Humboldt-Universität zu Berlin
Mathematisch-Naturwissenschaftliche Fakultät
Institut für Mathematik
AG *Geometrische Analysis und Spektraltheorie*

Masterarbeit

zur Erlangung des akademischen Grades
Master of Science (M. Sc.)
im Studienfach Mathematik

Dirac Eigenvalues of Homogeneous 3-Spheres

(Dirac-Eigenwerte homogener 3-Sphären)

eingereicht von

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geboren am 4. August 1996 in Berlin

eingereicht in/am

Berlin, den 9. September 2021

betreut durch

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Berlin, den 8. September 2021

Document version: 4th March 2024. *(See below for changelog.)*

Version	**Changes**
2023.03.22	removed deprecated university e-mail address; fixed $\dim_{\mathbb{R}}(\mathrm{Mat}_{\mathbb{C}}(n))$ (typo in §1); fixed Corollary 3.7 (credits to Peter Grabs)
2022.04.22	added new e-mail address; fixed broken link in title page; typos
2021.09.29	feedback from supervisors incorporated (only minor remarks; includes fixing a typo in an equation and misleading wording in two places)
2021.09.09	original version of this document (as handed in)

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Acknowledgements

I would like to thank my supervisor Dorothee Schüth, who has been mentoring me for many years and provided me with almost motherly advice in both mathematical and non-mathematical concerns. I am in awe of her knowledge, without which my work could not exist.

> *Sayın Dorothee Hanım,*
>
> *daha o zamanlar ben ilk dönem öğrencisiyken eşsiz hassasiyetinizle kalbimi kazandınız. Birçok sorunla karşılaşmak zorunda kaldığım son yıllarda bu konuda hiçbir şeyin değişmediğini bilmelisiniz, çünkü bana olan inancınız ve sürekli desteğiniz olmasaydı okulu başarıyla bitirebilir miydim gerçekten bilmiyorum. Kesinlikle hayatı renklendiren bu dünyanın güzel dilleriyle uğraşmayı ertelerdim...*
>
> *Benim için yaptığınız her şey için teşekkür etmek istiyorum, çünkü siz olmasaydınız şimdi olduğum yerde olmazdım.*

I am deeply grateful for the presence of my homies, who have been accompanying me through my life for many years of the quarter of a century that I already exist in this weird place. Your garden investments will not be forgotten, nor will the gaming sessions of supreme quality while vibing to the entrancing melodies of K-pop. *(Those were inducing creative efforts.)*

Last but certainly not least, my special thanks go to the following professors, with whom I had the pleasure of working during my time as a student assistant, for their contributions to this thesis:

- Helga Baum — a great teacher, a pleasant colleague to work with, and an expert in her field — for co-supervising this thesis despite her well-deserved retirement;
- Thomas Klose, for explaining to me why some people (in particular, theoretical physicists) might be interested in the Dirac operator apart from the underlying mathematical structures;
- Batu Güneysu, for confirming the grammatical correctness of my Turkish.

Thank you all!

This work is dedicated to my dog Biscuit for being the purest of all homies.

Introduction

The Dirac operator is used by theoretical physicists to write down the equation of motion of spin-$\frac{1}{2}$ particles (protons, electrons, ...) in the setting of relativistic quantum mechanics. Considering the Dirac equation on curved spaces is the attempt to include gravity, which Einstein recognised as the curvature of spacetime.

Viewed as a purely mathematical object (not caring too much about understanding the universe), the *Dirac operator* is a first-order elliptic linear differential operator living on a certain vector bundle (the *spinor bundle*) over an oriented Riemannian manifold endowed with some additional structure (a *spin structure*). An interesting question to investigate — at least from a mathematician's point of view — is in which ways the purely analytical information encoded in the Dirac operator (such as its spectral data) is related to the geometry of the underlying space.

In this thesis, we examine the situation where the underlying space is the harmless manifold $\mathbb{S}^3 = \{x \in \mathbb{R}^4 : |x| = 1\}$, endowed with the structure of a Lie group, which comes with a family of natural Riemannian metrics to consider — namely the *left-invariant* ones. These spaces are what the title of the thesis refers to as *homogeneous 3-spheres*, and each of them admits a unique and trivial spin structure (after an orientation has been fixed), for which the associated Dirac operator can be defined. We are particularly interested in the question whether two non-isometric homogeneous 3-spheres can be distinguished by their Dirac spectra.

Inspiration

The topic of this thesis, as kindly suggested to the author by his supervisor, was inspired by the following publications:

- Christian Bär's paper *"The Dirac operator on homogeneous spaces and its spectrum on 3-dimensional lens spaces"* (1992) [1], in which Bär (among other things) explicitly calculated the Dirac spectrum of a special subclass of homogeneous 3-spheres (the so-called *Berger spheres*); and
- Emilio Lauret's paper *"The smallest Laplace eigenvalue of homogeneous 3-spheres"* (2018) [2], in which Lauret (among other things) explicitly calculated the smallest Laplace eigenvalue of each homogeneous 3-sphere and proved that two homogeneous 3-spheres are Laplace isospectral if and only if they are isometric.

The main question of the thesis is whether a synthesis of these two results can be achieved; that is, whether it is possible to generalise Bär's calculation of the Dirac spectrum to *arbitrary* homogeneous 3-spheres and obtain a similar result as Lauret did for the Laplace operator.

Result

Indeed, we were able to obtain such a result, under the additional assumption that the underlying metric is of positive scalar curvature. More precisely, the goal of the thesis is to prove the following theorem (see §1 for notation):

Main Theorem *(Dirac Eigenvalues of Homogeneous 3-Spheres)*

Let $\mathbb{S}^3_{(a,b,c)}$ be a homogeneous 3-sphere of *positive scalar curvature*, endowed with either orientation.

Then the smallest absolute value of its Dirac eigenvalues is given by

$$\lambda_{*(a,b,c)} = a + b + c - \frac{1}{2}\Big(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}\Big) > 0,$$

and its Dirac spectrum determines the underlying metric up to isometry (within the class of homogeneous 3-spheres).

In order to *safely* guide the reader to the result stated above, the thesis follows the structure described below.

Structure

In section I (Preliminaries), we will clarify the geometric setting of the thesis (*"What is a homogeneous 3-sphere?"*) and give the necessary prerequisites so that every person with a background in differential geometry should be able to follow the proof of the Main Theorem.

- In §1 (The 3-Sphere as a Lie Group), we will introduce a Lie group structure on $\mathbb{S}^3$ (turning it into a homogeneous space) and give a classification of the arising left-invariant metrics up to isometry. Moreover, we will briefly discuss the representation theory of $\mathbb{S}^3 \cong \mathrm{SU}(2)$, which is eventually needed to extract the spectral data from the Dirac operator, using the *Peter–Weyl Theorem*.
- In §2 (Spin Geometry), we will introduce all objects and structures required to define the Dirac operator, and we will summarise its main analytic properties.

In section II (Dirac Eigenvalues of Homogeneous 3-Spheres), we will immerse ourselves right into the research question, with the result of the thesis shining bright at the end of the adventure.

- In §3 (Dirac Spectrum), we will clarify the spin geometry of $\mathbb{S}^3_{(a,b,c)}$ and derive a formula for the Dirac operator. We will then mimic Bär's approach to obtain a theoretical description of the Dirac spectrum via representation theory.
- In §4 (Smallest Eigenvalue and Spectral Invariance), we will discuss the (extensive but elementary) proof of the Main Theorem, mimicking Lauret's approach based on the *Gershgorin-Circle Theorem*. The paragraph will be split into several digestible parts, corresponding to different steps of the proof (including the heuristic).

There is nothing left to say except:

Have Fun!

The author hopes that reading this thesis is at least as joyful an experience as the writing process. *(The reader is allowed to deduce anything from this statement.)*

I Preliminaries

The purpose of this section is twofold: It serves to clarify the setting of the thesis (that is, to describe the geometry of homogeneous 3-spheres) as well as to discuss the necessary spin-geometric prerequisites for defining the corresponding Dirac operator and calculating its spectrum, the latter requiring a smidgen of representation theory.

The reader is assumed to be familiar with the fundamental notions of Riemannian geometry (such as the curvature of a Riemannian manifold), some basic theory of Lie groups, and fibre bundles (especially principal bundles, frame bundles, and associated vector bundles). Those topics that are unlikely to be covered in a typical three-semester course of differential geometry will be dealt with in the thesis. Note that once an expression for the Dirac operator is obtained and all the occuring objects are clear, the proof of the Main Theorem boils down to elementary calculations from analysis and linear algebra (which, however, does not imply that the result was easy to obtain).

We commence by introducing a Lie group structure on $\mathbb{S}^3$ (which only by itself is not too exciting a manifold), turning it into a homogeneous space acting on itself.

§1 — *The 3-Sphere as a Lie Group*

There are two common ways to endow the manifold $\mathbb{S}^3 = \{x \in \mathbb{R}^4 : |x| = 1\}$ with a group multiplication, turning it into a Lie group:

- Consider the special unitary group $\mathrm{SU}(n)$ consisting of all $U \in \mathrm{Mat}_{\mathbb{C}}(n)$ such that $UU^\dagger = 1$ and $\det(U) = 1$, where $U^\dagger$ denotes the conjugate transpose of U. It is a standard exercise in every elementary course of differential geometry to prove that $\mathrm{SU}(n)$ is an (n^2-1)-dimensional submanifold of $\mathrm{Mat}_{\mathbb{C}}(n) \cong \mathbb{R}^{2n^2}$ — hence, a Lie group — and that its Lie algebra $\mathfrak{su}(n) = \mathrm{T}_1\,\mathrm{SU}(n)$ consists of all $X \in \mathrm{Mat}_{\mathbb{C}}(n)$ such that $X + X^\dagger = 0$ and $\mathrm{tr}(X) = 0$. (This is an important application of the *Regular-Value Theorem*.)

  In the case $n = 2$, there is an evident diffeomorphism between $\mathbb{S}^3$ and $\mathrm{SU}(2)$, sending an element $(z, w) \in \mathbb{S}^3 \subset \mathbb{C}^2 \cong \mathbb{R}^4$ to $\left(\begin{smallmatrix} z & -\bar{w} \\ w & \bar{z} \end{smallmatrix}\right) \in \mathrm{SU}(2)$.
- Alternatively, we can view $\mathbb{S}^3 \subset \mathbb{R}^4 \cong \mathbb{H}$ as the set of unit quaternions equipped with the quaternion multiplication. This is clearly a Lie group, and its Lie algebra is the space $\mathrm{T}_1\mathbb{S}^3 = \{1\}^\perp = \mathrm{span}_{\mathbb{R}}\{\mathrm{i}, \mathrm{j}, \mathrm{k}\} \subset \mathbb{H}$.

By identifying $\mathbb{S}^3 \subset \mathbb{C}^2$ with $\mathbb{S}^3 \subset \mathbb{H}$ via $(z, w) \mapsto z - \mathrm{j}w$, both intepretations describe the same Lie group structure on $\mathbb{S}^3$. More precisely, we then have

$$1 \mathrel{\widehat{=}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathrm{i} \mathrel{\widehat{=}} \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \ \mathrm{j} \mathrel{\widehat{=}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \text{and } \mathrm{k} \mathrel{\widehat{=}} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$$

for which one can check $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$. In either case, the Lie bracket is simply given by $[X, Y] = XY - YX$, with the multiplication belonging to the respective algebra (that is, $\mathrm{Mat}_{\mathbb{C}}(2)$ or $\mathbb{H}$). Throughout the thesis, we will use whichever identification is more convenient for the current endeavour.

This settles the Lie group structure on the 3-sphere. The further procedure in this paragraph is as follows:

- In the first part (Left-Invariant Metrics), we introduce the natural geometries to consider on a general Lie group G and derive a classification (up to isometry) in our case $G = \mathbb{S}^3$.
- In the second part (Representation Theory), we state the *Peter–Weyl Theorem* for compact Lie groups G (which plays a key role in calculating the Dirac spectrum) and briefly describe the representation theory of $G = \mathbb{S}^3$ in this context.

Left-Invariant Metrics

Notational Remarks. For a Lie group G, we denote by $1 \in G$ its identity, by $\mathfrak{g} = \mathrm{T}_1 G$ its Lie algebra; for every $g \in G$, we denote by $\ell_g : G \ni h \mapsto gh \in G$ the left multiplication with g, and lastly, by ${\ell_g}_* : \mathfrak{g} \to \mathrm{T}_g G$ its differential at the identity. Given an inner product $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{g}$, we will *also* write $\langle \cdot\,, \cdot \rangle$ for the corresponding left-invariant metric on G (see below) obtained by propagation of $\langle \cdot\,, \cdot \rangle$ around G.

Definition 1.1 *(Left-Invariant Metric)*

A Riemannian metric $\langle \cdot\,, \cdot \rangle$ on a Lie group G is called **left-invariant** if for all $g \in G$, one has ${\ell_g}^*\langle \cdot\,, \cdot \rangle = \langle \cdot\,, \cdot \rangle$; that is, if $\langle {\ell_g}_* X, {\ell_g}_* Y \rangle_g = \langle X, Y \rangle_1$ holds for all $g \in G$ and $X, Y \in \mathfrak{g}$.

Define analogously **right-invariant** and **bi-invariant** Riemannian metrics.

In other words, the left multiplications of a Lie group G endowed with a left-invariant metric are isometries, and G is a Riemannian homogeneous space. This implies severe consequences concerning the geometry of G: Since the curvature tensor inherits left-invariance from the metric, so does the scalar curvature, which is a fancy way of saying that the latter is constant on G.

There is a one-to-one correspondence between the left-invariant metrics on G and the inner products on $\mathfrak{g}$, which are uncountably many. Fortunately, in the case $G = \mathbb{S}^3$, there is an easy way to describe the left-invariant metrics up to isometry using only three positive numbers as parameters. This will be useful later, since calculating the Dirac spectrum then boils down to a tangible multivariable problem.

To see this, consider the Killing form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on $\mathfrak{g} = \mathrm{T}_1\mathbb{S}^3$, that is, the Ad-invariant symmetric bilinear form defined by $B(X, Y) := \mathrm{tr}([X, [Y, \cdot\,]])$. Using the identification of $\mathbb{S}^3$ with the unit quaternions, one can easily compute that the matrix representation of B with respect to the basis $\{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$ of $\mathrm{T}_1\mathbb{S}^3$ is given by -8 times the identity. In other words, $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle := -\frac{1}{8}B$ defines a bi-invariant inner product on $\mathrm{T}_1\mathbb{S}^3$ such that $\{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$ becomes an orthonormal basis.

Definition 1.2 *(Inner Products on $\mathrm{T}_1\mathbb{S}^3$)*

1. We refer to $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle = -\frac{1}{8}B$ as the **bi-invariant standard inner product**.
2. Let $a, b, c > 0$. Define $\langle\,\cdot\,,\,\cdot\,\rangle_{(a,b,c)}$ as the inner product on $\mathrm{T}_1\mathbb{S}^3$ that has the matrix representation $\mathrm{diag}(\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2})$ with respect to the basis $\{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$ of $\mathrm{T}_1\mathbb{S}^3$, that is, such that $\{a\mathrm{i}, b\mathrm{j}, c\mathrm{k}\}$ becomes a $\langle\,\cdot\,,\,\cdot\,\rangle_{(a,b,c)}$-orthonormal basis.

As mentioned before, we will *also* write $\langle\,\cdot\,,\,\cdot\,\rangle_{(a,b,c)}$ for the left-invariant *metric* on $\mathbb{S}^3$ corresponding to the *inner product* $\langle\,\cdot\,,\,\cdot\,\rangle_{(a,b,c)}$ on $\mathrm{T}_1\mathbb{S}^3$. In addition, we will abbreviate the Riemannian manifold $(\mathbb{S}^3, \langle\,\cdot\,,\,\cdot\,\rangle_{(a,b,c)})$ to $\mathbb{S}^3_{(a,b,c)}$.

It turns out that it is sufficient to consider only the metrics $\langle\,\cdot\,,\,\cdot\,\rangle_{(a,b,c)}$ in order to fully describe the possible homogeneous geometries on $\mathbb{S}^3$. More precisely, we have the following well-known result:

Theorem 1.3 *(Classification of Left-Invariant Metrics on $\mathbb{S}^3$)*

Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be *any* left-invariant metric on $G = \mathbb{S}^3$. Then $(\mathbb{S}^3, \langle\,\cdot\,,\,\cdot\,\rangle)$ is isometric to $\mathbb{S}^3_{(a,b,c)} = (\mathbb{S}^3, \langle\,\cdot\,,\,\cdot\,\rangle_{(a,b,c)})$ for some $a \geq b \geq c > 0$.

A proof will be given for the sake of completeness, requiring some fundamental theory of Lie groups. For $g \in \mathbb{S}^3 \subset \mathbb{H}$, denote by ϕ_g the conjugation by g, that is, the inner automorphism defined by $\phi_g(h) := ghg^{-1}$. Consider the adjoint representation $\mathrm{Ad} : \mathbb{S}^3 \to \mathrm{Aut}(\mathrm{T}_1\mathbb{S}^3)$, sending g to $\mathrm{Ad}_g = \phi_{g*} : X \mapsto gXg^{-1}$. Then we have the following useful lemma:

Lemma 1.4 *(Adjoint Representation of $\mathbb{S}^3$)*

The group $\{\mathrm{Ad}_g\}_{g\in\mathbb{S}^3}$ of differentials of the inner automorphisms ϕ_g acts *transitively* on the set of $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$-orthonormal bases of $\mathrm{T}_1\mathbb{S}^3$ with the same orientation as $\{\mathrm{i},\mathrm{j},\mathrm{k}\}$.

In other words, one has $\mathrm{Ad}(\mathbb{S}^3) = \mathrm{SO}(\mathrm{T}_1\mathbb{S}^3, \langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle)$, where the latter denotes the group of orientation-preserving isometries of $\mathrm{T}_1\mathbb{S}^3$ with respect to $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$.

Proof. Since $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$ is bi-invariant, the inner automorphisms ϕ_g are isometries of $\mathrm{T}_1\mathbb{S}^3$, which means that Ad_g is an orthogonal map for each $g\in\mathbb{S}^3$. Since the continuous map Ad sends $1\in\mathbb{S}^3$ to $1\in\mathrm{SO}(\mathrm{T}_1\mathbb{S}^3,\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle)$ and $\mathbb{S}^3$ is connected, it follows that Ad_g preserves orientation; that is, $\mathrm{Ad}(\mathbb{S}^3)\subset\mathrm{SO}(\mathrm{T}_1\mathbb{S}^3,\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle)$. To see the other inclusion, let us recall the following fundamental result, carrying information from Lie algebras to Lie groups:

> Let $f : G\to H$ be a homomorphism of Lie groups. If H is connected and $\mathfrak{h}\subset f_*(\mathfrak{g})$, it follows that $H\subset f(G)$ (that is, $f(G)=H$).

