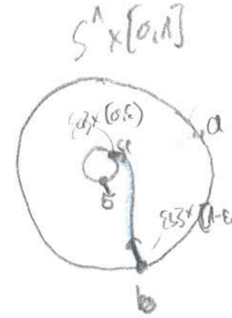
Example: ① Two points in S^1:

$N = \{a\}$, $N' = \{b\}$, $M = S^1$:

Obviously ⓐ: $\partial X = \{a\} \times \{0\} \cup \{b\} \times \{1\}$

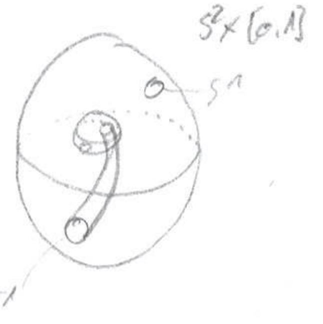
and ⓑ $X \cap (S^1 \times \{0\} \cup S^1 \times \{1\}) = \partial X$





② Two circles in S^2:

$N = S^1$, $N' = S^1$, $M = S^2$:



<u>Rmk:</u> Since cobordism defines an equivalence relation we could ask 'How does $\{M \mid M \text{ cpct}, \partial M = \emptyset\}/\sim$ look like?', where $A \sim B :\Longleftrightarrow \exists X$ cpct mf: $(X; A, B)$ is cobordism, i.e. classify all manifolds up to cobordism.

$\Rightarrow$ Go into co-/bordism theory!

<u>Pf:</u> • $A \sim A$ clear since $A \times [0,1]$ cpct.

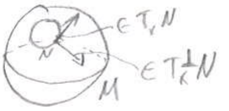
• $A \sim B \Rightarrow A \times [0,\varepsilon) \cup B \times (1-\varepsilon, 1]$ can be extended $\Rightarrow$

$B \times [0,\varepsilon) \cup A \times (1-\varepsilon, 1]$ —— " —— $\Rightarrow B \sim A$

• $A \sim B$ & $B \sim C \Rightarrow (X_1; A, B)$ & $(X_2; B, C)$ are cobordisms $\Rightarrow$

$X_1 \cup X_2$ is a cobordism for A & $C \Rightarrow A \sim C$. ∎

<u>Def:</u> (Framing, framed submanifold, framed cobordism) <u>Example:</u>



A smooth fct. $\nu : \begin{cases} N \longrightarrow T_x N^\perp \\ x \longmapsto (v^1(x), \dots, v^\delta(x)) \end{cases}$, where $(v^1(x), \dots, v^\delta(x))$ is a basis of normal vectors for $T_x N^\perp \subset T_x M$ is called a <u>framing</u> of N.

(N, ν) is a <u>framed submanifold</u> & $(N, \nu), (N', w)$ are <u>framed cobordant</u> if they are cobordant & $\exists$ framing u s.t. $u|_{N \times [0,\varepsilon)} = \nu \times \{0\}$ & $u|_{N' \times (1-\varepsilon, \varepsilon]} = w \times \{0\}$,

i.e. $u(x) = (u^1(x), \dots, u^\delta(x))$, $u^i(x,t) = \begin{cases} (v^i(x), 0) & : (x,t) \in N \times [0,\varepsilon) \\ (w^i(x), 0) & : (x,t) \in N' \times (1-\varepsilon, 1] \end{cases}$. Framed cobordism is an equivalence rel. $\sim$ too.

<u>Recall:</u> From Konstantin's talk / Lemma 1 p. 11 in Milnor:

$f : M \to N$ smooth, $y \in N$ reg. value $\Rightarrow f^{-1}(y)$ is smooth mf. of dim $\delta = m-n$.

and also (Lemma 2 p. 12 in Milnor)

The null space of $d_x f : T_x M \to T_y N$ is equal to the tangent space $T_x M' \subset T_x M$ of $M' = f^{-1}(y)$. $\Rightarrow d_x f(T_x M'^\perp) = T_y N$

<u>Def.</u> (Pontryagin mf.)

Let $f : M \to S^p$ be smooth, $y \in S^p$ a reg. value. Let further ν be a positively oriented base of $T_y S^p$, i.e. $w = d_x f^{-1}(\nu) =: f^* \nu$

—— " —— of $T_x f^{-1}(y)^\perp$ (Lemma 2). The framed submanifold $(f^{-1}(y), f^* \nu)$ is called a <u>Pontryagin mf.</u> of f.

