Proof of Whitney Graustein Theorem.

Let us consider curves with certain conditions in $\mathbb{R}^2$; more explicitly: regular closed curves in $\mathbb{R}^2$.

Def: A regular closed curve in $\mathbb{R}^2$ is a continuously differentiable map (*) $\gamma : \underline{I} \longrightarrow \mathbb{R}^2$ with the following properties:

i) $\gamma(0) = \gamma(1)$ $\rightsquigarrow$ the curve is closed $\left(\int_0^1 \gamma'(s)ds = 0 \right)$

ii) $\gamma'(t) \neq 0 \quad \forall t \in (0,1)$ $\rightsquigarrow$ $\exists$ tangentvector at every point of γ.

iii) $\gamma'(0) = \gamma'(1)$ $\rightsquigarrow$ the tangent vector turns to its initial direction

(*) $\rightsquigarrow$ without jumps, tangentvector moves smoothly and continuously.

ex:

now we want to deform one of these curves into another by obeying the following conditions:

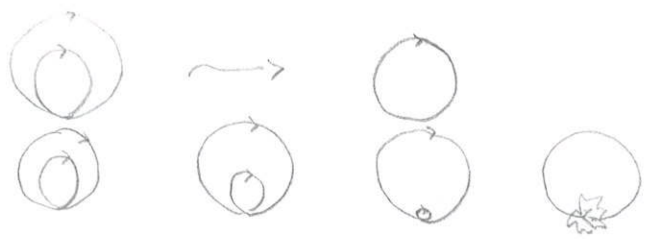
- not break or lift the curve and every intermediate curve
- the curves must not jump $\rightsquigarrow$ continuously throuout the def.
- each intermediate curve must be regular $\Rightarrow$ corners are not allowed
- tangentvector must turn and move contin.
  $\Rightarrow$ not jump or change direction

We call this a regular homotopy. Sum up the conditions to a Def

Def: A regular homotopy is a contin. differentiable homotopy $h: [0,1]^2 \longrightarrow \mathbb{R}^2$, $t \longmapsto h(s,t)$
s.t. $\forall s$ $h(s,t)$ is a regular closed curve and the partial derivative $\frac{\partial h}{\partial t}$ is a (free) homotopy between loops.

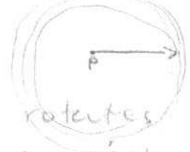
$\rightsquigarrow$ Two regular closed curves γ, δ are called reg. hom. if $\exists$ reg. hom. h with $h(0,t) = \gamma$, $h(1,t) = \delta$.

non-example



$\rightsquigarrow$ tangentvector doesnt turn cont. it jumps

Def the rotation number of γ $\text{rot}(\gamma)$ is defined as the degree of the map $\varphi: S^1 \longrightarrow \mathbb{R}^2 \setminus \{0\}$, $t \longmapsto \gamma'(s)$



remark: It is the number of times the tangent vector rotates (counterclockwise) traversing the closed curve once. It is always an integer and the direction of the curve affects $\text{rot}(\gamma)$ by ± 1

ex

$\text{rot}(\gamma) = 1$ $\quad$ $\text{rot}(\gamma) = 1$ $\quad$ $\text{rot}(\gamma) = 0$

<u>Theorem:</u> Two regular closed curves γ_0, γ_1 are regular homotopic if and only if $rot(\gamma) = rot(\delta)$

pf. „$\Rightarrow$" suppose two given r.c. curves are reg. hom.
$\Leftrightarrow$ $\exists$ a regular homotopy ~~between them~~ h
$\Rightarrow$ $\exists$ a (free) homotopy between γ_0' and γ_1' $\rightsquigarrow$ $\frac{\partial h}{\partial t}$ is a homotopy
$\Rightarrow$ γ' and δ' are homotopic
$\Rightarrow$ $deg(\gamma_0') = deg(\gamma_1')$
$\Leftrightarrow$ $rot(\gamma_0) = rot(\gamma_1)$

$\boxtimes$

non-example:
rk: • not true if tangentvector doesn't move cont. in a particular point
○ ∞ do not have the same rotation number
• ~~same with~~ turning ○ inside out.
It is only possible by violating the condition of
– the direction of the ~~tangentvector~~ must change cond.
– breaking the curve

„$\Leftarrow$" Now suppose two regular closed curves $\gamma_0, \gamma_1 : S^1 \to \mathbb{R}^2 \setminus \{0\}$ have the same rotation number. $r = rot(\gamma_0) = rot(\gamma_1)$.

• We want to find a regular homotopy between γ and ...