We certainly know that $H=\mathrm{SO}(\mathrm{T}_1\mathbb{S}^3,\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle)$ is connected. Now consider the differential $\mathrm{Ad}_* = \mathrm{ad} : \mathrm{T}_1\mathbb{S}^3\to\mathfrak{so}(\mathrm{T}_1\mathbb{S}^3,\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle)$ of the adjoint representation, sending X to $\mathrm{ad}_X=[X,\,\cdot\,]$. It is easy to see that Ad_* is a vector space isomorphism: In fact, the endomorphisms $[\mathrm{i},\,\cdot\,]$, $[\mathrm{j},\,\cdot\,]$, and $[\mathrm{k},\,\cdot\,]$ are linearly independent (plug $\{\mathrm{i},\mathrm{j},\mathrm{k}\}$ into $\alpha_1[\mathrm{i},\,\cdot\,]+\alpha_2[\mathrm{j},\,\cdot\,]+\alpha_3[\mathrm{k},\,\cdot\,]=0$ to obtain $\alpha_1=\alpha_2=\alpha_3=0$); hence, by comparing dimensions, they must form a basis of the $\frac{3\cdot(3-1)}{2}=3$-dimensional vector space $\mathfrak{so}(\mathrm{T}_1\mathbb{S}^3,\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle)$.

It follows from the boxed statement that $\mathrm{Ad}(\mathbb{S}^3)=\mathrm{SO}(\mathrm{T}_1\mathbb{S}^3,\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle)$. ♥

This observation is already enough to prove Theorem 1.3.

Proof (of Theorem 1.3). Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be an arbitrary left-invariant metric on $\mathbb{S}^3$. Since $\langle\,\cdot\,,\,\cdot\,\rangle$ defines an inner product on $\mathrm{T}_1\mathbb{S}^3$, we can find a $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$-orthonormal basis $\{u,v,w\}$ of $\mathrm{T}_1\mathbb{S}^3$ in which $\langle\,\cdot\,,\,\cdot\,\rangle$ has the form $\mathrm{diag}(\frac{1}{a^2},\frac{1}{b^2},\frac{1}{c^2})$ for some $a,b,c>0$. We may assume $a\geq b\geq c$ by rearranging the basis if necessary. We may further assume that $\{u,v,w\}$ has the same orientation as $\{\mathrm{i},\mathrm{j},\mathrm{k}\}$ (if not, replace u by $-u$).

Now, due to Lemma 1.4, there exists a Lie group automorphism ϕ of $\mathbb{S}^3$ such that ϕ_* maps $\{\mathrm{i},\mathrm{j},\mathrm{k}\}$ to $\{u,v,w\}$. Then the pulled-back metric $\phi^*\langle\,\cdot\,,\,\cdot\,\rangle$ is again left-invariant, since

$$\ell_g{}^*(\phi^*\langle\,\cdot\,,\,\cdot\,\rangle)=\phi^*(\ell_{\phi(g)}{}^*\langle\,\cdot\,,\,\cdot\,\rangle)=\phi^*\langle\,\cdot\,,\,\cdot\,\rangle$$

⋮

holds for all $g \in \mathbb{S}^3$ (use the chain rule and the fact that ϕ is a homomorphism; that is, $\phi \circ \ell_g = \phi(g\,\cdot\,) = \phi(g)\,\phi(\,\cdot\,) = \ell_{\phi(g)} \circ \phi$).

However, we have $\phi^*\langle\,\cdot\,,\,\cdot\,\rangle = \langle\,\cdot\,,\,\cdot\,\rangle_{(a,b,c)}$ for $a, b, c > 0$ as above by construction, from which the claim promptly follows. ♥

Now that we know what the left-invariant metrics on $\mathbb{S}^3$ look like, it makes sense to seek expressions for the curvature in terms of the parameters a, b, and c. Specifically, we are interested in a formula for the scalar curvature, since it will turn out later that the calculation of the smallest Dirac eigenvalue corresponding to $\langle\,\cdot\,,\,\cdot\,\rangle_{(a,b,c)}$ gives rise to the natural restriction $\mathrm{scal}_{\mathbb{S}^3_{(a,b,c)}} > 0$. We will derive such an expression directly by calculating the sectional curvatures from the Levi–Civita connection.

Proposition 1.5 *(Scalar Curvature of Homogeneous 3-Spheres)*

The constant scalar curvature of $\mathbb{S}^3_{(a,b,c)}$ is given by

$$\mathrm{scal}_{(a,b,c)} \coloneqq \mathrm{scal}_{\mathbb{S}^3_{(a,b,c)}} \equiv 4(a^2+b^2+c^2) - 2\Big(\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2}\Big).$$

Proof. Let $\langle\,\cdot\,,\,\cdot\,\rangle = \langle\,\cdot\,,\,\cdot\,\rangle_{(a,b,c)}$ be a left-invariant metric on $\mathbb{S}^3$. Recall that the Levi–Civita connection ∇ is uniquely determined by the Koszul formula

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X(\langle Y, Z\rangle) + Y(\langle X, Z\rangle) - Z(\langle X, Y\rangle) \\ &\quad + \langle [X,Y], Z\rangle + \langle [Z,X], Y\rangle - \langle [Y,Z], X\rangle. \end{aligned}$$

Denote by $\{\mathrm{i},\mathrm{j},\mathrm{k}\}$ also the left-invariant *vector fields* on $\mathbb{S}^3$ corresponding to our standard basis of $\mathrm{T}_1\mathbb{S}^3$. Using that $\langle\,\cdot\,,\,\cdot\,\rangle$ has the form $\mathrm{diag}(\frac{1}{a^2},\frac{1}{b^2},\frac{1}{c^2})$ with respect to $\{\mathrm{i},\mathrm{j},\mathrm{k}\}$ as well as $[\mathrm{i},\mathrm{j}] = 2\mathrm{k}$ (plus cyclic permutations), it follows that

$$\langle \nabla_{\mathrm{i}}\,\mathrm{j}, \mathrm{i}\rangle = \langle \nabla_{\mathrm{i}}\,\mathrm{j}, \mathrm{j}\rangle = 0 \quad\text{and}\quad \langle \nabla_{\mathrm{i}}\,\mathrm{j}, \mathrm{k}\rangle = \frac{1}{c^2} + \frac{1}{b^2} - \frac{1}{a^2};$$

that is (using that ∇ is torsion-free in the second equation),

$$\nabla_{\mathrm{i}}\,\mathrm{j} = \Big(1 + \frac{c^2}{b^2} - \frac{c^2}{a^2}\Big)\mathrm{k} \quad\text{and}\quad \nabla_{\mathrm{j}}\,\mathrm{i} = \nabla_{\mathrm{i}}\,\mathrm{j} - [\mathrm{i},\mathrm{j}] = \Big(-1 + \frac{c^2}{b^2} - \frac{c^2}{a^2}\Big)\mathrm{k}.$$

One also finds $\nabla_{\mathrm{i}}\,\mathrm{i} = \nabla_{\mathrm{j}}\,\mathrm{j} = \nabla_{\mathrm{k}}\,\mathrm{k} = 0$. We can use this (plus cyclic permutations) to calculate the Riemannian curvature tensor

⋮

$$
\begin{aligned}
\langle \mathcal{R}(\mathrm{i},\mathrm{j})\mathrm{j},\mathrm{i}\rangle &= \langle \nabla_{\mathrm{i}} \nabla_{\mathrm{j}} \mathrm{j} - \nabla_{\mathrm{j}} \nabla_{\mathrm{i}} \mathrm{j} - \nabla_{[\mathrm{i},\mathrm{j}]} \mathrm{j}, \mathrm{i}\rangle \\
&= \Big\langle 0 - \Big(1+\frac{c^2}{b^2}-\frac{c^2}{a^2}\Big) \nabla_{\mathrm{j}} \mathrm{k} - 2\nabla_{\mathrm{k}} \mathrm{j}, \mathrm{i}\Big\rangle \\
&= -\Big\langle \Big(1+\frac{c^2}{b^2}-\frac{c^2}{a^2}\Big)\Big(1+\frac{a^2}{c^2}-\frac{a^2}{b^2}\Big)\mathrm{i} + 2\Big(-1+\frac{a^2}{c^2}-\frac{a^2}{b^2}\Big)\mathrm{i}, \mathrm{i}\Big\rangle \\
&= -\frac{1}{a^2}\Big[\Big(1+\frac{c^2}{b^2}-\frac{c^2}{a^2}\Big)\Big(1+\frac{a^2}{c^2}-\frac{a^2}{b^2}\Big) + 2\Big(-1+\frac{a^2}{c^2}-\frac{a^2}{b^2}\Big)\Big].
\end{aligned}
$$

Expand this expression to see that the sectional curvatures are given by

$$
\kappa(\mathrm{i},\mathrm{j}) = \frac{\langle \mathcal{R}(\mathrm{i},\mathrm{j})\mathrm{j},\mathrm{i}\rangle}{\langle \mathrm{i},\mathrm{i}\rangle\langle \mathrm{j},\mathrm{j}\rangle} = a^2b^2\langle \mathcal{R}(\mathrm{i},\mathrm{j})\mathrm{j},\mathrm{i}\rangle = 2a^2+2b^2-2c^2-\frac{3a^2b^2}{c^2}+\frac{b^2c^2}{a^2}+\frac{c^2a^2}{b^2}
$$

plus analogous expressions for $\kappa(\mathrm{j},\mathrm{k})$ and $\kappa(\mathrm{k},\mathrm{i})$ obtained by cyclically permuting the pairs $\{(\mathrm{i},a),(\mathrm{j},b),(\mathrm{k},c)\}$.

The claim eventually follows from $\mathrm{scal} = 2(\kappa(\mathrm{i},\mathrm{j})+\kappa(\mathrm{j},\mathrm{k})+\kappa(\mathrm{k},\mathrm{i}))$. ♥

We will see in the first part of §3 that it is easily possible to *write down* a formula in terms of a, b, and c for the Dirac operator associated with $\langle\,\cdot\,,\,\cdot\,\rangle_{(a,b,c)}$ and a fixed orientation. However, that alone is not enough, and in order to retrieve its spectral data we first have to deal with the representation theory of $\mathbb{S}^3$ (roughly speaking, converting abstract objects into numbers).

Representation Theory

The *actual* aim of this part of the paragraph is to state the *Peter–Weyl Theorem* for (arbitrary) compact Lie groups, which is a powerful tool required to obtain a description of the Dirac spectrum later and which justifies (only in hindsight) *why* we are interested in the representation theory of $\mathbb{S}^3$. In this sense, the reader is kindly requested to first endure a lot of preparatory definitions (taken from [3, Chapter 2.2]). The connection to $\mathbb{S}^3$ will be made at the end.

Let G be a Lie group, and let V be a real or complex (possibly infinite-dimensional) vector space. In the infinite-dimensional case, we would like V to be a Banach space. Denote by $\mathrm{GL}(V)$ the group of (in the infinite-dimensional case, bounded) invertible linear operators on V, equipped with the operator norm topology.

Definition 1.6 *(Representation of a Lie Group on a Vector Space)*

1. A **representation** of G on V is a continuous[1] homomorphism $\pi : G \to \mathrm{GL}(V)$, which we will abbreviate to (G, π, V).

2. A representation (G, π, V) is called **irreducible** if it has no proper invariant subspaces. (That is, the only subspaces $W \subset V$ such that $\pi(g)W \subset W$ holds for all $g \in G$ are $V = \{0\}$ and $W = V$.)

3. Let (G, π, V) and $(G, \tilde{\pi}, \tilde{V})$ be two representations. A bounded linear operator $\Phi : V \to \tilde{V}$ is called an **intertwining map** if $\Phi \circ \pi(g) = \tilde{\pi}(g) \circ \Phi$ holds for all $g \in G$. We denote by $\mathrm{Hom}_G(V, \tilde{V})$ the space of maps intertwining π and $\tilde{\pi}$.

4. Two representations (G, π, V) and $(G, \tilde{\pi}, \tilde{V})$ are called **equivalent** if there is an isomorphism $\Phi : V \to \tilde{V}$ intertwining π and $\tilde{\pi}$.

5. A representation (G, π, V) is called the **direct sum** of a countable collection $\{(G, \pi_n, V_n)\}_{n \in \mathbb{N}_0}$ of representations if for all $n \in \mathbb{N}_0$, there is an injective intertwining map $\Phi_n \in \mathrm{Hom}_G(V_n, V)$ such that the sum $W := \bigoplus_{n \in \mathbb{N}_0} \Phi_n(V_n)$ is direct and $W \subset V$ is a dense subspace. We will then write
$$V = \overline{\bigoplus_{n \in \mathbb{N}_0} V_n}.$$

The completely unbiased reader might ask themselves: *"Why though?"* — It is not clear at all *why* it should be important to know how, say, $G = \mathbb{S}^3 \cong \mathrm{SU}(2)$ acts on some unspecific vector spaces, given that the elements of $\mathrm{SU}(2)$ are already acting on $V = \mathbb{C}^2$ *by default* (with the choice of a basis in mind).

To give just a vague explanation, consider that the (extension of the) Dirac operator will live on the Hilbert space of square-integrable sections of a certain associated vector bundle, which are quite abstract objects. Luckily, in our case, this function space turns out to be identifiable with a pretty simple space (namely the L^2-functions from $\mathbb{S}^3$ to some finite-dimensional complex vector space); however, it is of course still infinite-dimensional, and the Dirac operator still involves rather complicated operations. The punch line is that, with the means of representation theory, we can find a decomposition of this space of L^2-functions into subspaces of *finite* dimension on each of which the Dirac

[1] In the case $\dim_{\mathbb{K}}(V) = n < \infty$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), one has $\mathrm{GL}(V) \cong \mathrm{GL}(\mathbb{K}^n) \cong \mathrm{GL}_{\mathbb{K}}(n)$, which is a harmless matrix Lie group. Interestingly, continuity in this case already implies smoothness, which is a consequence of the *Closed-Subgroup Theorem*.

operator acts as a vector space endomorphism. This ultimately allows us to obtain some information about the spectral data of the Dirac operator, using our precious knowledge of elementary linear algebra.

Now let us focus on the case where the representation spaces $(V, \langle\,\cdot\,,\,\cdot\,\rangle)$ are complex inner product spaces. (As suggested above, we are interested in a functional-analytic result, which makes the most sense if $V = \mathcal{H}$ is a complex Hilbert space.)

Definition 1.7 *(Unitary Analogues)*

1. A representation (G, π, V) is called **unitary** if $\langle \pi(g)v, \pi(g)w\rangle = \langle v, w\rangle$ holds for all $v, w \in V$ and $g \in G$.
2. Two representations are called **unitarily equivalent** if there exists a unitary operator intertwining them.
3. A representation (G, π, V) is called the **unitary direct sum** of a countable collection $\{(G, \pi_n, V_n)\}_{n\in\mathbb{N}_0}$ of representations if $V = \overline{\bigoplus_{n\in\mathbb{N}_0} V_n}$ and the spaces V_n are pairwise orthogonal (after identifying each V_n with its image in V).

Let us unpack these definitions step by step, starting with which space exactly shall take the place of the Hilbert space $V = \mathcal{H}$ in the Peter–Weyl Theorem.

Recall that every compact oriented Riemannian manifold comes with a natural integral, which is particularly easy to describe if it carries the structure of a Lie group G endowed with a left-invariant metric. Indeed, simply pick your favourite orthonormal basis $\{X_1, X_2, \ldots, X_n\}$ of $\mathfrak{g}$, and consider the associated left-invariant vector fields (which yield at each point $g \in G$ an orthonormal basis of $\mathrm{T}_g G$, since the metric is left-invariant), also denoted by $X_1, X_2, \ldots, X_n$. Then the associated Riemannian volume form ω on G is determined by $\omega_g\big(X_1(g), X_2(g), \ldots, X_n(g)\big) = 1$. For $\varphi \in \mathrm{C}^0(G, \mathbb{R})$, this leads to the usual definition $\int_G \varphi := \int_G \varphi\,\omega$, where G is endowed with the orientation provided by ω.

If $\varphi = \mathrm{Re}(\varphi) + \mathrm{i}\,\mathrm{Im}(\varphi) \in \mathrm{C}^0(G, \mathbb{C})$, we set $\int_G \varphi := \int_G \mathrm{Re}(\varphi) + \mathrm{i}\int_G \mathrm{Im}(\varphi)$ and define $\mathrm{L}^2(G, \mathbb{C})$ as the completion of $\mathrm{C}^0(G, \mathbb{C})$ with respect to the Hermitian standard inner product $\langle \varphi_1, \varphi_2\rangle_{\mathrm{L}^2} := \int_G \varphi_1 \bar{\varphi}_2$. This leads us to consider $\mathcal{H} = \mathrm{L}^2(G, \mathbb{C})$, and we will now define an action of the group $G \times G$ on this space.

Consider the natural unitary action of $G \times G$ on $\mathrm{C}^0(G, \mathbb{C})$ described by the representation $\rho(g, h)\varphi := \varphi(g^{-1}(\,\cdot\,)h)$. The following result states that ρ extends to a unitary representation of $G \times G$ on $\mathrm{L}^2(G, \mathbb{C})$:

Proposition 1.8 *(Regular Representation)*

For all $g, h \in G$, the linear map $\rho(g,h)$ extends to a unitary operator on $\mathrm{L}^2(G,\mathbb{C})$, and $\rho : G \times G \to \mathrm{GL}(\mathrm{L}^2(G,\mathbb{C}))$ is continuous.

We refer to ρ as the **regular representation** of $G \times G$ on $\mathrm{L}^2(G,\mathbb{C})$.

Proof. See [3, Proposition 2.6.2 (2)]. ♥

We would like to recognise the infinite-dimensional representation ρ of $G\times G$ on $\mathrm{L}^2(G,\mathbb{C})$ as a unitary direct sum of *finite*-dimensional representations, on which we can perform linear algebra. The way in which these spaces will be embedded into $\mathrm{L}^2(G,\mathbb{C})$ requires one last preparatory definition.

Definition 1.9 *(New Representations from Old)*

Let V and $\tilde{V}$ each be a complex inner product space of finite dimension.