The three following theorems will make the connection between homotopy classes & framed cobordism classes and are to be proven:

Thm. A: Let y, y' be reg. values of $f: M \to S^p$ (smooth) and ν & ν' be positively oriented bases for $T_x S^p$ then
$(f^{-1}(y), f^*\nu) \sim_f (f^{-1}(y'), f^*\nu')$.

Thm. B: $f, g: M \to S^p$ are smoothly homotopic iff $(f^{-1}(y), f^*\nu) \sim_f (g^{-1}(y), g^*\nu)$.

Thm. C: Any framed subm. (N, w) of codimension p in M occurs as a Pontryagin mf. for some smooth $f: M \to S^p$.

⊛ Hoc Hopf thm.

To prove Thm A. we'll need three lemmas:

Lemma 1: If ν & ν' are different positively oriented bases at y, then
$(f^{-1}(y), f^*\nu) \sim_f (f^{-1}(y), f^*\nu')$.

Pf: Let $\gamma: [0,1] \to GL^+(p, \mathbb{R})$ smooth with $\gamma(0) = \nu$, $\gamma(1) = \nu'$.
Such a path exists since $GL^+(p, \mathbb{R})$ is an open subset of $\mathbb{R}^{p^2}$, thus connectedness $\Leftrightarrow$ path-connectedness, and $GL^+(p, \mathbb{R})$ is connected.
Now $f^{-1}(y) \times [0,1]$ is a cobordism between $f^{-1}(y)$ and $f^{-1}(y)$ and γ induces a framing of that cobordism. ∎

Lemma 2: If y is a reg. value of f, and z is sufficiently close to y, then $(f^{-1}(z), f^*\nu) \sim_f (f^{-1}(y), f^*\nu)$ for some ν.

Pf: Analogous to the proofs in §4,5, where we showed that the degree is locally constant.
Idea: $\exists \varepsilon > 0$: $B_\varepsilon(y)$ contains only reg values.

- Choose rotations $r: \begin{cases} I \times S^p \to S^p \\ (t, x) \mapsto r(t,x) \end{cases}$ s.t. $r(1, y) = z$, $r(t,x) = id$ for $t \in [0, \varepsilon')$, $r(t,x) = r(1,x)$ for $t \in [1-\varepsilon', 1]$
- Def: $F: \begin{cases} M \times I \to S^p \\ (x, t) \mapsto r(t, f(x)) \end{cases}$
- $F^{-1}(z)$ is framed cobordism between $f^{-1}(z)$ and $f^{-1}(y)$

∎

Lemma 3: If f & g are smoothly homotopic, then $(f^{-1}(y), f^*\nu) \sim_f (g^{-1}(y), g^*\nu)$ for any common reg. value and framing ν.

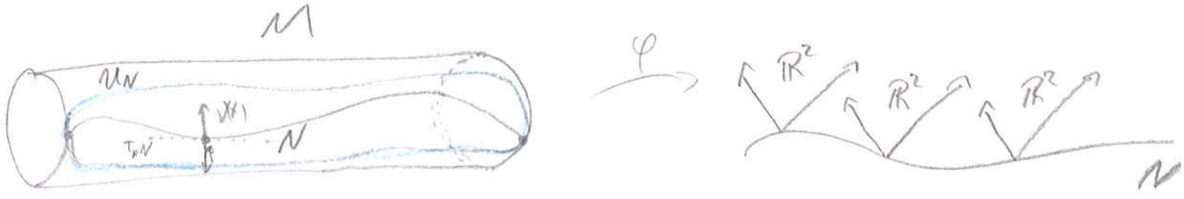
Proof: $\exists\, F: M \times I \to S^p$, $F(x,t) = f(x)$ for $t \in [0,\varepsilon)$
$F(x,t) = g(x)$ " $t \in (1-\varepsilon, 1]$

- Choose reg val. $z \in B_\varepsilon(y)$ s.t. $f^{-1}(z) \sim_f f^{-1}(y)$ and $g^{-1}(z) \sim_f g^{-1}(y)$.
- Then $F^{-1}(z) = \{(x,t) \in M \times I : F(x,t) = z\}$ is a framed cobordism between $f^{-1}(z)$ and $g^{-1}(z)$ since $F^{-1}(z)$ cpct as closed subset of cpct space M and $F^*\nu$ provides the framing. ∎