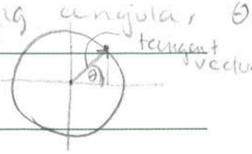
(Preparation: Shrink or magnify each given curve to a standard length)

• Since $rot(\gamma_0) = rot(\gamma_1)$, the maps $\gamma_0' : S^1 \to \mathbb{R}^2 \setminus \{0\}$, $\gamma_1' : S^1 \to \mathbb{R}^2 \setminus \{0\}$ have the same degree. By Hopf's theorem it follows that $\exists$ a homotopy between γ_0' and γ_1' $h : I \times S^1 \to \mathbb{R}^2 \setminus \{0\}$
$(s,t) \mapsto h(s,t)$

<u>Idea</u> Integrate h to get a regular homotopy between γ_0 and γ_1.
recall: a regular homotopy $H : I \times S^1 \to \mathbb{R}^2 \setminus \{0\}$ must fulfill
$(s,t) \mapsto H(s,t)$
the condition that $\forall s \in I$ $H(s,\cdot)$ is a regular closed curve.
$\rightsquigarrow$ 1. $\int_0^1 H'(s,t)\,dt = 0$ i.e. $H(s,0) = H(s,1)$ $\forall s$
2. $H'(s,t) \neq 0$ $\forall t$

We must define $h(s,\cdot)$ and alter it properlly such that by integration 1. and 2. is fulfilled.

• <u>Define h</u>
1. Turn the curves such that the initial tangent vectors point in the same direction $\rightsquigarrow$ $\gamma_0'(0) = \gamma_1'(0) = (\cos(0), \sin(0))$
2. now as $\gamma_0', \gamma_1' : I \to \mathbb{R}^2 \setminus \{0\}$ $\exists$ a corresponding angular $\Theta_i(t)$ with $\gamma_i'(t) = (\cos(\Theta_i(t)), \sin(\Theta_i(t)))$
$\rightsquigarrow$ $\Theta_i(0) = 0$, $\Theta_i(1) = r \cdot 2\pi$ $i = 1, 2$.



3. Set $\Theta_s(t) := s\,\Theta_1(t) + (1-s)\,\Theta_0(t)$
and define $h(s,t) := (\cos(\Theta_s(t)), \sin(\Theta_s(t)))$
this is well defined since $h(0,t) = (\cos(\Theta_0(t)), \sin(\Theta_0(t))) = \gamma_0'(t)$
$h(1,t) = \gamma_1'(t)$
obviously contin.

- alter h

Set $\tilde{h}(s,t) = h(s,t) - \int_0^1 h(s,u)\,du$

well defined: $\tilde{h}(0,t) = h(0,t) - \underbrace{\int_0^1 \underbrace{h(0,u)}_{\gamma_0'(u)}\,du}_{=0} = \gamma_0'(t)$ same with $\tilde{h}(1,t)$

- Integration

- Integrations leads to

$H: I \times S^1 \longrightarrow \mathbb{R}^2 \qquad H(s,t) = \int_0^t \tilde{h}(s,t)\,dt$

claim 1. H is a homotopy between γ and δ

pf $H(0,t) = \int_0^t \tilde{h}(0,u)\,du = \int_0^t \gamma'(u)\,du = \gamma(t) - \underbrace{\gamma(0)}_{0}$

$H(1,t) = \int_0^t \tilde{h}(1,u)\,du = \int_0^t \delta'(u)\,du = \delta(t) - \underbrace{\delta(0)}_{0 ?}$ ok

claim 2. $\forall s$ $H(s,\cdot)$ is closed

pf: $\forall s$ $H(s,0) = H(s,1)$

$H(s,0) = \underbrace{\int_0^0 \tilde{h}(s,u)\,du}_{=0} = \underbrace{\int_0^1 \tilde{h}(s,u)\,du}_{\int_0^1 h(s,u)\,du - \int_0^1 \underbrace{\int_0^1 h(s,u)\,du}_{\text{fixed}}\,du} = 0$

claim 3. $H(s,\cdot)$ is regular $\forall s$ $\rightsquigarrow$ $H'(s,t) \neq 0$

$H'(s,t) = \tilde{h}(s,t) = h(s,t) - \int_0^1 h(s,t)\,dt \neq 0$

$\iff h(s,t) \neq \int_0^1 h(s,t)\,dt \quad \forall s,t$

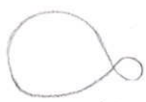
This holds $\forall \gamma$ with $\operatorname{rot}(\gamma) \neq 0$

bc. then $h(s,t)$ passes over all of S^1, at least once. By def.

 $h(s,t)$ of $h(s,t)$. Therefore the average $\int_0^1 h(s,t)\,dt$ lies in the interior of S^1  $\Rightarrow \int_0^1 h(s,t)\,dt \neq h(s,t)$

- For $\operatorname{rot}(\gamma) = 0$ We may alter $\Theta_s(t)$ s.t. $h(s,t)$ is not constant

Then $\tilde{h}(s,t) \neq 0$.

δ $\rightsquigarrow$  $\tilde{\delta}$ $\Rightarrow \int_0^1 \tilde{\delta}\,dt \neq \tilde{\delta}$