1. The **dual** of a representation (G, π, V) is the representation (G, π^*, V^*) defined by $\pi^*(g)f \coloneqq f \circ \pi(g^{-1})$ for all $g \in G$ and $f \in V^*$. *(The inner product on V^* is such that the dual basis of an orthonormal basis is orthonormal.)*

2. The **tensor product** of two representations (G, π, V) and $(\tilde{G}, \tilde{\pi}, \tilde{V})$ is the representation $(G\times\tilde{G}, \pi\otimes\tilde{\pi}, V\otimes\tilde{V})$ defined by $(\pi\otimes\tilde{\pi})(g,\tilde{g})(v\otimes\tilde{v}) \coloneqq \pi(g)v\otimes\tilde{\pi}(\tilde{g})\tilde{v}$ for all $g \in G$, $\tilde{g} \in \tilde{G}$ and $v \in V, \tilde{v} \in \tilde{V}$. *(The inner product on $V \otimes \tilde{V}$ is such that tensor products of orthonormal bases form orthonormal bases.)*

We have now gathered everything needed to state the Peter–Weyl Theorem, on which we will rely in the second part of §3, where we describe the Dirac spectrum of $\mathbb{S}^3_{(a,b,c)}$.

Theorem 1.10 *(Peter–Weyl Theorem)*

Let G be a compact Lie group. Denote by $\hat{G}$ the set of equivalence classes of irreducible unitary representations of G. For each class $[\pi]_\sim \in \hat{G}$, fix a representative (G, π, V_π), and define $\Phi_\pi \in \mathrm{Hom}_{G\times G}(V_\pi \otimes {V_\pi}^*, \mathrm{L}^2(G,\mathbb{C}))$ by $\Phi_\pi(v \otimes f)(g) \coloneqq f(\pi(g^{-1})v)$. Then the regular representation ρ of $G \times G$ on $\mathrm{L}^2(G,\mathbb{C})$ is unitarily equivalent to the unitary direct sum of the $V_\pi \otimes {V_\pi}^*$ embedded via Φ_π, that is,

$$\mathrm{L}^2(G,\mathbb{C}) \cong \overline{\bigoplus_{[\pi]_\sim \in \hat{G}} V_\pi \otimes {V_\pi}^*}.$$

Proof. See [3, Theorem 2.8.2]. ♥

Let us now return from the general situation to our special case $G = \mathbb{S}^3 \cong \mathrm{SU}(2)$. The Peter–Weyl Theorem explains *why* it is so important to know the irreducible representations of $\mathbb{S}^3$. Luckily, the representation theory of $\mathbb{S}^3$ is well-known, and easy to state without further preparation.

Definition 1.11 *(Representations of $\mathbb{S}^3$)*

① For each $n \in \mathbb{N}_0$, denote by V_n the complex vector space of homogeneous polynomials in two complex variables; that is, $V_n = \operatorname{span}_{\mathbb{C}}\{P_k : 0 \leq k \leq n\}$ with $P_k(z,w) = z^{n-k}w^k$. In particular, $\dim_{\mathbb{C}}(V_n) = n+1$.

② Let the group $\mathbb{S}^3 \cong \mathrm{SU}(2)$ act on V_n via $(\pi_n(g)P)(z,w) \coloneqq P((z,w)g)$.

One can easily verify that $(\mathrm{SU}(2), \pi_n, V_n)$ indeed defines a representation for all $n \in \mathbb{N}_0$. Much more complicated to prove (but easy to state nonetheless) is that the collection $\{\pi_n\}_{n\in\mathbb{N}_0}$ describes the irreducible representations of $\mathrm{SU}(2)$ up to equivalence.

Theorem 1.12 *(Classification of Irreducible Representations of $\mathbb{S}^3$)*

Every irreducible (unitary) representation of $\mathrm{SU}(2)$ is (unitarily) equivalent to one of the $(\mathrm{SU}(2), \pi_n, V_n)$.

Proof. See for example [4, Chapter II, Proposition 5.3]. ♥

The attentive reader will certainly have noticed that the word "unitary" in the above statement does not make any sense, since we have never specified an inner product on V_n. This, however, is only a technical requirement, since every representation π of a compact group G on a finite-dimensional vector space V can be made unitary: Indeed, pick any inner product $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$ on V and a right-invariant volume form ω on G (endowed with the corresponding orientation), and let $\langle v, w \rangle \coloneqq \int_G \langle\!\langle \pi(\cdot)v, \pi(\cdot)w \rangle\!\rangle\, \omega$. Then (G, π, V) is unitary with respect to $\langle \cdot\,, \cdot \rangle$. One can further show that two irreducible unitary finite-dimensional representations are unitarily equivalent if and only if they are equivalent (see [3, Corollary 2.3.9]), so we need not worry about that.

This is everything we want to know about homogeneous 3-spheres — for now, not including their spin geometry, the general concepts of which first have to be introduced in the next paragraph.

§2 — *Spin Geometry*

The aim of this subsection is to give the reader a very condensed overview of the notions required to define the Dirac operator and of its most important analytic properties. We closely follow Habermann's lecture notes [5] for reference, with some minimal structural tweaks and enrichments inspired by the lecture on spin geometry that the author attended as a student.

The *Dirac operator* is a linear differential operator that lives on the sections of a certain vector bundle (the *spinor bundle*) equipped with a notion of taking covariant derivatives (the *spin connection*), the former of which is associated with a certain principal bundle (a *spin structure*) over suitable oriented Riemannian manifolds, whose structure group is the *spin group*, which in turn requires constructing the *Clifford algebra*. We will need to deal with all these notions recursively, focusing only on the most important definitions and statements.

The Clifford Algebra

In the more general setting, let V be a real finite-dimensional vector space equipped with a quadratic form $Q : V \to \mathbb{R}$.

Definition 2.1 *(Clifford Algebra)*

① The **Clifford algebra** $\mathcal{C}(V, Q)$ is the real associative algebra with multiplicative identity obtained by modding out from the tensor algebra $\bigoplus_{k=0}^{\infty} V^{\otimes k}$ the two-sided ideal generated by $\{v \otimes v + Q(v)\}_{v \in V}$.

② We write $\mathcal{C}_n := \mathcal{C}(\mathbb{R}^n, \langle \cdot\, , \cdot \rangle)$ in the Euclidean case.

The latter is the main definition that we will use. Simply put, one can think of an element of $\mathcal{C}_n$ as a finite linear combination of products of elements of $\mathbb{R}^n$ satisfying the identity $x^2 = -|x|^2$ (or, equivalently, $xy + yx = -2\langle x, y\rangle$). Rephrasing this in terms of the standard basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{R}^n$, one obtains

$$ {e_k}^2 = -1 \quad \text{and} \quad e_k e_\ell = -e_\ell e_k \text{ whenever } k \neq \ell. \tag{$*$} $$

We will only use this interpretation of elements of $\mathcal{C}_n$ because of its simplicity (rather than equivalence classes of formal linear combinations of tensor products), but one should of course keep the rigorous definition in mind when dealing with these objects.

Proposition 2.2 *(Dimension of the Clifford Algebra)*

There exists a vector space isomorphism between the Clifford algebra $\mathcal{C}_n$ and the exterior algebra $\bigwedge \mathbb{R}^n$. In particular, $\dim(\mathcal{C}_n) = 2^n$.

Proof. See [6, Chapter I, Proposition 1.3]. ♥

Although the rigorous proof based on the algebraic Definition 2.1 is somewhat technical, the statement is intuitively clear, since we can rearrange every product of canonical basis vectors such that they are in increasing order (up to a sign) by repeatedly applying ($*$). More precisely, the increasing products $e_{k_1}e_{k_2}\dots e_{k_m}$ (where $m \in \mathbb{N}_0$ and $1 \leq k_1 < k_2 < \dots < k_m \leq n$) form a basis of $\mathcal{C}_n$, and the map $e_{k_1}e_{k_2}\dots e_{k_m} \mapsto e_{k_1} \wedge e_{k_2} \wedge \dots \wedge e_{k_m}$ plus linear extension defines a natural vector space isomorphism $\mathcal{C}_n \cong \bigwedge \mathbb{R}^n$ (which, however, is of course *not* an isomorphism of algebras).

It is desirable to have identifications with already familiar algebras in order to better understand $\mathcal{C}_n$. For small n, one quickly recognises $\mathcal{C}_1 \cong \mathbb{C}$ (via $e_1 \mapsto \mathrm{i}$) and $\mathcal{C}_2 \cong \mathbb{H}$ (via $e_1 \mapsto \mathrm{i}$ and $e_2 \mapsto \mathrm{j}$). One proceeds by defining $\mathcal{C}'_n := \mathcal{C}(\mathbb{R}^n, -\langle\,\cdot\,,\,\cdot\,\rangle)$ and establishing the isomorphisms $\mathcal{C}_{n+2} \cong \mathcal{C}'_n \otimes \mathcal{C}_2$ and $\mathcal{C}'_{n+2} \cong \mathcal{C}_n \otimes \mathcal{C}'_2$. Then, starting from $\mathcal{C}'_1 \cong \mathbb{R} \oplus \mathbb{R}$ and $\mathcal{C}'_2 \cong \mathrm{Mat}_{\mathbb{R}}(2)$, one can advance inductively to obtain $\mathcal{C}_3 \cong \mathcal{C}'_1 \otimes \mathcal{C}_2 \cong \mathbb{H} \oplus \mathbb{H}$, $\mathcal{C}_4 \cong \mathcal{C}'_2 \otimes \mathcal{C}_2 \cong \mathrm{Mat}_{\mathbb{H}}(2)$, and so forth (see for example [6, Chapter I, Theorem 4.1]).

A less practical but nonetheless crucial statement (for *our* purposes) is the following classification of the *complexified* Clifford algebras $\mathcal{C}_n^{\mathbb{C}} = \mathcal{C}_n \otimes_{\mathbb{R}} \mathbb{C}$:

Theorem 2.3 *(Classification of the Complexified Clifford Algebras)*

1. If $n = 2m$ is even, then one has $\mathcal{C}_n^{\mathbb{C}} \cong \mathrm{Mat}_{\mathbb{C}}(2^m)$.
2. If $n = 2m+1$ is odd, then one has $\mathcal{C}_n^{\mathbb{C}} \cong \mathrm{Mat}_{\mathbb{C}}(2^m) \oplus \mathrm{Mat}_{\mathbb{C}}(2^m)$.

Proof. See [5, Satz 1.1.6]. ♥

We briefly discuss the idea of the proof because we will need the objects that appear in it for later reference. In order to define isomorphisms ϕ_n from $\mathcal{C}_n$ to the respective right-hand side, consider the complex basis $\{E, U, V, T\}$ of $\mathrm{Mat}_{\mathbb{C}}(2)$ given by

$$E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U := \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad V := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad T := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note $E \mathrel{\hat{=}} 1$, $U \mathrel{\hat{=}} \mathrm{i}$, $V \mathrel{\hat{=}} \mathrm{j}$, and $\mathrm{i}T \mathrel{\hat{=}} \mathrm{k}$, following the identification $\mathrm{Mat}_{\mathbb{C}}(2) \cong \mathbb{H}$ described at the beginning of §1.

Now define $\phi_n(e_k)$ for all $k \in \{1, 2, \dots, n\}$:

- If $n = 2m$ is even, set $\phi_n(e_{2j-1}) \coloneqq T^{\otimes j-1} \otimes U \otimes E^{\otimes m-j}$ and $\phi_n(e_{2j}) \coloneqq T^{\otimes j-1} \otimes V \otimes E^{\otimes m-j}$ for all $j \in \{1, 2, \dots, m\}$.
- If $n = 2m + 1$ is odd, set $\phi_n(e_k) \coloneqq \big(\phi_{2m}(e_k), \phi_{2m}(e_k)\big)$ for all $k \in \{1, 2, \dots, 2m\}$, and set $\phi_n(e_n) \coloneqq (\mathrm{i}T^{\otimes m}, -\mathrm{i}T^{\otimes m})$.

One then has to verify that, in both cases, the canonical extension of ϕ_n to the increasing products of the e_k plus linear extension yields a well-defined algebra homomorphism ϕ_n that is surjective (and, hence, bijective by comparing dimensions).

This leads us to the following definitions that we will need later:

Definition 2.4 *(Spinors, Spin Representation, and Clifford Multiplication)*

Let $m \coloneqq \lfloor \frac{n}{2} \rfloor$.

1. We write $\Sigma_n \coloneqq \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2$ (m times) and refer to its elements $\mathbf{w} \in \Sigma_n$ as complex n-**spinors**. Furthermore, we define a complex inner product on Σ_n by $\langle \mathbf{w}, \tilde{\mathbf{w}} \rangle = \langle w_1 \otimes \dots \otimes w_m, \tilde{w}_1 \otimes \dots \otimes \tilde{w}_m \rangle \coloneqq \langle w_1, \tilde{w}_1 \rangle \dots \langle w_m, \tilde{w}_m \rangle$, where $\langle \cdot\,, \cdot \rangle$ on the right-hand side denotes the standard Hermitian inner product on $\mathbb{C}^2$.

2. Let $\sigma_n \coloneqq \phi_n$ for even n and $\sigma_n \coloneqq \mathrm{pr}_1 \circ \phi_n$ for odd n. The resulting algebra homomorphism $\sigma_n : \mathcal{C}_n^{\mathbb{C}} \to \mathrm{End}(\Sigma_n)$ is called the **spin representation** of $\mathcal{C}_n^{\mathbb{C}}$.

3. The **Clifford multiplication** assigns to each vector $x \in \mathbb{R}^n \subset \mathcal{C}_n \subset \mathcal{C}_n^{\mathbb{C}}$ and each spinor $\mathbf{w} \in \Sigma_n$ the spinor $x \cdot \mathbf{w} \coloneqq \sigma_n(x)\mathbf{w}$.

Lastly, we discuss another property of $\mathcal{C}_n$ required to define the spin group: $\mathbb{Z}_2$-grading. Denote by $\alpha : \mathcal{C}_n \to \mathcal{C}_n$ the automorphism induced by the linear map $x \mapsto -x$ on $\mathbb{R}^n$, and denote by $\beta : \mathcal{C}_n \to \mathcal{C}_n$ the antiautomorphism that reverses the multiplication (that is, $\beta : e_{k_1} e_{k_2} \dots e_{k_m} \mapsto e_{k_m} \dots e_{k_2} e_{k_1}$ plus linear extension).

Proposition 2.5 *($\mathbb{Z}_2$-Grading of the Clifford Algebra)*

The Clifford algebra $\mathcal{C}_n$ can be decomposed into an **even** and an **odd** part

$$\mathcal{C}_n^0 \coloneqq \{u \in \mathcal{C}_n : \alpha(u) = u\} \quad \text{and} \quad \mathcal{C}_n^1 \coloneqq \{u \in \mathcal{C}_n : \alpha(u) = -u\}, \text{ respectively.}$$

More precisely, one has $\mathcal{C}_n = \mathcal{C}_n^0 \oplus \mathcal{C}_n^1$ and $\mathcal{C}_n^i \mathcal{C}_n^j \subset \mathcal{C}_n^{i+j}$ (mod 2).

Proof. See [5, Lemma 1.1.9]. ♥

We will now move forward a little from the *purely* algebraic basics towards differential geometry (the keyword being *Lie groups*).

The Spin Group

Observe that every $x \in \mathbb{S}^{n-1} \subset \mathbb{R}^n \subset \mathcal{C}_n$ admits the multiplicative inverse $x^{-1} = -x$. It follows that $u := x_1 x_2 \ldots x_m \in \mathcal{C}_n$ (where $x_1, x_2, \ldots, x_m \in \mathbb{S}^{n-1}$ for some $m \in \mathbb{N}_0$) is invertible with $u^{-1} = (-1)^m x_m \ldots x_2 x_1 = \alpha(\beta(u))$.

Definition 2.6 *(Pin and Spin Group)*

1. The **pin group** $\mathrm{Pin}(n) \subset \mathcal{C}_n$ consists of all elements of the form $x_1 x_2 \ldots x_m$, where $m \in \mathbb{N}_0$ and $x_1, x_2, \ldots, x_m \in \mathbb{S}^{n-1}$.

2. The **spin group** is the even part of the Pin group, $\mathrm{Spin}(n) := \mathrm{Pin}(n) \cap \mathcal{C}_n^0$.

We would like to define a homomorphism $\theta : \mathrm{Pin}(n) \to \mathrm{O}(n)$. One can easily check $uy\beta(u) = \pm uyu^{-1} \in \mathbb{R}^n$ for all $y \in \mathbb{R}^n$ and $u \in \mathrm{Pin}(n)$ (see [5, Lemma 1.2.1]); that is, for each $u \in \mathrm{Pin}(n)$, one has an endomorphism $\theta(u) : y \mapsto uy\beta(u)$ of $\mathbb{R}^n$, which one can further prove to be orthogonal (see [5, Satz 1.2.2 (i)]).

We canonically identify $\mathrm{O}(\mathbb{R}^n)$ with $\mathrm{O}(n)$ via the standard basis of $\mathbb{R}^n$. One then has the desired map $\theta : \mathrm{Pin}(n) \to \mathrm{O}(n)$, which has the following propetries:

Proposition 2.7 *(Spin Cover)*

The map $\theta : \mathrm{Pin}(n) \to \mathrm{O}(n)$ is a continuous and surjective group homomorphism with $\ker \theta = \{\pm 1\}$ and $\theta^{-1}(\mathrm{SO}(n)) = \mathrm{Spin}(n)$.

One can conclude that $\mathrm{Spin}(n)$ is a double cover of $\mathrm{SO}(n)$ for all $n \geq 2$.

Proof. See [5, Satz 1.2.2], [5, Satz 1.2.3], and [5, Bemerkung 1.2.4].

Continuity is meant with respect to the natural topology on $\mathcal{C}_n \cong \mathbb{R}^{2^n}$. ♥

Since we have not talked too much about differential geometry yet, let us have a look at the topological properties of $\mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$ that one can conclude with the aid of θ. What is particularly interesting for us is that these groups are Lie groups.

Proposition 2.8 *(Topology of the Pin and Spin Group)*

1. For all $n \geq 2$, $\mathrm{Spin}(n)$ is path-connected.

2. Both $\mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$ are compact. In particular, they are closed subgroups (within the open group of invertible elements of $\mathcal{C}_n$), which implies that they are Lie groups by the Closed-Subgroup Theorem.

Proof.

1. See [5, Satz 1.2.3 (ii)].

2. For each $u \in \mathrm{Pin}(n)$, the map $\theta(u) \in \mathrm{O}(\mathbb{R}^n)$ is the composition of at most n reflections. It follows that $u = x_1 x_2 \dots x_m$ for certain $x_1, x_2, \dots, x_m \in \mathbb{S}^{n-1}$ with $m \leq n$. Then $\mathrm{Pin}(n) = M_1 \cup M_2 \cup \dots \cup M_n$ is the *finite* union of the compact sets $M_m = \{x_1 x_2 \dots x_m : x_1, x_2, \dots, x_m \in \mathbb{S}^{n-1}\}$; hence, it is compact. The spin group is then compact as a closed subset of a compact set (since it is the intersection of a closed set and a closed subspace).