Proof of Thm A: Given reg. values y, z of $f: M \to S^p$ choose rotations $r: S^p \times I \to S^p$ s.t. $r(x,0) = id$, $r(y,1) = z$. Then $f = r(f,0)$ is homotopic to $r(f(x),1) \overset{\text{Lemma 3}}{\Longrightarrow} f^{-1}(z)$ is framed cobordant to $f^{-1}(y)$ ∎
(The statement for the framings follows from Lemma 1 & transitivity of $\sim_f$)

To prove thm C we'll need the product nbhd. thm. which I'll just state w/o. proof:

Product Nbhd. Thm: (N,ν) framed submf. of codim. p, N closed. $\exists$ nbhd $U_N \subset M$ of N s.t. $\varphi \in \text{Diffeo}(U_N, N \times \mathbb{R}^p)$. Furthermore one can choose φ s.t. $\forall x \in N$ $\varphi(x) = (x,0) \in N \times \mathbb{R}^p$, i.e. every normal frame $\nu(x)$ corresponds the the standard base of $\mathbb{R}^p$.



An unframeable submf:

for this mf $\nexists$ Prod. nbhd.

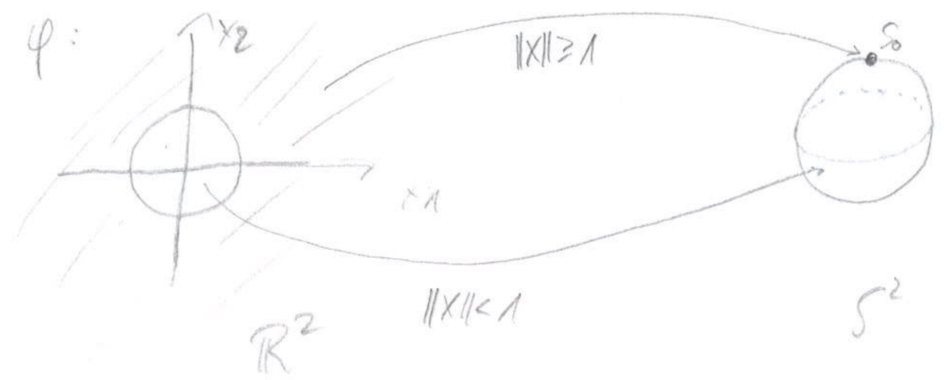
Proof of thm C: Let $(N,\nu) \subset M$ be closed, framed submf. Let $U_N \subset M$ be a nbhd. of N.
Then the above thm. provides a diffeo:

$g: N \times \mathbb{R}^p \to U_N \subset M$.

Now def. $\pi: \begin{cases} U_N \to \mathbb{R}^p \\ g(x,y) \mapsto y \end{cases}$, i.e. the projection onto $\mathbb{R}^p$.

Then since 0 is a reg. value $\pi^{-1}(0) = \{g(x,y) \in U_N : \pi(g(x,y)) = y = 0\} = N$.

Choose $\varphi: \mathbb{R}^p \to S^p$ s.t. if $\|x\| \geq 1$; $\varphi(x) = s_0$ for some s_0 and if $\|x\| < 1$
$\varphi: S^{p-1} \xrightarrow{\sim} S^p \setminus \{s_0\}$ diffeo.



Define $f : M \to S^p$ by $f(x) = \begin{cases} \varphi(\pi(x)) & \text{if } x \in U_N \\ s_0 & \text{if } x \notin U_N \end{cases}$. f is smooth everywhere and $\varphi(0)$ is reg. val. of f. Thus $(f^{-1}(\varphi(0)), f^*\nu)$ is a Pontryagin mf. But since $f^{-1}(\varphi(0)) = \pi^{-1}(0) = N$ and $f^*\nu = w$ we've shown the claim. ∎

Now comes in a way the main result of the talk that relates homotopy-classes & framed cobordism classes. But since the proof is slightly technical I'll skip over the proof and only do it a the end of the talk if there's enough time left.

<u>Lemma 4</u>: Let $f, g : M \to S^p$ smooth with y reg. val. of f <u>and</u> g.

If $(f^{-1}(y), f^*\nu) = (g^{-1}(y), g^*\nu)$ then $f \sim_h g$.