♥

Now knowing that $\mathrm{Spin}(n) \subset \mathcal{C}_n \subset \mathcal{C}_n^{\mathbb{C}}$ is a Lie group, the following definition is more reasonable than before:

Definition 2.9 *(Spin Representation)*

The **spin representation** is the restriction $\sigma := \sigma_n|_{\mathrm{Spin}(n)} : \mathrm{Spin}(n) \to \mathrm{GL}(\Sigma_n)$, where σ_n is the spin representation of $\mathcal{C}_n^{\mathbb{C}}$ from Definition 2.4 (2).

Note that σ is now a representation in the sense of Definition 1.6 (1) with some interesting properties (some of which are needed at a later moment to justify that certain things are well-defined).

Proposition 2.10 *(Properties of the Spin Representation)*

1. If n is odd, then σ is irreducible. If, on the other hand, n is even, then σ can be decomposed into two irreducible representations.

2. One has $\sigma(u)(x \cdot \mathbf{w}) = \theta(u)x \cdot \sigma(u)\mathbf{w}$ for all $u \in \mathrm{Spin}(n)$, $x \in \mathbb{R}^n$, $\mathbf{w} \in \Sigma_n$.

3. The spin representation is unitary with respect to the inner product on Σ_n defined in Definition 2.4 1.

Proof. See [5, Abschnitt 1.4], [5, Satz 1.5.4], and [5, Satz 1.5.7]. ♥

Now we can devote ourselves entirely to differential geometry. We will commence by introducing some additional structure on certain oriented Riemannian manifolds.

Spin Structures on Oriented Riemannian Manifolds

Let $\mathcal{M}^n = (\mathcal{M}^n, \langle\,\cdot\,,\,\cdot\,\rangle)$ be an *oriented* Riemannian manifold. Then it makes sense to consider the principal $\mathrm{SO}(n)$-bundle of positively oriented orthonormal frames of $\mathcal{M}$, denoted by $\mathrm{SO}(\mathcal{M}) = \mathrm{SO}(\mathcal{M}, \langle\,\cdot\,,\,\cdot\,\rangle)$.

We would like to define an object that (in some sense) relates to $\mathrm{SO}(\mathcal{M})$ as $\mathrm{Spin}(n)$ relates to $\mathrm{SO}(n)$. This leads to the following quite plausible definition:

Definition 2.11 *(Spin Structure)*

A **spin structure** on $\mathcal{M}$ consists of a principal $\mathrm{Spin}(n)$-bundle $\mathcal{S}$ over $\mathcal{M}$ and a bundle map $\Theta : \mathcal{S} \to \mathrm{SO}(\mathcal{M})$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{S} \times \mathrm{Spin}(n) \arrow[r] \arrow[d, "\Theta\times\theta"'] & \mathcal{S} \arrow[d, "\Theta"] \arrow[dr, "\pi_{\mathcal{S}}"] & \\
\mathrm{SO}(\mathcal{M}) \times \mathrm{SO}(n) \arrow[r] & \mathrm{SO}(\mathcal{M}) \arrow[r, "\pi_{\mathrm{SO}(\mathcal{M})}"'] & \mathcal{M}
\end{tikzcd}$$

The horizontal arrows depict the right actions of $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$ on the principal bundles $\mathrm{SO}(\mathcal{M})$ and $\mathcal{S}$, respectively.

A manifold endowed with a spin structure is called a **spin manifold**.

A natural question to ask is whether spin structures of $\mathcal{M}$ always exist; and if they do, then how to distinguish between them? Let us first agree that two spin structures should be declared equivalent if their respective principal bundles are isomorphic, and if the corresponding covering maps are compatible with each other.

Definition 2.12 *(Isomorphic Spin Structures)*

Two spin structures $(\mathcal{S}, \Theta)$ and $(\tilde{\mathcal{S}}, \tilde{\Theta})$ on $\mathcal{M}$ are said to be **isomorphic** if there exists a principal bundle isomorphism $\Phi : \mathcal{S} \to \tilde{\mathcal{S}}$ such that $\Theta = \tilde{\Theta} \circ \Phi$.

Regarding the answer to the above question, there is the following topological result:

Theorem 2.13 *(Classification of Spin Structures)*

1. A spin structure on $\mathcal{M}$ exists if and only if the second Stiefel–Whitney class $\mathrm{w}_2(\mathcal{M}) \in \mathrm{H}^2(\mathcal{M}; \mathbb{Z}_2)$ vanishes.

2. If this is the case, then there is a one-to-one correspondence between the spin structures of $\mathcal{M}$ up to isomorphism and the cohomology group $\mathrm{H}^1(\mathcal{M}; \mathbb{Z}_2)$.

Proof. See [6, Chapter II, Theorem 1.7]. ♥

To each spin manifold, we can associate the corresponding spinor bundle; and to each spinor bundle, we can associate the spin connection as a notion of taking derivatives. This is the last piece of the puzzle needed to define the Dirac operator.

The Associated Spinor Bundle and the Spin Connection

Let $\mathcal{M}^n$ be a spin manifold endowed with the spin structure $(\mathcal{S}, \Theta)$. Then we would like to associate to $\mathcal{M}$ a vector bundle on which the Dirac operator will live later.

Definition 2.14 *(Spinor Bundle Associated with a Spin Manifold)*

The **spinor bundle** associated with $\mathcal{M}$ with respect to $(\mathcal{S}, \Theta)$ is the associated complex vector bundle $\Sigma\mathcal{M} := \mathcal{S} \times_\sigma \Sigma_n$.

Local sections $\varphi : U \to \Sigma\mathcal{M}$ of $\Sigma\mathcal{M}$ on open neighbourhoods $U \subset \mathcal{M}$ are called **spinor fields** and denoted by $\varphi \in \Gamma_U(\Sigma\mathcal{M})$. We also write $\Gamma(\Sigma\mathcal{M}) := \Gamma_\mathcal{M}(\Sigma\mathcal{M})$.

The elements of $\Sigma\mathcal{M}$ are equivalence classes of the form $[p, \mathbf{w}]_\sim$ (where $p \in \mathcal{S}$ and $\mathbf{w} \in \Sigma_n$) with respect to the equivalence relation

$$(p, \mathbf{w}) \sim (\tilde{p}, \tilde{\mathbf{w}}) \iff (\tilde{p}, \tilde{\mathbf{w}}) = (p, \mathbf{w})u := (pu, \sigma(u^{-1})\mathbf{w}) \text{ for some } u \in \mathrm{Spin}(n),$$

endowed with the vector space structure $[p, \mathbf{w}]_\sim + \alpha[p, \tilde{\mathbf{w}}]_\sim := [p, \mathbf{w} + \alpha\tilde{\mathbf{w}}]_\sim$ for $\alpha \in \mathbb{C}$. We further equip $\Sigma\mathcal{M}$ with a complex bundle metric $\{\langle\,\cdot\,,\,\cdot\,\rangle_x\}_{x\in\mathcal{M}}$ as follows:

Definition 2.15 *(Bundle Metric on the Spinor Bundle)*

Let $x \in \mathcal{M}$, $p \in \mathcal{S}_x$, and $[p, \mathbf{w}]_\sim, [p, \tilde{\mathbf{w}}]_\sim \in \Sigma\mathcal{M}$. Then we define

$$\langle [p, \mathbf{w}]_\sim, [p, \tilde{\mathbf{w}}]_\sim \rangle_x := \langle \mathbf{w}, \tilde{\mathbf{w}} \rangle \in \mathbb{C},$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the inner product on Σ_n from Definition 2.4 1.

This is well-defined because σ is unitary (see Proposition 2.10 3).

In order to define the Dirac operator, we need to extend the Clifford multiplication to bundles. This is done by identifying the tangent bundle $\mathrm{T}\mathcal{M}$ with a vector bundle associated with $\mathcal{S}$ of the fibre type $\mathbb{R}^n$.

Lemma 2.16 *(Identification of the Tangent Bundle)*

Denote by $\iota : \mathrm{SO}(n) \hookrightarrow \mathrm{GL}(\mathbb{R}^n)$ the inclusion. Then we have the following canonical isomorphims of vector bundles:

$$\begin{array}{ccccc}
\mathcal{S} \times_{\iota\circ\theta} \mathbb{R}^n & \to & \mathrm{SO}(\mathcal{M}) \times_\iota \mathbb{R}^n & \to & \mathrm{T}\mathcal{M} \\
\Uplus & & \Uplus & & \Uplus \\
[p, y]_\sim & \mapsto & [\Theta(p), y]_\sim & & \\
 & & [\{e_1, e_2, \ldots, e_n\}, y]_\sim & \mapsto & \sum_{k=1}^n y_k e_k
\end{array}$$

Proof. See [5, Lemma 2.2.7] and [5, Lemma 2.2.8]. ♥

This allows the following definition, which is correct due to Proposition 2.10 2:

Definition 2.17 *(Clifford Multiplication on Bundles)*

Let $X \in \Gamma_U(\mathrm{T}\mathcal{M})$ be a vector field, and let $\varphi \in \Gamma_U(\Sigma\mathcal{M})$ be a spinor field. Then the Clifford product of X and φ is the spinor field $X \cdot \varphi \in \Gamma_U(\Sigma\mathcal{M})$ defined by

$$(X \cdot \varphi)(x) := \mu_x\big(X(x) \otimes \varphi(x)\big) := [p, y \cdot \mathbf{w}]_\sim \text{ for all } x \in U,$$

where $p \in \mathcal{S}_x$, $\varphi(x) = [p, \mathbf{w}]_\sim$ for some $\mathbf{w} \in \Sigma_n$, and $X(x) \in \mathrm{T}_x\mathcal{M}$ is identified with $[p, y]_\sim$ for some $y \in \mathbb{R}^n$ in the sense of Lemma 2.16.

The map $\mu : \Gamma(\mathrm{T}\mathcal{M} \otimes \Sigma\mathcal{M}) \to \Gamma(\Sigma\mathcal{M})$ is called the **Clifford multiplication**.

With the Clifford multiplication on bundles, we can talk about taking the derivative of a spinor field along a vector field. More precisely, the Levi–Civita connection ∇ on $\mathrm{T}\mathcal{M}$ induces a connection ∇^{Σ} on $\Sigma\mathcal{M}$.

Theorem 2.18 *(Spin Connection)*

Let $X \in \Gamma(\mathrm{T}\mathcal{M})$ and $\varphi \in \Gamma(\Sigma\mathcal{M})$. Define the **spinor derivative** $\nabla^{\Sigma}_X\varphi \in \Gamma(\Sigma\mathcal{M})$ of φ along X as follows: For each point $x \in \mathcal{M}$, pick a local section $s \in \Gamma_U(\mathcal{S})$ on a neighbourhood $U \subset \mathcal{M}$ of x such that $\varphi|_U = [s, \varphi_s]_\sim$ for some $\varphi_s \in \mathrm{C}^\infty(U, \Sigma_n)$, and let $\{e_1, e_2, \ldots, e_n\} := \Theta \circ s$.

Then the expression

$$\nabla^{\Sigma}_X\varphi := [s, X(\varphi_s)]_\sim + \frac{1}{4}\sum_{j=1}^{n} e_j \cdot \nabla_X e_j \cdot \varphi$$

evaluated at $x \in U$ does not depend on the particular choice of s. Furthermore, it defines a covariant derivative on the spinor bundle.

The map $\nabla^{\Sigma} : \Gamma(\Sigma\mathcal{M}) \to \Gamma(\mathrm{T}\mathcal{M}^* \otimes \Sigma\mathcal{M})$ is called the **spin connection**.

Proof. See [5, Satz 2.3.6]. ♥

With all these objects, we can *finally* define the very thing that we will be talking about in the rest of this thesis.

The Dirac Operator

Let $\mathcal{M}^n$ be a spin manifold, and let $\Sigma\mathcal{M}$ be the associated spinor bundle.

Definition 2.19 *(Dirac Operator)*

The **Dirac operator** $\mathscr{D} : \Gamma(\Sigma\mathcal{M}) \to \Gamma(\Sigma\mathcal{M})$ is the map

$$\mathscr{D} : \Gamma(\Sigma\mathcal{M}) \xrightarrow{\nabla^{\Sigma}} \Gamma(\mathrm{T}\mathcal{M}^* \otimes \Sigma\mathcal{M}) \xrightarrow{\sharp} \Gamma(\mathrm{T}\mathcal{M} \otimes \Sigma\mathcal{M}) \xrightarrow{\mu} \Gamma(\Sigma\mathcal{M}),$$

where ∇^{Σ} is the spin connection, $\sharp : \mathrm{T}\mathcal{M}^* \to \mathrm{T}\mathcal{M}$ is the musical isomorphism given by the Riemannian metric, and μ is the Clifford multiplication on bundles.

The following local description of $\mathscr{D}$ is immediate from the definition:

Proposition 2.20 *(Local Description of the Dirac Operator)*

Let $\{e_1, e_2, \ldots, e_n\}$ be a local orthonormal frame of $\mathcal{M}$, and let $\varphi \in \Gamma(\Sigma\mathcal{M})$. Then

$$\mathscr{D}\,\varphi = \sum_{i=1}^{n} e_i \cdot \nabla^{\Sigma}_{e_i}\varphi.$$

Proof. See [5, Satz 2.4.2]. ♥

This also implies that the Dirac operator is a first-order linear differential operator.

From now on, let $\mathcal{M}$ be closed (compact and without boundary). We would like to summarise the most important analytic properties of the Dirac operator together with their consequences for the Dirac spectrum.

For $\varphi, \tilde{\varphi} \in \Gamma(\Sigma\mathcal{M})$, consider the complex inner product $\langle \varphi, \tilde{\varphi} \rangle_{\mathrm{L}^2} \coloneqq \int_{\mathcal{M}} \langle \varphi, \tilde{\varphi} \rangle$, where $\langle \varphi, \tilde{\varphi} \rangle : \mathcal{M} \to \mathbb{C}$ is defined by $\langle \varphi, \tilde{\varphi} \rangle(x) \coloneqq \langle \varphi(x), \tilde{\varphi}(x) \rangle_x$ (see Definition 2.15), and denote by $\mathrm{L}^2(\Sigma\mathcal{M})$ the completion of $\Gamma(\Sigma\mathcal{M})$ with respect to $\| \cdot \|_{\mathrm{L}^2} \coloneqq \sqrt{\langle \cdot\,, \cdot \rangle_{\mathrm{L}^2}}$. Then the Dirac operator extends to an unbounded operator with dense domain in $\mathrm{L}^2(\Sigma\mathcal{M})$ with the following properties:

Proposition 2.21 *(Properties of the Dirac Operator)*

The Dirac operator is

1. elliptic *(its principal symbol $\sigma_\xi(\mathscr{D})$ is an isomorphism for all $\xi \in \mathrm{T}\mathcal{M}^*$, $\xi \neq 0$)*,
2. formally self-adjoint *($\langle \mathscr{D}\,\varphi, \tilde{\varphi} \rangle_{\mathrm{L}^2} = \langle \varphi, \mathscr{D}\,\tilde{\varphi} \rangle_{\mathrm{L}^2}$ holds for all $\varphi, \tilde{\varphi} \in \Gamma(\Sigma\mathcal{M})$)*.

Proof. See [6, Chapter II, Lemma 5.1] and [6, Chapter II, Proposition 5.3]. ♥

Since $\mathscr{D}$ is an elliptic and formally self-adjoint linear differential operator, it follows from the general theory of these operators on vector bundles over closed manifolds:

Theorem 2.22 *(Properties of the Dirac Spectrum)*

1. The eigenvalues of $\mathscr{D}$ form a discrete set $\mathrm{spec}(\mathscr{D}) = \{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$ such that $0 \leq |\lambda_1| \leq |\lambda_2| \leq \ldots \to \infty$.
2. The corresponding eigenspaces $\mathrm{E}_\lambda(\mathscr{D}) \coloneqq \{\varphi \in \mathrm{L}^2(\Sigma\mathcal{M}) : \mathscr{D}\,\varphi = \lambda\varphi\}$ (whose non-zero elements we refer to as **eigenspinors** of $\mathscr{D}$) are of finite dimension, and $\mathrm{E}_\lambda(\mathscr{D}) \perp \mathrm{E}_\mu(\mathscr{D})$ whenever $\lambda \neq \mu$.

⋮

③ The set $\bigoplus_{\lambda\in\operatorname{spec}(\mathscr{D})} \mathrm{E}_\lambda(\mathscr{D})$ is dense in the Hilbert space $\mathrm{L}^2(\Sigma\mathcal{M})$.

④ In particular (combining ② and ③), there exists an orthonormal basis of $\mathrm{L}^2(\Sigma\mathcal{M})$ consisting of smooth eigenspinors of $\mathscr{D}$.

Proof. See [6, Chapter III, Theorem 5.8]. ♥

Unlike the Laplace spectrum, the Dirac spectrum generally consists of both positive and negative eigenvalues, which makes estimating eigenvalues a bit more difficult. However, one has $\operatorname{spec}(\mathscr{D}) = -\operatorname{spec}(\mathscr{D})$ unless $n = 3 \pmod 4$.

Proposition 2.23 *(Symmetry of the Dirac Spectrum)*

If n is even or $n = 1 \pmod 4$, then $\operatorname{spec}(\mathscr{D})$ is symmetric about zero, and the eigenspaces corresponding to the eigenvalues $\pm\lambda$ are of the same dimension.

Proof. See [7, Chapter 4, Exercise 5]. ♥

In the case $n = 3 \pmod 4$ though, anything can happen, and it is just our luck that $\dim(\mathbb{S}^3) = 3$ will *not* provide us with some nice extra symmetry. Bummer.

On the plus side, the Main Theorem holds for either choice of an orientation, since the Dirac spectrum of $\mathbb{S}^3$ only changes signs if we reverse the orientation.

Lemma 2.24 *(Dirac Spectrum under Change of Orientation)*

Let $\mathcal{M}^n$ (n odd) be endowed with the spin structure $(\mathcal{S}, \Theta)$ and the Dirac operator $\mathscr{D}$. Reverse the orientation of $\mathcal{M}$. Let $\tilde{\Theta} \coloneqq -\Theta$, and denote by $\tilde{\mathscr{D}}$ the Dirac operator associated with the new spin structure $(\mathcal{S}, \tilde{\Theta})$. Then $\operatorname{spec}(\tilde{\mathscr{D}}) = -\operatorname{spec}(\mathscr{D})$.

Proof. Conclude in this order: $\tilde{\mu} = -\mu$, $\tilde{\nabla}^\Sigma = \nabla^\Sigma$, and $\tilde{\mathscr{D}} = -\mathscr{D}$. ♥

Lastly, one also has a lower bound on $\operatorname{spec}(\mathscr{D}^2)$ in terms of the scalar curvature of $\mathcal{M}$.

Theorem 2.25 *(Lichnerowicz–Friedrich)*

Let λ be an eigenvalue of $\mathscr{D}$, and let $\operatorname{scal}_* \coloneqq \min\{\operatorname{scal}_\mathcal{M}(x)\}_{x\in\mathcal{M}}$. Then we have the sharp lower bound

$$\lambda^2 \geq \frac{n}{n-1}\frac{\operatorname{scal}_*}{4}.$$

In particular, $\operatorname{scal}_\mathcal{M} > 0$ implies $0 \notin \operatorname{spec}(\mathscr{D})$.

Proof. The slightly weaker (non-optimal) result without the factor $\frac{n}{n-1}$ is due to Lichnerowicz, which was later improved by Friedrich to the above (sharp) estimate. For reference, see [5, Satz 3.2.7] and [5, Satz 3.2.8], respectively. ♥

There are of course many more things to say (about the relation of the Dirac operator to the geometry of $\mathcal{M}$), but they reach *far* beyond the topic of this thesis.

This concludes the preliminaries. The reader is advised to fasten their seat belt, since we will now apply the theory developed up to this point.

II Dirac Eigenvalues of Homogeneous 3-Spheres

Now that all prerequisites have been clarified and the ambitious reader is equipped with knowledge about all the scary objects that might appear in the tall grass, it is time to explore the spin geometry of homogeneous 3-spheres, unveil the associated Dirac operator, and extract its spectral data.

§3 — *Dirac Spectrum*

The goal of this paragraph is to obtain a theoretical description of the Dirac spectrum of $\mathbb{S}^3_{(a,b,c)}$ as an infinite set whose element of the smallest absolute value is to be determined in order to prove the Main Theorem.

- In the first part (Deriving the Dirac Formula), we will describe the spin geometry of $\mathbb{S}^3_{(a,b,c)}$ by inspecting all the occurring fibre bundles and derive only from that (by elementary means) a formula for the associated Dirac operator.
- In the second part (Dirac Eigenvalues via Representation Theory), we will use the Peter–Weyl Theorem (Theorem 1.10) to obtain a decomposition of the space of L^2-spinor fields into finite-dimensional subspaces. We will then mimic the approach given by Bär in [1, Section 5] in order to recover the Dirac spectrum of $\mathbb{S}^3_{(a,b,c)}$ as the eigenvalues of an infinite collection of linear maps living on these subspaces.

Deriving the Dirac Formula

A general description of the spin structures of an arbitrary oriented Riemannian homogeneous space with a simply-connected structure group was given by Bär in [1]. Bär also calculated a formula for the Dirac operator in this much more complicated setting.

Instead of applying Bär's general formula to our special setting, we choose the didactically more valuable approach and derive an expression for the Dirac operator directly from its local description stated in Proposition 2.20. This turns out to be a quite effortless calculation due to the distinct symmetry of our setting. In order to use this formula, we first have to understand what the relevant bundles look like. Luckily, all of them are so kind to be trivial.

Let $\mathbb{S}^3$ be endowed with a left-invariant metric $\langle \cdot\,, \cdot \rangle = \langle \cdot\,, \cdot \rangle_{(a,b,c)}$ (recall the notation from Definition 1.2). View a, b, and c as fixed parameters, and further abbreviate $\mathbb{S}^3_{(a,b,c)}$ to $\mathbb{S}^3$ for the sake of readability.

Unless stated otherwise, we will from now on always consider the "standard" orientation of $\mathbb{S}^3$, with respect to which $\{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$ (in this order) is a positively oriented basis of $\mathrm{T}_1\mathbb{S}^3$. In particular, $\{a\mathrm{i}, b\mathrm{j}, c\mathrm{k}\} \in \mathrm{SO}(\mathbb{S}^3)$. Recall that reversing the orientation of $\mathbb{S}^3$ leads to the Dirac operator changing its sign (see Lemma 2.24).

Proposition 3.1 *(Trivial Bundles over $\mathbb{S}^3$)*

We have the following isomorphisms of bundles over $\mathbb{S}^3 = \mathbb{S}^3_{(a,b,c)}$:

1. $\mathrm{T}\mathbb{S}^3 \cong \mathbb{S}^3 \times \mathbb{R}^3$ *(The metric is not required here.)*
   In particular, $\Gamma(\mathrm{T}\mathbb{S}^3) \cong \mathrm{C}^\infty(\mathbb{S}^3, \mathbb{R}^3)$.
2. $\mathrm{SO}(\mathbb{S}^3) \cong \mathbb{S}^3 \times \mathrm{SO}(3)$
3. There exists exactly one spin structure on $\mathbb{S}^3$ (up to isomorphism), namely $\mathrm{Spin}(\mathbb{S}^3) := \mathbb{S}^3 \times \mathrm{Spin}(3)$, together with the obvious covering map Θ defined by $\Theta(g, u) := (g, \theta(u))$.
4. $\Sigma\mathbb{S}^3 \cong \mathbb{S}^3 \times \Sigma_3 = \mathbb{S}^3 \times \mathbb{C}^2$ via $\Phi : [(g, u), \mathbf{w}]_\sim \mapsto (g, \sigma(u)\mathbf{w})$
   In particular, $\Gamma(\Sigma\mathbb{S}^3) \cong \mathrm{C}^\infty(\mathbb{S}^3, \Sigma_3)$ and $\mathrm{L}^2(\Sigma\mathbb{S}^3) \cong \mathrm{L}^2(\mathbb{S}^3, \Sigma_3)$.

Note that the identifications stated in 1 and 2 require the choice of a (in the latter case, positive orthonormal) basis of $\mathrm{T}_1\mathbb{S}^3$ — see the proof below. Of course, we pick the obvious candidate $\{a\mathrm{i}, b\mathrm{j}, c\mathrm{k}\}$.

Proof.

1. This is because $\mathbb{S}^3$ is a Lie group. Indeed, one generally has $\mathrm{T}G \cong G \times \mathfrak{g}$, since for any basis $\{X_1, X_2, \ldots, X_n\}$ of $\mathfrak{g}$, the corresponding left-invariant vector fields define a global frame $g \mapsto \{X_1(g), X_2(g), \ldots, X_n(g)\}$ of $\mathrm{T}G$.
2. If G is an oriented Lie group endowed with a *left-invariant* metric $\langle \cdot\,, \cdot \rangle$, one has $\mathrm{SO}(G, \langle \cdot\,, \cdot \rangle) \cong G \times \mathrm{SO}(n)$ for the same reason: If $\{X_1, X_2, \ldots, X_n\}$ is a positive orthonormal basis of $\mathfrak{g}$, the global frame defined in 1 is a global section of $\mathrm{SO}(G, \langle \cdot\,, \cdot \rangle)$, since $\langle \cdot\,, \cdot \rangle$ is left-invariant.
3. Obviously, $(\mathrm{Spin}(\mathbb{S}^3), \Theta)$ defines a spin structure on $\mathbb{S}^3$. Uniqueness follows from the fact that $\mathrm{H}^1(\mathbb{S}^3; \mathbb{Z}_2)$ is the trivial group (see Theorem 2.13).
4. The map Φ is indeed an isomorphism of vector bundles:
   It is well-defined, since $\Phi([(g, u\tilde{u}), \sigma(\tilde{u}^{-1})\mathbf{w}]_\sim)$ would have been defined as $(g, \sigma(u\tilde{u})\sigma(\tilde{u}^{-1})\mathbf{w}) = (g, \sigma(u)\mathbf{w})$. It is certainly smooth and maps fibres to fibres over the same base point. Also, its restrictions to the fibres are vector space isomorphisms (since $\sigma(u)$ is one), from which the claim follows.

♥

Given that all occurring bundles are trivial and the unique spin structure is somewhat canonical, it is actually very easy (compared to Bär's general calculation) to obtain an explicit formula for the Dirac operator.

Notation 3.2 *(Left-Invariant Vector Fields)*

Denote by $X_1, X_2, X_3 \in \Gamma(\mathrm{T}\mathbb{S}^3)$ *also* the left-invariant vector fields corresponding to the positive orthonormal basis $\{X_1, X_2, X_3\} := \{a\mathrm{i}, b\mathrm{j}, c\mathrm{k}\}$ of $\mathrm{T}_1\mathbb{S}^3$.

The Dirac formula then looks as follows:

Theorem 3.3 *(Dirac Formula)*

The associated Dirac operator $\mathscr{D} : \Gamma(\Sigma\mathbb{S}^3) \to \Gamma(\Sigma\mathbb{S}^3)$ is given by the formula

$$\mathscr{D}\varphi = \sum_{i=1}^{3} X_i \cdot X_i(\varphi) + \frac{1}{2}\Big(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}\Big) X_1 \cdot X_2 \cdot X_3 \cdot \varphi.$$

However, before we prove the Dirac formula, the following lemma shall grant some insight on how the Clifford multiplication has to be understood if we want to view $\varphi \in \Gamma(\Sigma\mathbb{S}^3)$ as an element of $\mathrm{C}^\infty(\mathbb{S}^3, \Sigma_3)$, according to Proposition 3.1 4:

Lemma 3.4 *(Clifford Multiplication)*

Let $\varphi \in \mathrm{C}^\infty(\mathbb{S}^3, \Sigma_3)$. Then the Clifford product $X_i \cdot \varphi \in \mathrm{C}^\infty(\mathbb{S}^3, \Sigma_3)$ is given by

$$(X_i \cdot \varphi)(g) = e_i \cdot \varphi(g),$$

where $\{e_1, e_2, e_3\}$ denotes the standard basis of $\mathbb{R}^3$.

In other words, the left-invariant vector field $X_i \in \Gamma(\mathrm{T}\mathbb{S}^3)$ has to be identified with the constant map $e_i \in \mathrm{C}^\infty(\mathbb{S}^3, \mathbb{R}^3)$ in order for the Clifford multiplication to yield the correct result.

Proof. Recall the Clifford multiplication $\mu : \mathrm{T}M \otimes \Sigma M \to \Sigma M$ from Definition 2.17, sending $[p, y]_\sim \otimes [p, \mathbf{w}]_\sim$ to $[p, y \cdot \mathbf{w}]_\sim$.

On the one hand, Lemma 2.16 and Proposition 3.1 3 yield the identifications

$$\begin{array}{ccccc} \mathrm{T}\mathbb{S}^3 & \cong & \mathrm{SO}(\mathbb{S}^3) \times_\iota \mathbb{R}^3 & \cong & \mathrm{Spin}(\mathbb{S}^3) \times_{\iota\circ\theta} \mathbb{R}^3 \\ \cup & & \cup & & \cup \\ X_i(g) & \widehat{=} & [\{X_1(g), X_2(g), X_3(g)\}, e_i]_\sim & \widehat{=} & [(g,1), e_i]_\sim. \end{array}$$

$\vdots$

On the other hand, Proposition 3.1 4 gives us $(g, \varphi(g)) \mathrel{\widehat{=}} [(g,1), \varphi(g)]_\sim$.

Hence, $X_i(g) \cdot (g, \varphi(g)) = (g, e_i \cdot \varphi(g))$. ♥

Now we can safely view $\mathscr{D}$ as an operator living on $\mathrm{C}^\infty(\mathbb{S}^3, \Sigma_3)$. In particular, we have $X_i \cdot X_j \cdot \varphi = -X_j \cdot X_i \cdot \varphi$ for $i \neq j$, which we will need in the proof of Theorem 3.3 below. Recall also $\nabla_{X_i} X_i = 0$ from the proof of Proposition 1.5.

Proof (of Theorem 3.3). We use the local description of $\mathscr{D}$ from Proposition 2.20. Consider the trivial global section $s : g \mapsto (g, 1)$ of $\mathrm{Spin}(\mathbb{S}^3) = \mathbb{S}^3 \times \mathrm{Spin}(3)$. Then $\Theta \circ s$ is the global frame $\{X_1, X_2, X_3\}$.

The spinor derivative $\nabla^\Sigma_X \varphi \in \mathrm{C}^\infty(\mathbb{S}^3, \Sigma_3)$ of $\varphi \in \mathrm{C}^\infty(\mathbb{S}^3, \Sigma_3)$ along $X \in \mathrm{C}^\infty(\mathbb{S}^3, \mathbb{R}^3)$ (recall Theorem 2.18) is then given by

$$\nabla^\Sigma_X \varphi = X(\varphi) + \frac{1}{4} \sum_{j=1}^{3} X_j \cdot \nabla_X X_j \cdot \varphi,$$

from which we conclude (using $\nabla_{X_i} X_i = 0$ to kill the diagonal of the double sum below; and using that ∇ is torsion-free together with $X_j \cdot X_i \cdot \varphi = -X_i \cdot X_j \cdot \varphi$ for $i \neq j$ as well as $[X_i, X_j] = -[X_j, X_i]$ in order to reduce the summation over $i \neq j$ to the cases where $i < j$ via elementary index manipulations)

$$\begin{aligned}
\mathscr{D}\varphi &= \sum_{i=1}^{3} X_i \cdot \nabla^\Sigma_{X_i} \varphi \\
&= \sum_{i=1}^{3} X_i \cdot \Big(X_i(\varphi) + \frac{1}{4} \sum_{j=1}^{3} X_j \cdot \nabla_{X_i} X_j \cdot \varphi \Big) \\
&= \sum_{i=1}^{3} X_i \cdot X_i(\varphi) + \frac{1}{4} \sum_{i,j=1}^{3} X_i \cdot X_j \cdot \nabla_{X_i} X_j \cdot \varphi \\
&= \sum_{i=1}^{3} X_i \cdot X_i(\varphi) + \frac{1}{4} \Big(\sum_{i=1}^{3} X_i \cdot X_i \cdot \underbrace{\nabla_{X_i} X_i}_{=0} \cdot \varphi + \sum_{i \neq j} X_i \cdot X_j \cdot \nabla_{X_i} X_j \cdot \varphi \Big) \\
&= \sum_{i=1}^{3} X_i \cdot X_i(\varphi) + \frac{1}{8} \Big(\sum_{i \neq j} X_i \cdot X_j \cdot \nabla_{X_i} X_j \cdot \varphi + \sum_{i \neq j} X_i \cdot X_j \cdot \nabla_{X_i} X_j \cdot \varphi \Big) \\
&= \sum_{i=1}^{3} X_i \cdot X_i(\varphi) + \frac{1}{8} \Big(\sum_{i \neq j} X_i \cdot X_j \cdot \nabla_{X_i} X_j \cdot \varphi + \sum_{j \neq i} X_j \cdot X_i \cdot \nabla_{X_j} X_i \cdot \varphi \Big)
\end{aligned}$$

$\vdots$

$$
\begin{aligned}
&= \sum_{i=1}^{3} X_i \cdot X_i(\varphi) + \frac{1}{8} \sum_{i \neq j} X_i \cdot X_j \cdot (\nabla_{X_i} X_j - \nabla_{X_j} X_i) \cdot \varphi \\
&= \sum_{i=1}^{3} X_i \cdot X_i(\varphi) + \frac{1}{8} \sum_{i \neq j} X_i \cdot X_j \cdot [X_i, X_j] \cdot \varphi \\
&= \sum_{i=1}^{3} X_i \cdot X_i(\varphi) + \frac{1}{8} \Big(\sum_{i<j} X_i \cdot X_j \cdot [X_i, X_j] \cdot \varphi + \sum_{i>j} X_i \cdot X_j \cdot [X_i, X_j] \cdot \varphi \Big) \\
&= \sum_{i=1}^{3} X_i \cdot X_i(\varphi) + \frac{1}{8} \Big(\sum_{i<j} X_i \cdot X_j \cdot [X_i, X_j] \cdot \varphi + \sum_{i<j} X_j \cdot X_i \cdot [X_j, X_i] \cdot \varphi \Big) \\
&= \sum_{i=1}^{3} X_i \cdot X_i(\varphi) + \frac{1}{4} \sum_{i<j} X_i \cdot X_j \cdot [X_i, X_j] \cdot \varphi.
\end{aligned}
$$

The claimed formula eventually follows from

$$
\begin{aligned}
X_1 \cdot X_2 \cdot [X_1, X_2] \cdot \varphi &= \frac{2ab}{c} X_1 \cdot X_2 \cdot X_3 \cdot \varphi, \\
X_2 \cdot X_3 \cdot [X_2, X_3] \cdot \varphi &= \frac{2bc}{a} X_2 \cdot X_3 \cdot X_1 \cdot \varphi, \text{ and} \\
X_1 \cdot X_3 \cdot [X_1, X_3] \cdot \varphi &= -\frac{2ca}{b} X_1 \cdot X_3 \cdot X_2 \cdot \varphi.
\end{aligned}
$$

♥

This was surprisingly easy, thanks to the identifications all being canonical in some sense. Note how left-invariance has played a key role in every calculation so far, allowing such straightforward results. However, the Dirac formula *by itself* is not of much use to us, since it tells nothing about the spectrum. This is where representation theory finally comes into play:

Recall from Theorem 1.12 the well-known fact that every irreducible representation of $\mathbb{S}^3 \cong \mathrm{SU}(2)$ is equivalent to $\pi_n : \mathbb{S}^3 \to \mathrm{GL}(V_n)$ for some $n \in \mathbb{N}_0$, where π_n acts on $V_n = \mathrm{span}_{\mathbb{C}}\{P_k : 0 \leq k \leq n\}$, $P_k(z, w) = z^{n-k} w^k$ via $(\pi_n(g)P)(z, w) = P((z, w)g)$. Applying the Peter–Weyl Theorem (Theorem 1.10) to $\mathrm{L}^2(\mathbb{S}^3, \Sigma_3) = \mathrm{L}^2(\mathbb{S}^3, \mathbb{C}) \otimes \Sigma_3$ (and using the canonical identification $V^* \otimes W \cong \mathrm{Hom}(V, W)$) leads to the decomposition

$$
\mathrm{L}^2(\mathbb{S}^3, \Sigma_3) \cong \overline{\bigoplus_{n \in \mathbb{N}_0} V_n \otimes \mathrm{Hom}(V_n, \Sigma_3)},
$$

where the embedding $V_n \otimes \mathrm{Hom}(V_n, \Sigma_3) \hookrightarrow \mathrm{L}^2(\mathbb{S}^3, \Sigma_3)$ is achieved by identifying

$$v \otimes f \mathrel{\widehat{=}} \Phi_n(v \otimes f) : g \mapsto f(\pi_n(g^{-1})v).$$

This enables us to explicitly calculate some Dirac eigenvalues in the next step (in fact, we will get a *theoretical* description of all eigenvalues; but as usual, things tend to get complicated very quickly).