<u>Idea</u>:
- $f^*\nu = g^*\nu \Rightarrow d_x f = d_x g \;\; \forall x \in f^{-1}(y)$
- Assume $f = g$ on a nbhd. $U_{f^{-1}(y)} \subset M \Rightarrow$ Construct homotopy ✓
- choose again product reps. of $U_{f^{-1}(y)}$ and constr. new homotopy

$$\tilde{F} : \begin{cases} N \times \mathbb{R}^p \times [0,1] \to \mathbb{R}^p \\ (x, u, t) \mapsto (1 - \lambda(u) t) F(x,u) + \lambda(u) t \, G(x,u) \end{cases}$$ ✓

<u>Proof of the B</u>: $\Rightarrow$: Let $f \sim_h g \overset{\text{Lemma 3}}{\Longrightarrow} (f^{-1}(y), f^*\nu) \sim_f (g^{-1}(y), g^*\nu)$

$\Leftarrow$: Let (X, w) be a framed cobordism between $f^{-1}(y)$ and $g^{-1}(y)$. Construct a homotopy $F : M \times [0,1] \to S^p$ analogously to the proof of thm. C. Then $(F^{-1}(y), F^*\nu) = (X, w)$. Since $F(0,x) = f(x)$, $F(1,x) = g(x)$, $F(0,x)$ and $f(x)$ have the same Pontryagin mf and $F(1,x)$, $g(x)$ have the same.

Then Lemma 4 provides that $F(1,x) \sim_h g(x)$, and
$F(0,x) \sim_h f \Rightarrow g \sim_h f$

$\blacksquare$

The Hopf Theorem:

If now M is connected, oriented (and closed) mf. of dim p, a framed submf. N of codim. p is a finite set of pt's with a base at each of them.

Let $\operatorname{sgn}(x) = \begin{cases} +1 & \text{if the base @ } x \text{ is positively oriented} \\ -1 & \text{— " — negatively — " —} \end{cases}$.

Then $\sum_{x\in N} \operatorname{sgn}(x) = \deg(f)$, where $f: M \to S^p$ induces N.

Thus the framed cobordism class of N is uniquely* determined by $\sum_{x\in N} \operatorname{sgn}(x)$.

Hence we just proved

Hopf's thm: If M is connected, oriented and closed, then: $f,g: M \to S^m$ are smoothly homotopic $\Leftrightarrow \deg(f) = \deg(g)$

Pf: $\Leftarrow$: $\deg(f) = \deg(g) \Rightarrow \sum_{x\in f^{-1}(y)} \operatorname{sgn}(x)$ determines $[(f^{-1}(y), f^*\nu)]_f$ uniquely

$\Rightarrow (g^{-1}(y), g^*\nu) \in [(f^{-1}(y), f^*\nu)]_f \overset{\text{Thm 3}}{\Longrightarrow} f \sim_h g$.

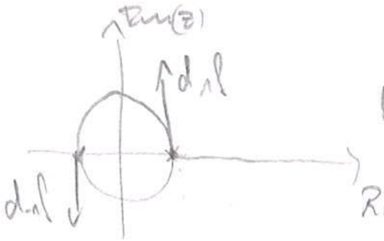
$\Rightarrow$: $f \sim_h g \Rightarrow (f^{-1}(y), f^*\nu) \sim_f (g^{-1}(y), g^*\nu) \Rightarrow \sum_{x\in f^{-1}(y)} \operatorname{sgn}(x) = \deg(f) = \deg(g)$.

$\blacksquare$

---

* $f: \begin{cases} S^1 \overset{\mathbb{C}}{\to} S^1 \overset{\mathbb{C}}{} \\ z \mapsto z^2 \end{cases}$ smooth, $y = 1$ reg. val. of f, $f^{-1}(y) = \{z \in S^1 : f(z) = z^2 = 1\} = \{1, -1\}$

$d_z f = 2z$, $d_{f^{-1}(1)} f = \{2, -2\}$



both positively oriented $\Rightarrow$

$\sum_{x\in f^{-1}(y)} \operatorname{sgn}(x) = 2$

If M is not orientable one can reverse the orientation of the tangent space T_xM smoothly. Thus instead of the degree the mod 2-degree determines the the framed cobordism class of a framed submf uniquely.

Thm: If M is closed, connected, but non-orientable, then: $f, g : M \to S^m$

$f \sim_h g \iff \deg_2(f) = \deg_2(g)$.