However, before we move forward to the Dirac spectrum, let us just squeeze in a brief definition for the mere sake of readability.

Notation 3.5 *(An Important Constant)*

In the Dirac formula, we abbreviate $C \coloneqq \dfrac{1}{2}\Big(\dfrac{ab}{c} + \dfrac{bc}{a} + \dfrac{ca}{b}\Big)$.

The constant C will play an important role later on (too important to be defined only in passing, which is why it gets its own fancy box).

Dirac Eigenvalues via Representation Theory

The following procedure is the same as in [1, Sections 3–5]. The goal is to find out how $\mathscr{D} : \mathrm{L}^2(\mathbb{S}^3, \Sigma_3) \to \mathrm{L}^2(\mathbb{S}^3, \Sigma_3)$ acts on each finite-dimensional subspace $V_n \otimes \mathrm{Hom}(V_n, \Sigma_3)$ as an endomorphism $\mathcal{D}_n \in \mathrm{End}(\mathrm{Hom}(V_n, \Sigma_3))$ in the second factor of the tensor product, whose matrix representation should be as nice as possible with respect to a suitably chosen basis, so that we can have the easiest possible access to its eigenvalues. Luckily, Bär already did the math, which we can use virtually unchanged in our setting. The downside, however, is that we will not be able to get an explicit expression for the entire spectrum (unlike Bär did in the special case $b = c$), but instead have to face a whole §4 of computations.

Let us commence by finding out what $\mathcal{D}_n$ looks like (see [1, Proposition 1]).

Proposition 3.6 *(Dirac Restriction to Embedded Subspaces)*

The Dirac operator acts on each $V_n \otimes \mathrm{Hom}(V_n, \Sigma_3)$ via $\mathscr{D}(v \otimes f) = v \otimes \mathcal{D}_n(f)$ (abbreviating $\Phi_n(v \otimes f) \mathrel{\widehat{=}} v \otimes f$), where $\mathcal{D}_n(f) \in \mathrm{Hom}(V_n, \Sigma_3)$ is the map given by

$$\mathcal{D}_n(f)P = -\sum_{i=1}^{3} e_i \cdot f(\pi_{n*}(X_i)P) + C e_1 \cdot e_2 \cdot e_3 \cdot f(P). \quad \textit{(for each } P \in V_n\textit{)}$$

In particular, $\mathscr{D}$ leaves each subspace $V_n \otimes \mathrm{Hom}(V_n, \Sigma_3)$ invariant.

Note that for this formula to make sense, each element of $\{X_1, X_2, X_3\} \subset \mathrm{T}_1\mathbb{S}^3 \subset \mathbb{H}$ (see Notation 3.2) has to be identified with the corresponding matrix in $\mathfrak{su}(2)$, that is,

$$X_1 \mathrel{\hat=} a \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \; X_2 \mathrel{\hat=} b \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ and } X_3 \mathrel{\hat=} c \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$$

as described at the beginning of §1 (now with coefficients a, b, and c).

Proof. Plug $\Phi_n(v \otimes f)$ into the Dirac formula (Theorem 3.3). A quick calculation reveals that the derivative in the formula is given by

$$\begin{aligned} X_i(\Phi_n(v \otimes f))(g) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\, \Phi_n(v \otimes f)(g\mathrm{e}^{tX_i}) && \textit{(calculate the derivative)} \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\, f(\pi_n(\mathrm{e}^{-tX_i}g^{-1})v) && \textit{(definition of } \Phi_n\textit{)} \\ &= f(\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\, (\pi_n(\mathrm{e}^{-tX_i}) \circ \pi_n(g^{-1}))v) && (f \textit{ is linear}) \\ &= f(\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\, (\mathrm{e}^{-t\pi_{n*}(X_i)} \circ \pi_n(g^{-1}))v) && (\mathrm{e}^{\cdot} \textit{ commutes with } \pi_n) \\ &= -(f \circ \pi_{n*}(X_i) \circ \pi_n(g^{-1}))v && (\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\, \mathrm{e}^{tX} = X) \\ &= \Phi_n(v \otimes -f \circ \pi_{n*}(X_i))(g). && \textit{(definition of } \Phi_n\textit{)} \end{aligned}$$

The claim follows because everything else appearing in the formula is linear and due to Lemma 3.4. ♥

The fact that the Dirac operator leaves each subspace $V_n \otimes \mathrm{Hom}(V_n, \Sigma_3)$ invariant allows us to describe the Dirac spectrum considering only the linear maps $\mathcal{D}_n$ for each $n \in \mathbb{N}_0$, using the Peter–Weyl Theorem.

Corollary 3.7 *(Theoretical Description of the Dirac Spectrum)*

The Dirac spectrum can be expressed as

$$\operatorname{spec}(\mathscr{D}) = \{\lambda \in \mathbb{R} : \lambda \text{ is an eigenvalue of } \mathcal{D}_n \text{ for some } n \in \mathbb{N}_0\},$$

and each $\mathcal{D}_n$ with eigenvalue λ of multiplicity m contributes $m \dim_{\mathbb{C}}(V_n) = m(n+1)$ to the total multiplicity of λ as an eigenvalue of $\mathscr{D}$.

It follows, by the way, that each eigenvalue λ of $\mathscr{D}$ can appear as an eigenvalue of $\mathcal{D}_n$ for only *finitely* many $n \in \mathbb{N}_0$, since the eigenspace of $\mathscr{D}$ corresponding to the eigenvalue λ is always finite-dimensional (see Theorem 2.22 2).

Proof.

($\supset$) Let $f \in \mathrm{Hom}(V_n, \Sigma_3)$ be an eigenfunction of $\mathcal{D}_n$ corresponding to an eigenvalue λ of multiplicity m. Then, by Proposition 3.6, we have $\mathscr{D}(v \otimes f) = v \otimes \mathcal{D}_n(f) = v \otimes \lambda f = \lambda(v \otimes f)$ for all $v \in V_n$; that is, $v \otimes f$ is an eigenspinor of $\mathscr{D}$ with eigenvalue λ for all $v \in V_n$. In particular, the multiplicity of λ contributed by $\mathcal{D}_n$ is $m \dim_{\mathbb{C}}(V_n) = m(n+1)$.

($\subset$) Since $\bigoplus_{n \in \mathbb{N}_0} V_n \otimes \mathrm{Hom}(V_n, \Sigma_3)$ is a dense subspace of $\mathrm{L}^2(\mathbb{S}^3, \Sigma_3)$, we can write each $\varphi \in \mathrm{L}^2(\mathbb{S}^3, \Sigma_3)$ as $\varphi = \sum_{n=0}^{\infty} \varphi_n$, where φ_n is a linear combination of some $v_{n,i} \otimes f_{n,i} \in V_n \otimes \mathrm{Hom}(V_n, \Sigma_3)$, $n \in \mathbb{N}_0$. If $\varphi \neq 0$ is an eigenspinor of $\mathscr{D}$ with eigenvalue λ, it follows from the direct and orthogonal decomposition (using again Proposition 3.6)

$$\begin{array}{lll} \mathscr{D}\,\varphi & = \displaystyle\sum_{n=0}^{\infty} \mathscr{D}\,\varphi_n & = \displaystyle\sum_{n=0}^{\infty} \text{some linear combination of } v_{n,i} \otimes \mathcal{D}_n(f_{n,i}) \\ \parallel & & \\ \lambda\varphi & = \displaystyle\sum_{n=0}^{\infty} \lambda\varphi_n & = \displaystyle\sum_{n=0}^{\infty} \text{some linear combination of } v_{n,i} \otimes \lambda f_{n,i} \end{array}$$

that λ is an eigenvalue of $\mathcal{D}_n$ for all $n \in \mathbb{N}_0$ with $\varphi_n \neq 0$. Since there has to be at least one $n \in \mathbb{N}_0$ with $\varphi_n \neq 0$, it follows that at least one $\mathcal{D}_n$ has the eigenvalue λ.

♥

The further strategy boils down to elementary linear algebra: Pick a basis $\mathscr{B}$ of V_n and a basis $\tilde{\mathscr{B}}$ of Σ_3. Then each linear map $f \in \mathrm{Hom}(V_n, \Sigma_3)$ can be identified with its matrix representation $A \in \mathrm{Mat}_{\mathbb{C}}(2, n+1)$ with respect to $\mathscr{B}$ and $\tilde{\mathscr{B}}$. By again choosing a convenient basis of $\mathrm{Mat}_{\mathbb{C}}(2, n+1)$, we obtain an identification of $\mathrm{Mat}_{\mathbb{C}}(2, n+1)$ with $\mathbb{C}^{2n+2}$, and the endomorphism $\mathcal{D}_n : \mathrm{Hom}(V_n, \Sigma_3) \to \mathrm{Hom}(V_n, \Sigma_3)$ can be viewed as an element of $\mathrm{Mat}_{\mathbb{C}}(2n+2)$.

We commence by expressing $\pi_{n*}(X_i)$ in coordinates (see Proposition 3.6). Note that Bär's default basis of $\mathfrak{su}(2)$ corresponds to $\{\mathrm{k}, -\mathrm{j}, \mathrm{i}\}$ rather than our $\{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$ due to the particular identification of $\mathbb{S}^3 \subset \mathbb{H}$ with $\mathbb{S}^3 \cong \mathrm{SU}(2)$ that we use, so the computations differ slightly. In the end, of course, this does not matter.

Lemma 3.8 *(Differentials of π_n in Coordinates)*

With respect to the complex basis $\mathscr{B} = \{P_k : 0 \le k \le n\}$ of V_n, the endomorphism $\pi_{n*}(X_i) \in \mathrm{End}(V_n)$ has the matrix representation $\Pi_n^i \in \mathrm{Mat}_{\mathbb{C}}(n+1)$, where

$$\Pi_n^1 = a\mathrm{i}\ \mathrm{diag}(n, n-2, \dots, -n),$$

$$\Pi_n^2 = b \begin{pmatrix} 0 & 1 & & & \\ -n & 0 & 2 & & \\ & -n+1 & 0 & \ddots & \\ & & \ddots & \ddots & n \\ & & & -1 & 0 \end{pmatrix}, \text{ and } \Pi_n^3 = c\mathrm{i} \begin{pmatrix} 0 & 1 & & & \\ n & 0 & 2 & & \\ & n-1 & 0 & \ddots & \\ & & \ddots & \ddots & n \\ & & & 1 & 0 \end{pmatrix}.$$

Proof. Recall $P_k(z,w) = z^{n-k}w^k$ and $(\pi_n(g)P)(z,w) = P((z,w)g)$.

We compute the entries of the matrices Π_n^i by observing how $\pi_{n*}(X_i)$ acts on the basis elements $P_k \in V_n$.

- If $i = 1$, we have $\mathrm{e}^{tX_1} = \left(\begin{smallmatrix} \exp(ait) & 0 \\ 0 & \exp(-ait) \end{smallmatrix}\right)$; that is,

$$\begin{aligned} (\pi_{n*}(X_1)P_k)(z,w) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\pi_n(\mathrm{e}^{tX_1})P_k)(z,w) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P_k\Big((z,w)\left(\begin{smallmatrix} \exp(ait) & 0 \\ 0 & \exp(-ait) \end{smallmatrix}\right)\Big) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P_k\big(z\exp(ait), w\exp(-ait)\big) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} z^{n-k}w^k \exp(a\mathrm{i}(n-2k)t) \\ &= a\mathrm{i}(n-2k)P_k(z,w). \end{aligned}$$

- If $i = 2$, we have $\mathrm{e}^{tX_2} = \left(\begin{smallmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{smallmatrix}\right)$; that is,

$$\begin{aligned} (\pi_{n*}(X_2)P_k)(z,w) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\pi_n(\mathrm{e}^{tX_2})P_k)(z,w) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P_k\Big((z,w)\left(\begin{smallmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{smallmatrix}\right)\Big) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P_k\big(z\cos(bt) - w\sin(bt), z\sin(bt) + w\cos(bt)\big) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big(z\cos(bt) - w\sin(bt)\big)^{n-k}\big(z\sin(bt) + w\cos(bt)\big)^k \\ &= (n-k)z^{n-k-1}(-bw)w^k + z^{n-k}kw^{k-1}bz \\ &= -b(n-k)P_{k+1}(z,w) + bkP_{k-1}(z,w). \end{aligned}$$

$\vdots$

Note that this also holds if $k \in \{0, n\}$, since the hypothetical terms P_{-1} and P_{n+1} are then multiplied with zero.

- If $i = 3$, we have $\mathrm{e}^{tX_3} = \left(\begin{smallmatrix} \cos(ct) & \mathrm{i}\sin(ct) \\ \mathrm{i}\sin(ct) & \cos(ct) \end{smallmatrix}\right)$, and the calculation is almost the same as in the case $i = 2$.

This proves the claim. ♥

The next thing to deal with in the formula from Proposition 3.6 is the Clifford multiplication with $e_i \in \mathbb{R}^3$. However, given our notion of the Clifford multiplication and our choice of $\{X_1, X_2, X_3\}$, we need not choose a particularly fancy basis of Σ_3. In fact, the standard basis will do just fine.

Lemma 3.9 *(Clifford Multiplication in Coordinates)*

With respect to the complex standard basis $\tilde{\mathscr{B}} = \{\binom{1}{0}, \binom{0}{1}\}$ of $\Sigma_3 = \mathbb{C}^2$, the Clifford multiplication with $e_k \in \mathbb{R}^3$ (which is an endomorphism of Σ_3) has the matrix representation E_k, where

$$E_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \ E_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ and } E_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

Proof. With σ_n as in Definition 2.4 (2) and $U, V, T \in \mathrm{Mat}_{\mathbb{C}}(2)$ as in the sketch of the proof of Theorem 2.3, we have

$$\begin{aligned} \sigma_3(e_1) &= \phi_2(e_{2\cdot 1-1}) &&= U &&= E_1, \\ \sigma_3(e_2) &= \phi_2(e_{2\cdot 1}) &&= V &&= E_2, \text{ and} \\ \sigma_3(e_3) &= \phi_3(e_3) &&= \mathrm{i}T &&= E_3, \end{aligned}$$

which already proves the claim. ♥

Note that $\{E_1, E_2, E_k\} \mathrel{\widehat{=}} \{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$ according to our identification of $\mathrm{Mat}_2(\mathbb{C})$ with $\mathbb{H}$; in particular, $E_1E_2E_3 \mathrel{\widehat{=}} \mathrm{ijk} = -1$. Hence, $e_1 \cdot e_2 \cdot e_3 \cdot \mathbf{w} = -\mathbf{w}$ for all $\mathbf{w} \in \Sigma_3$.

Putting everything together, we obtain a matrix representation of $\mathcal{D}_n(f)$ if f is expressed in coordinates with respect to the bases $\mathscr{B}$ and $\tilde{\mathscr{B}}$.

Corollary 3.10 *(Dirac Restriction in Coordinates)*

Let $f \in \mathrm{Hom}(V_n, \Sigma_3)$, and let $A \in \mathrm{Mat}_{\mathbb{C}}(2, n+1)$ be the matrix representation of f with respect to the bases $\mathscr{B}$ and $\tilde{\mathscr{B}}$. Then the matrix representation of $\mathcal{D}_n(f)$ with respect to the same bases is given by

$$\mathcal{D}_n(A) = -\sum_{i=1}^{3} E_i \cdot A \cdot \Pi_n^i - CA,$$

where the dot of course means the matrix multiplication.

Proof. This follows from Proposition 3.6, Lemma 3.8, and Lemma 3.9. ♥

The second summand of $\mathcal{D}_n$ only shifts the spectrum by $-C$, so we will drop it in later calculations and adopt the following notation from Bär:

Notation 3.11 *(Shifting the Spectrum)*

We write $\mathcal{D}'_n(A) := -\sum_{i=1}^{3} E_i \cdot A \cdot \Pi_n^i$; that is, $\mathcal{D}_n = \mathcal{D}'_n - C$.

It remains to choose a basis of $\mathrm{Mat}_{\mathbb{C}}(2, n+1)$ such that the map $\mathcal{D}'_n : A \mapsto \mathcal{D}'_n(A)$ has a nice matrix representation, which we will *also* denote by $\mathcal{D}'_n \in \mathrm{Mat}_{\mathbb{C}}(2n+2)$ (forgive the abuse of notation). Consider Bär's choice $\{A_0, A_1, \ldots, A_n, B_0, B_1, \ldots, B_n\}$, where

$$A_0 = \begin{pmatrix} 1 & 0 & \ldots \\ 0 & 0 & \ldots \end{pmatrix}, \; A_1 = \begin{pmatrix} 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \end{pmatrix}, \; \ldots ;$$

$$B_0 = \begin{pmatrix} 0 & 0 & \ldots \\ 1 & 0 & \ldots \end{pmatrix}, \; B_1 = \begin{pmatrix} 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & \ldots \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \end{pmatrix}, \; \ldots$$

Then $\mathcal{D}'_n$ is a tridiagonal matrix consisting of an upper left and a lower right block $\mathcal{A}'_n$ and $\mathcal{B}'_n$, respectively. More precisely, the matrices are of the following form:

Theorem 3.12 *(Primed Dirac Matrices)*

With respect to the basis $\{A_0, A_1, \ldots, A_n, B_0, B_1, \ldots, B_n\}$ of $\mathrm{Mat}_{\mathbb{C}}(2, n+1)$, the primed restricted Dirac operator $\mathcal{D}'_n \in \mathrm{End}(\mathrm{Mat}_{\mathbb{C}}(2, n+1))$ has the matrix representation $\mathcal{D}'_n = \mathrm{diag}(\mathcal{A}'_n, \mathcal{B}'_n) \in \mathrm{Mat}_{\mathbb{C}}(2n+2)$, where $\mathcal{A}'_n, \mathcal{B}'_n \in \mathrm{Mat}_{\mathbb{C}}(n+1)$ have the following entries $M[k, \ell] := M_{k+1,\ell+1}$ (defined whenever $k, \ell \in \{0, 1, \ldots, n\}$; entries

⋮

at illegal indices should of course be ignored):

If $k \in \{0, 1, \ldots, n\}$ is even:

$$\begin{aligned} \mathcal{A}'_n[k-1,k] &= (c-b)(n-k+1) \\ \mathcal{A}'_n[k,k] &= a(n-2k) \\ \mathcal{A}'_n[k+1,k] &= (c+b)(k+1) \end{aligned}$$

If $k \in \{0, 1, \ldots, n\}$ is odd:

$$\begin{aligned} \mathcal{A}'_n[k-1,k] &= (c+b)(n-k+1) \\ \mathcal{A}'_n[k,k] &= -a(n-2k) \\ \mathcal{A}'_n[k+1,k] &= (c-b)(k+1) \end{aligned}$$

All other entries are zero. The expressions for the entries of $\mathcal{B}'_n$ are identical, one only has to swap "k even" and "k odd".

Proof. If $k \in \{0, 1, \ldots, n\}$ is even, we have (for example)

$$\begin{aligned} E_1 \cdot A_k \cdot \Pi_n^1 &= \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \cdot \begin{pmatrix} \cdots & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \end{pmatrix} \cdot a\mathrm{i} \begin{pmatrix} \ddots & & \\ & n-2k & \\ & & \ddots \end{pmatrix} \\ &= a\mathrm{i} \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \cdot \begin{pmatrix} \cdots & 0 & n-2k & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \end{pmatrix} \\ &= -a(n-2k)A_k \end{aligned}$$

and, in a similar fashion,

$$\begin{aligned} E_2 \cdot A_k \cdot \Pi_n^2 &= b(n-k+1)A_{k-1} - b(k+1)A_{k+1}, \\ E_3 \cdot A_k \cdot \Pi_n^3 &= -c(n-k+1)A_{k-1} - c(k+1)A_{k+1} \end{aligned}$$

(of course, we have to set $A_{-1} \coloneqq A_{n+1} \coloneqq 0$ for this to hold).

If, on the other hand, k is odd, we have

$$\begin{aligned} E_1 \cdot A_k \cdot \Pi_n^1 &= a(n-2k)A_k, \\ E_2 \cdot A_k \cdot \Pi_n^2 &= -b(n-k+1)A_{k-1} + b(k+1)A_{k+1}, \text{ and} \\ E_3 \cdot A_k \cdot \Pi_n^3 &= -c(n-k+1)A_{k-1} - c(k+1)A_{k+1}. \end{aligned}$$

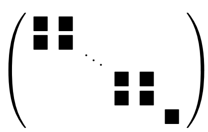
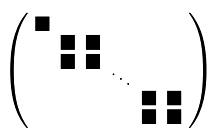
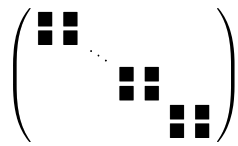
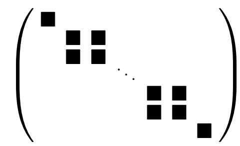
Plugging these into $\mathcal{D}'_n(A_k) = -\sum_{i=1}^3 E_i \cdot A_k \cdot \Pi_n^i$ yields

$$\begin{aligned} \mathcal{D}'_n(A_k) &= (c-b)(n-k+1)A_{k-1} + (c+b)(k+1)A_{k+1} + a(n-2k)A_k, \quad (k \text{ even}) \\ \mathcal{D}'_n(A_k) &= (c+b)(n-k+1)A_{k-1} + (c-b)(k+1)A_{k+1} - a(n-2k)A_k. \quad (k \text{ odd}) \end{aligned}$$

$\vdots$

The expressions with "B_k" instead of "A_k" are the same, one only has to swap the cases where k is even or odd. This proves the claim. ♥

Under the assumption $b = c$ (Bär's setting), the blocks $\mathcal{A}'_n$ and $\mathcal{B}'_n$ can be further decomposed into 2×2- and 1×1-blocks, with the concrete shape depending on the parity of n as illustrated below.

$\mathcal{A}'_n$ (n even)	$\mathcal{B}'_n$ (n even)	$\mathcal{A}'_n$ (n odd)	$\mathcal{B}'_n$ (n odd)
			

In this special case, it is possible to directly calculate the eigenvalues of each $\mathcal{D}'_n$ as the collection of eigenvalues of the small blocks, obtaining an explicit description of the entire Dirac spectrum (after shifting everything by $-C$). For the parameters $a = \frac{1}{T}$ and $b = c = 1$, Bär obtained

$$\operatorname{spec}(\mathscr{D}) = \left\{ -\frac{m}{T} - \frac{T}{2} \right\}_{m \geq 1} \cup \left\{ -\frac{T}{2} \pm \sqrt{\left(\frac{1}{T^2} - 1\right)(m - 2k)^2 + m^2} \right\}_{\substack{m \geq 2,\\ 1 \leq k \leq m-1}}, \qquad (1)$$

where $m = n + 1 = \dim_{\mathbb{C}}(V_n)$ (see [1, Theorem 3]).

Unfortunately, in the general setting (where $a, b, c > 0$ are arbitrary), it is not feasible to have such an explicit result, since $\mathcal{D}'_n$ assumes its general tridiagonal form and does not break down into smaller blocks. In fact, we only have easy-to-calculate eigenvalues of $\mathcal{D}'_n$ for $n \in \{0, 1\}$ (see §4, Low-Dimensional Subspaces). Bummer.

If we cannot have the whole spectrum, it would be nice to know an expression for the smallest absolute non-zero eigenvalue λ_* depending on a, b, and c instead (in analogy to what Lauret did in his paper [2]). This, however, will *not* work without the a priori restriction $\mathrm{scal}_{(a,b,c)} > 0$:

Note that Bär's spectrum contains eigenvalues of the form $-\frac{T}{2} + m$ (the case $m = 2k$ in the second set). Start at $T = 1$, and observe the behaviour of each eigenvalue $-\frac{T}{2} + m$ as T tends to infinity: At the moment $T = 2m$, the expression passes through zero, and then its magnitude increases again. This way, subspaces of arbitrarily high dimension contribute the smallest eigenvalue for a short period of time. The restriction $\mathrm{scal}_{(a,b,c)} > 0$ prevents this from happening, since T is then bounded from above, and the

Lichnerowicz–Friedrich estimate (Theorem 2.25) also guarantees that the eigenvalues cannot continuously pass through zero like that.

This leads us to the final paragraph of this thesis, which solely deals with hunting down a symbolic expression for the smallest Dirac eigenvalue $\lambda_*(\mathbb{S}^3_{(a,b,c)})$ under the additional assumption $\mathrm{scal}_{(a,b,c)} > 0$.

§ 4 —— *Smallest Eigenvalue and Spectral Invariance* ——

This final paragraph is devoted to the proof of the Main Theorem:

Main Theorem *(Dirac Eigenvalues of Homogeneous 3-Spheres)*

Let $\mathbb{S}^3_{(a,b,c)}$ be a homogeneous 3-sphere of *positive scalar curvature*, endowed with either orientation.

Then the smallest absolute value of its Dirac eigenvalues is given by

$$\lambda_{*(a,b,c)} = a + b + c - \frac{1}{2}\Big(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}\Big) > 0,$$

and its Dirac spectrum determines the underlying metric up to isometry (within the class of homogeneous 3-spheres).

Recall that we only consider the "standard" orientation of $\mathbb{S}^3$ (with respect to which $\{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$ is positively oriented). In this case, $\lambda_{*(a,b,c)}$ will be an eigenvalue of $\mathscr{D}$. In the other case, $\mathscr{D}$ just changes its sign (see Lemma 2.24), and $\lambda_{*(a,b,c)}$ stays the same. **(!)**

Let us briefly summarise the situation we find ourselves in: Given a left-invariant metric $\langle\,\cdot\,,\,\cdot\,\rangle_{(a,b,c)}$ on the 3-sphere, we know from the last paragraph that the spectrum of the associated Dirac operator $\mathscr{D}$ consists exactly of the eigenvalues of an infinite family of tridiagonal matrices $\mathcal{D}_n = \mathcal{D}'_n - C = \operatorname{diag}(\mathcal{A}'_n, \mathcal{B}'_n) - C$ of the size $2n+2$ (where $n \in \mathbb{N}_0$), whose entries depend on the parameters $a, b, c > 0$. It is not possible to explicitly calculate the entire spectrum in this slightly more general setting (compared to Bär's case, where $b = c$), since certain off-diagonal entries do not vanish anymore. However, under the additional assumption $\operatorname{scal}_{(a,b,c)} > 0$, it makes sense to search for a closed expression for the smallest eigenvalue $\lambda_{*(a,b,c)}$ of $\mathscr{D}$, of which we hope that it will also determine $\langle\,\cdot\,,\,\cdot\,\rangle_{(a,b,c)}$ up to isometry, together with the aid of the known spectral invariants $\operatorname{vol}_{(a,b,c)}$ and $\operatorname{scal}_{(a,b,c)}$.

Since the proof of the Main Theorem is spun from many threads, it will be split into several digestible steps that arose quite naturally during the puzzle phase. Remember that we will **always** assume $\operatorname{scal}_{(a,b,c)} > 0$.

- In the first part (Heuristic), we will briefly discuss the overall idea of the proof and present the approach used to find a candidate for λ_* in the first place, together with the problems that arose immediately.
- In the second part (Low-Dimensional Subspaces), we will prove that our candidate λ_* has the smallest absolute value among all eigenvalues of $\mathcal{D}_n$ with $n \leq 4$. This

will be done by going through the individual cases, either directly comparing the eigenvalues or arguing analytically with the characteristic polynomial.

- In the third part (Dirac Squares and Centrosymmetry), we will derive explicit expressions for the entries of the squares $\mathcal{D}_n^2 = (\mathcal{D}_n' - C)^2$ (as a preparation for the next step) and inspect certain symmetries exhibited by the matrices $\mathcal{D}_n$ and $\mathcal{D}_n^2$ depending on the parity of n.
- In the fourth part (Triangle Induction), we will prove that λ_* has the smallest absolute value among all eigenvalues of $\mathcal{D}_n$ with $n \geq 5$ (and thus proving the first statement of the Main Theorem) by showing that λ_*^2 is the smallest eigenvalue of $\mathcal{D}_n^2$, using an inductive argument based on the *Gershgorin-Circle Theorem*.
- In the fifth and last part (Spectral Invariance), we will conclude the proof of the Main Theorem by showing that the triple $\{\mathrm{vol}_{(a,b,c)}, \mathrm{scal}_{(a,b,c)}, \lambda_{*(a,b,c)}\}$ determines the underlying metric $\langle\,\cdot\,,\,\cdot\,\rangle_{(a,b,c)}$ up to isometry.

As threatened, we commence by discussing some initial ideas that lead to the general proof strategy, as well as a few preparations.

Heuristic

It is understandable to be overwhelmed by the problem at the first moment, since there is an infinite family of matrices over a three-dimensional parameter space, increasing in size, and staring menacingly at us. That's *a lot* of data for sure.

Luckily, we can simplify things a bit by taking advantage of the behaviour of the problem under rescaling: If $\mathscr{D}$ is the Dirac operator associated with $\langle\,\cdot\,,\,\cdot\,\rangle_{(a,b,c)}$ and $\tilde{\mathscr{D}}$ is the Dirac operator associated with $\langle\,\cdot\,,\,\cdot\,\rangle_{(\alpha a,\alpha b,\alpha c)}$ for a scaling factor $\alpha > 0$, then we see from the Dirac formula (Theorem 3.3) together with Lemma 3.4 that $\tilde{\mathscr{D}} = \alpha\,\mathscr{D}$; that is, if φ is an eigenspinor of $\mathscr{D}$ with eigenvalue λ, then φ is also an eigenspinor of $\tilde{\mathscr{D}}$ with eigenvalue $\alpha\lambda$. This allows us to assume $a = 1$ without loss of generality by considering $(a, b, c) = a\,(1, \frac{b}{a}, \frac{c}{a})$. Another simplification comes directly from the classification theorem of left-invariant metrics (Theorem 1.3), whose exact wording stated that we may further assume $a \geq b \geq c$ without loss of generality. Putting these together, we may reduce the three-dimensional parameter space $\{(a, b, c) : a, b, c > 0\}$ to the two-dimensional space $\{(b, c) : 1 \geq b \geq c > 0\}$. This is a useful simplification, since it allows plotting real-valued functions on the now two-dimensional parameter space, which is nice for building hypotheses.

Note that if $\langle\,\cdot\,,\,\cdot\,\rangle = \langle\,\cdot\,,\,\cdot\,\rangle_{(1,1,1)}$ is the standard metric on $\mathbb{S}^3$, we obtain

$$\mathrm{spec}(\mathscr{D}) = \Big\{ -\frac{3}{2} - n \Big\}_{n\geq 0} \cup \Big\{ \frac{1}{2} + n \Big\}_{n\geq 1} = \Big\{ \pm\Big(\frac{3}{2} + \ell\Big) \Big\}_{\ell\geq 0}$$

by plugging $T = 1$ into Bär's formula (1); that is, we have the smallest eigenvalue $\mp\frac{3}{2}$ contributed by the subspaces V_0 (to the first set) and V_1 (to the second set), respectively. By viewing the general case $\langle\,\cdot\,,\,\cdot\,\rangle = \langle\,\cdot\,,\,\cdot\,\rangle_{(a,b,c)}$ as a "perturbation" of the special case $\langle\,\cdot\,,\,\cdot\,\rangle = \langle\,\cdot\,,\,\cdot\,\rangle_{(1,1,1)}$, we hope that the inspection of V_0 and V_1 also yields a promising candidate λ_* for the smallest eigenvalue $\lambda_{*(a,b,c)}$ in general.

Now, suppose we have found such a candidate λ_* by inspecting the eigenvalues of $\mathcal{D}_0$ and $\mathcal{D}_1$ (namely $\lambda_* = a+b+c-\frac{1}{2}(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b})$, as it soon turns out). Then the real work is to prove that λ_* has in fact the smallest absolute value among *all* eigenvalues of *all* $\mathcal{D}_n$ (where $n \in \mathbb{N}_0$). Since there are no explicit formulas, we have to resort to estimates. The idea is the same as in the paper of Lauret [2], who made use of the Gershgorin-Circle Theorem.

Theorem 4.1 *(Gershgorin-Circle Theorem)*

Each eigenvalue of a square matrix $M = (M_{k,\ell}) \in \mathrm{Mat}_{\mathbb{C}}(n)$ lies in at least one of the discs centered at $M_{k,k}$ with radius $\sum_{\ell\neq k} |M_{k,\ell}|$ (defined for each $k \in \{1,2,\ldots,n\}$), which we will refer to as the k-th **Gershgorin disc** of M.

Proof. This is a classical result, and the proof is elementary. ♥

Unfortunately, in contrast to the Laplace spectrum, the Dirac spectrum consists of both positive and negative eigenvalues; and on top of that, we need not have symmetry about zero, since the dimension of our manifold is of the class 3 (mod 4) (see Proposition 2.23). As a consequence, it does not look hopeful (and, in fact, turns out to be useless) to apply the theorem to $\mathcal{D}_n$. Instead, we will be looking at the squared matrices $\mathcal{D}_n^2$, whose eigenvalues are guaranteed to be non-negative (and non-zero under the assumption $\mathrm{scal}_{(a,b,c)} > 0$, see Theorem 2.25), at the cost of more complicated expressions.

Notation 4.2 *(Gershgorin Expressions)*

Fix $a = 1$ and (b,c) with $1 = a \geq b \geq c > 0$ and $\mathrm{scal}_{(1,b,c)} > 0$. Let $n \in \mathbb{N}_0$ and $k \in \{0,1,\ldots,n\}$. Then we write

$$\Delta_{\mathcal{A}}(n,k) := (\text{left boundary point of } k\text{-th Gershgorin disc of } \mathcal{A}_n^2) - \lambda_*^2 \quad \text{and}$$
$$\Delta_{\mathcal{B}}(n,k) := (\text{left boundary point of } k\text{-th Gershgorin disc of } \mathcal{B}_n^2) - \lambda_*^2,$$

where $\mathcal{D}_n^2 = \mathrm{diag}(\mathcal{A}_n^2, \mathcal{B}_n^2)$ and λ_* is our promising candidate. Again, the indices of $\mathcal{A}_n^2, \mathcal{B}_n^2 \in \mathrm{Mat}_{\mathbb{C}}(n+1)$ are counted from zero.

In order to prove that λ_*^2 is the smallest of all eigenvalues of all $\mathcal{D}_n^2$ (where $n \in \mathbb{N}_0$), we would have to show for all fixed (b, c) in the parameter space that $\Delta_{\mathcal{A}}(n, k) \geq 0$ and $\Delta_{\mathcal{B}}(n, k) \geq 0$ hold for all pairs (n, k) with $n \in \mathbb{N}_0$ and $k \in \{0, 1, \ldots, n\}$. The only problem is that this claim is not true, as a numerical experiment quickly reveals:

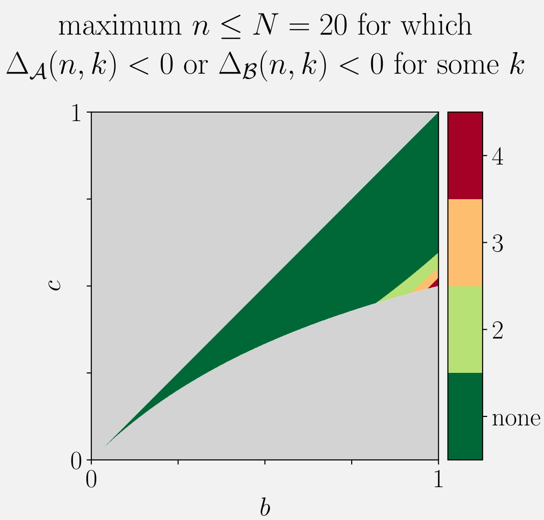


Figure 1: For each point (b, c) in the parameter space $\{(b, c) : 1 \geq b \geq c > 0,\ \mathrm{scal}_{(1,b,c)} > 0\}$ (after discretisation), it was checked numerically for which values of $n \leq N = 20$ there exists some $k \in \{0, 1, \ldots, n\}$ such that one of the estimates $\Delta_{\mathcal{A}}(n, k) \geq 0$ or $\Delta_{\mathcal{B}}(n, k) \geq 0$ fails to hold.

The plot shows for each (b, c) the maximum value of all n that failed the test. "None" means that the estimates held for all $n \leq N$.

We see that as (b, c) approaches the point $(1, \frac{1}{2})$, the maximum value rises from $n = 2$ up to $n = 4$. We will have to come up with something other than Gershgorin for these small values of n.

The plot in Figure 1 also suggests that our parameter space lies above the line $b \mapsto \frac{1}{2}b$. In fact, the set of points of zero scalar curvature (in the region $1 \geq b \geq c > 0$) turns out to be the graph of a quite harmless function, as stated in the lemma below.

Lemma 4.3 *(Parameter Space Contained in Triangle)*

The set $\{(b, c) : 1 \geq b \geq c > 0,\ \mathrm{scal}_{(1,b,c)} = 0\}$ is the graph of the function

$$c : (0, 1] \ni b \mapsto \frac{b}{1 + b} \in (0, \tfrac{1}{2}],$$

and the parameter space $\{(b, c) : 1 \geq b \geq c > 0,\ \mathrm{scal}_{(1,b,c)} > 0\}$ is contained in the triangle $\{(b, c) : 1 \geq b \geq c > \frac{1}{2}b > 0\}$. In particular, we have $c > \frac{1}{2}b$.

Proof. If $a = 1$, we have

$$\mathrm{scal}_{(1,b,c)} = 4(1 + b^2 + c^2) - 2\Big(\frac{b^2}{c^2} + b^2c^2 + \frac{c^2}{b^2}\Big)$$

from Proposition 1.5. We factorise the expression

$$-\tfrac{1}{2}b^2c^2\, \mathrm{scal}_{(1,b,c)} = -2b^2c^2 - 2b^4c^2 - 2b^2c^4 + b^4 + b^4c^4 + c^4$$

$\vdots$

$$\begin{aligned}
&= b^4(1-2c^2+c^4) - 2b^2c^2(1+c^2) + c^4 \\
&= b^4(1-c)^2(1+c)^2 - b^2c^2\big((1-c)^2+(1+c)^2\big) + c^4 \\
&= \left(b^2(1-c)^2 - c^2\right)\left(b^2(1+c)^2 - c^2\right).
\end{aligned}$$

The second factor is positive, since $b^2(1+c)^2 - c^2 > b^2 - c^2 \geq 0$ holds for all (b,c) with $b \geq c > 0$. It follows that

$$\begin{aligned}
\mathrm{scal}_{(1,b,c)} \geq 0 &\iff b^2(1-c)^2 - c^2 \leq 0 \\
&\iff b^2(1-c)^2 \leq c^2 \\
&\iff b - bc = b(1-c) \leq c \\
&\iff b \leq (1+b)c
\end{aligned}$$

holds for all (b,c) with $1 \geq b \geq c > 0$. The equality $\mathrm{scal}_{(1,b,c)} = 0$ is true if and only if $b = (1+b)c$; that is, the set $\{(b,c) : 1 \geq b \geq c > 0,\ \mathrm{scal}_{(1,b,c)} = 0\}$ is the graph of the function

$$c : (0,1] \ni b \mapsto \frac{b}{1+b} \in (0, \tfrac{1}{2}].$$

On the other hand, $\mathrm{scal}_{(1,b,c)} > 0$ implies $c > \dfrac{b}{1+b} \geq \dfrac{b}{1+1} = \dfrac{b}{2}$. ♥

Having secured the useful estimate $c > \frac{1}{2}b$ in our loot bag, let us return to the numerical experiment (see again Figure 1). We conclude that we should adapt our proof strategy as follows:

- For all $n \leq 4$, we should try to prove by case distinction that our candidate λ_* satisfies $|\lambda_*| \leq |\lambda|$ for all eigenvalues λ of $\mathcal{D}_n$.
- For all $n \geq 5$, we should try to prove that $\Delta_{\mathcal{A}}(n,k) \geq 0$ and $\Delta_{\mathcal{B}}(n,k) \geq 0$ hold for all $k \in \{0,1,\dots,n\}$. For this, we will have to calculate $\mathcal{D}_n^2 = \mathrm{diag}(\mathcal{A}_n^2, \mathcal{B}_n^2)$.

This justifies the structure given at the beginning of this paragraph. However, before we embark on this adventure, it would be nice to gather a few bounds on $C = \frac{1}{2}(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b})$, since both subproofs involve a lot of estimates, and it would be more than impractical to write out the definition of C every time.

Recall once again from Proposition 1.5 the formula for the scalar curvature of $\mathbb{S}^3_{(a,b,c)}$, which we can rewrite in terms of a, b, c, and C as

$$\mathrm{scal}_{(a,b,c)} = 4(a^2+b^2+c^2) - 2\Big(\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2}\Big) = 8(a^2+b^2+c^2-C^2). \tag{2}$$

Then we have the following useful estimates:

Lemma 4.4 *(Lower and Upper Bounds on C)*

We have the following lower bounds on C in terms of a, b, and c:

1. $C > \max\{a, b, c\}$ *(there is a stronger estimate that we will not use)*[2]
2. $2C \geq a + b + c$, "=" if and only if $a = b = c$

Under the additional assumption $\mathrm{scal}_{(a,b,c)} > 0$, we also have an upper bound:

3. $C^2 < a^2 + b^2 + c^2$ (in particular, $C < a + b + c$)

Proof.

1. Since $x + \frac{1}{x} \geq 2$ holds for all $x > 0$, we have

$$C = \frac{1}{2}\Big(\Big(\frac{b}{c} + \frac{c}{b}\Big)a + \frac{bc}{a}\Big) > \frac{1}{2}(2a + 0) = a. \quad \textit{(The same holds for } b \textit{ and } c.\textit{)}$$

2. Apply the AM–GM inequality to

$$\begin{aligned}(ab)^2 + (bc)^2 + (ca)^2 &= \frac{(ab)^2 + (bc)^2}{2} + \frac{(bc)^2 + (ca)^2}{2} + \frac{(ca)^2 + (ab)^2}{2} \\ &\geq ab \cdot bc + bc \cdot ca + ca \cdot ab \\ &= abc(a + b + c).\end{aligned}$$

Divide by abc to obtain $2C \geq a + b + c$. Equality occurs if and only if $ab = bc = ca$, that is, if $a = b = c$.

3. This follows immediately from $a^2 + b^2 + c^2 - C^2 = \frac{1}{8}\,\mathrm{scal}_{(a,b,c)} > 0$, see (2).

♥

With this, we are well-equipped for the proof of the Main Theorem. Recall that for the rest of this thesis, we will **always** assume

$$\mathrm{scal}_{(a,b,c)} > 0, \quad \textbf{(!)}$$

[2] We even have the stronger estimate $C^2 \geq \max\{a^2 + b^2, b^2 + c^2, c^2 + a^2\}$. This can be seen by comparing C^2 with $\tilde{C}^2$, where $\tilde{C} := \frac{1}{2}(-\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b})$. Then $C^2 - (a^2 + b^2) = \tilde{C}^2 \geq 0$ (plus the analogous statements). We will not use this estimate, but it might still be worth mentioning.

as stated in the Main Theorem.

Low-Dimensional Subspaces

As promised, let us first have a look at the eigenvalues of $\mathcal{D}_n$ for $n \leq 4$. Recall that it is our aim to obtain a candidate λ_* by inspecting $\mathcal{D}_0$ and $\mathcal{D}_1$ and to prove that λ_* has the smallest absolute value among all eigenvalues contributed up to $\mathcal{D}_4$.

For purely aesthetic reasons, we will *not* use the assumption $1 = a \geq b \geq c$ in this subproof, but only assume $a, b, c > 0$ instead (and of course $\mathrm{scal}_{(a,b,c)} > 0$).

- **Case $\boldsymbol{n = 0}$:** We have the 1×1-blocks $\mathcal{A}'_0 = \mathcal{B}'_0 = (0)$; that is, $\mathcal{D}_0 = \mathcal{D}'_0 - C$ contributes two times the eigenvalue $-C$.
- **Case $\boldsymbol{n = 1}$:** The 2×2-blocks are given by

$$\mathcal{A}'_1 = \begin{pmatrix} a & c+b \\ c+b & a \end{pmatrix} \quad \text{and} \quad \mathcal{B}'_1 = \begin{pmatrix} -a & c-b \\ c-b & -a \end{pmatrix};$$

  that is, $\mathcal{D}_1 = \mathcal{D}'_1 - C$ contributes the four eigenvalues

$$a+b+c-C, \; a-b-c-C \quad \text{and} \quad -a+b-c-C, \quad -a-b+c-C.$$

  Note that we have the distinguished element $a+b+c-C$ due to its symmetry in a, b, and c (in contrast to the other values). *(How suspicious...)*
- **Case $\boldsymbol{n = 2}$:** Sadly, the joy of having easy-to-calculate eigenvalues ends here, since the characteristic polynomials of the 3×3-blocks

$$\mathcal{A}'_2 = \begin{pmatrix} 2a & 2(c+b) & 0 \\ c+b & 0 & c-b \\ 0 & 2(c-b) & -2a \end{pmatrix} \quad \text{and} \quad \mathcal{B}'_2 = \begin{pmatrix} -2a & 2(c-b) & 0 \\ c-b & 0 & c+b \\ 0 & 2(c+b) & 2a \end{pmatrix}$$

  are cubic and their roots look suitably awful. (Indeed, by feeding the matrices to a computer algebra system, one *does* obtain symbolic expressions for their eigenvalues, but these are anything but human-readable.)
- **Case $\boldsymbol{n = 3}$:** Interestingly, in contrast to the case $n = 2$, the 4×4-blocks

$$\mathcal{A}'_3 = \begin{pmatrix} 3a & 3(c+b) & 0 & 0 \\ c+b & -a & 2(c-b) & 0 \\ 0 & 2(c-b) & -a & c+b \\ 0 & 0 & 3(c+b) & 3a \end{pmatrix} \quad \text{and} \quad \mathcal{B}'_3 = \begin{pmatrix} -3a & 3(c-b) & 0 & 0 \\ c-b & a & 2(c+b) & 0 \\ 0 & 2(c+b) & a & c-b \\ 0 & 0 & 3(c-b) & -3a \end{pmatrix}$$

do yield human-readable eigenvalues, which will be listed as soon as they are needed (see the proof of Lemma 4.7).

- **Case $n = 4$:** It does not seem to be possible at all to have explicit expressions for the eigenvalues of the 5×5-blocks

$$\mathcal{A}'_4 = \begin{pmatrix} 4a & 4(c+b) & 0 & 0 & 0 \\ c+b & -2a & 3(c-b) & 0 & 0 \\ 0 & 2(c-b) & 0 & 2(c+b) & 0 \\ 0 & 0 & 3(c+b) & 2a & c-b \\ 0 & 0 & 0 & 4(c-b) & -4a \end{pmatrix} \text{ and } \mathcal{B}'_4 = \begin{pmatrix} -4a & 4(c-b) & 0 & 0 & 0 \\ c-b & 2a & 3(c+b) & 0 & 0 \\ 0 & 2(c+b) & 0 & 2(c-b) & 0 \\ 0 & 0 & 3(c-b) & -2a & c+b \\ 0 & 0 & 0 & 4(c+b) & 4a \end{pmatrix}.$$

  This is because the characteristic polynomials are of degree five, and quintic equations need not be solvable.

Now let us start with the inspection of $\mathcal{D}_0$ and $\mathcal{D}_1$.

Lemma 4.5 *(Dirac Eigenvalues — Cases $n = 0$ and $n = 1$)*

Out of the easy-to-calculate eigenvalues of $\mathscr{D}$ contributed by $\mathcal{D}_0$ and $\mathcal{D}_1$, the number

$$\lambda_* := a + b + c - C > 0$$

has the smallest absolute value. This is strict (that is, there is no other eigenvalue λ with $|\lambda| = |\lambda_*|$) whenever $a \neq b$ or $b \neq c$ or $c \neq a$.

In the case $a = b = c$ (and only in this case), we have $-C = -\lambda_*$, which then of course has the same absolute value as λ_*.

Proof. The assertion $\lambda_* > 0$ follows from Lemma 4.4 (3) and $\mathrm{scal}_{(a,b,c)} > 0$.

- $\lambda_* \leq |-C|$: This simplifies to $a+b+c-C \leq C$, which holds by Lemma 4.4 (2). Equality occurs if and only if $a = b = c$.
- $\lambda_* < |a-b-c-C|$: It suffices to show that $a-b-c-C < -\lambda_* < 0$. However, this is equivalent to $2a < 2C$, which holds by Lemma 4.4 (1).
- The claims $\lambda_* < |-a-b+c-C|$ and $\lambda_* < |-a+b-c-C|$ of course follow the same reasoning.

♥

We now have a promising candidate λ_* for the Dirac eigenvalue of the smallest absolute value under the assumption $\mathrm{scal}_{(a,b,c)} > 0$. Let us see whether λ_* also prevails in the cases $n \in \{2, 3, 4\}$.

Lemma 4.6 *(Dirac Eigenvalues — Case $n = 2$)*

For all eigenvalues λ of $\mathcal{D}_2 = \mathcal{D}_2' - C$, we have $\lambda_* < |\lambda|$.

Since we do not have human-readable expressions for the eigenvalues of $\mathcal{D}_2$, we have to resort to an argument based on the characteristic polynomial.

Proof. We want to show that no eigenvalue of $\mathcal{D}_2$ lies in the interval $[-\lambda_*, \lambda_*]$; or, equivalently, that $\mathcal{D}_2' = \mathcal{D}_2 + C$ has no eigenvalues in $[2C - a - b - c, a + b + c]$. The latter interval is contained in $[0, a + b + c]$ due to Lemma 4.4 (2).

The characteristic polynomials of $\mathcal{A}_2'$ and $\mathcal{B}_2'$ are both given by

$$\chi_2(x) \coloneqq \chi_{\mathcal{A}_2'}(x) = \chi_{\mathcal{B}_2'}(x) = x^3 - 4(a^2 + b^2 + c^2)x - 16abc.$$

Since its second derivative $\chi_2''(x) = 6x$ is non-negative for all $x \geq 0$, it follows that χ_2 is convex on $[0, a + b + c]$. Furthermore, at the boundary points, we have $\chi_2(0) = -16abc < 0$ and that $\chi_2(a + b + c)$ expands to some non-trivial polynomial in a, b, and c with coefficients in $-\mathbb{N}_0$ (hence, $\chi_2(a + b + c) < 0$).

Convexity implies $\chi_2(x) < 0$ for all $x \in [0, a + b + c]$; in particular, χ_2 has no roots in this interval, from which the claim follows. ♥

Lemma 4.7 *(Dirac Eigenvalues — Case $n = 3$)*

For all eigenvalues λ of $\mathcal{D}_3 = \mathcal{D}_3' - C$, we have $\lambda_* < |\lambda|$.

Since we *do* have rather simple eigenvalues in the case $n = 3$ (see below), we can argue directly by comparison.

Eigenvalues of $\mathcal{A}_3 = \mathcal{A}_3' - C$:

(1) $a+b-c-2\sqrt{a^2 + b^2 + c^2 - ab + bc + ca}-C$

(2) $a+b-c+2\sqrt{a^2 + b^2 + c^2 - ab + bc + ca}-C$

(3) $a-b+c-2\sqrt{a^2 + b^2 + c^2 + ab + bc - ca}-C$

(4) $a-b+c+2\sqrt{a^2 + b^2 + c^2 + ab + bc - ca}-C$

Eigenvalues of $\mathcal{B}_3 = \mathcal{B}_3' - C$:

(5) $-a-b-c-2\sqrt{a^2 + b^2 + c^2 - ab - bc - ca}-C$

(6) $-a-b-c+2\sqrt{a^2 + b^2 + c^2 - ab - bc - ca}-C$

(7) $-a+b+c-2\sqrt{a^2 + b^2 + c^2 + ab - bc + ca}-C$

(8) $-a+b+c+2\sqrt{a^2 + b^2 + c^2 + ab - bc + ca}-C$

The proof only makes use of Lemma 4.4 (1).

Proof. Again, we want to show that these eigenvalues do *not* lie in $[-\lambda_*, \lambda_*]$.

(1) Let us prove that this expression is less than $-\lambda_* = C - a - b - c$. This is

⋮

equivalent to $C - a - b + \sqrt{a^2 + b^2 + c^2 - ab + bc + ca} > 0$. Rewrite the term under the square root as

$$T \coloneqq \frac{1}{2}\big((a-b)^2 + (b+c)^2 + (c+a)^2\big) > \frac{1}{2}(0 + b^2 + a^2) \geq \min\{a^2, b^2\}.$$

It follows that $C - a - b + \sqrt{T} > C - a - b + (a \text{ or } b) > 0$ (due to Lemma 4.4 1).

2 We prove in a similar fashion that this is greater than $\lambda_* = a + b + c - C$. This is equivalent to $\sqrt{T} - c > 0$, which is true, since $T > \frac{1}{2}(0 + c^2 + c^2) = c^2$.

3 This is of the same type as 1.

4 This is of the same type as 2.

5 This is less than 6, which we will prove to be less than $-\lambda_*$.

6 This is less than $-\lambda_*$ if and only if $\sqrt{a^2 + b^2 + c^2 - ab - bc - ca} < C$, which is equivalent to $a^2 + b^2 + c^2 - ab - bc - ca < C^2$. Assume for a moment that $a \geq b \geq c$. Then the left-hand side is less than or equal to $a^2 + b^2 + c^2 - b^2 - c^2 - c^2 = a^2 - c^2 < a^2 < C^2$ (due to Lemma 4.4 1). The same holds for the other permutations due to the symmetry in a, b, and c.

7 This is of the same type as 1.

8 This is of the same type as 2.

♥

Lemma 4.8 *(Dirac Eigenvalues — Case $n = 4$)*

For all eigenvalues λ of $\mathcal{D}_4 = \mathcal{D}_4' - C$, we have $\lambda_* < |\lambda|$.

The proof is very similar to the case $n = 2$ (the argument is applied twice).

Proof. Just like in the case $n = 2$, we will even show that no eigenvalue of $\mathcal{D}_4'$ lies in the interval $[0, a + b + c]$. The characteristic polynomials of $\mathcal{A}_4'$ and $\mathcal{B}_4'$ are again identical and given by $\chi_4 \coloneqq \chi_{\mathcal{A}_4'} = \chi_{\mathcal{B}_4'}$, where

$$\begin{aligned}\chi_4(x) = x^5 &- 20(a^2 + b^2 + c^2)x^3 - 80abc\,x^2 \\ &+ 64(a^4 + b^4 + c^4 + 4a^2b^2 + 4b^2c^2 + 4c^2a^2)x + 768(a^3bc + ab^3c + abc^3).\end{aligned}$$

$\vdots$

We compute its second derivative

$$\chi_4''(x) = 20x^3 - 120(a^2 + b^2 + c^2)x - 160abc.$$

Then we have $\chi_4''(x) < 0$ for all $x \in [0, a + b + c]$ with the same reasoning as for $n = 2$. (Let $f \coloneqq \chi_4''$. Then $f(0) = -160abc < 0$, $f(a + b + c)$ expands to a bunch of negative terms, and $f''(x) = 120x$ implies convexity on $[0, a + b + c]$.) It follows that χ_4 is concave on $[0, a + b + c]$.

Since $\chi_4(0) > 0$, and $\chi_4(a+b+c)$ expands to a non-trivial polynomial in a, b, and c with coefficients in $\mathbb{N}_0$, concavity implies that $\chi_4(x) > 0$ holds for all $x \in [0, a+b+c]$. In particular, $\mathcal{D}_4'$ has no eigenvalues in this interval. ♥

By combining these results, we eventually obtain the first piece of our Main Theorem.

Fragment 1 / 3 of Main Theorem *(Completion of Subproof for $n \leq 4$)*

The positive number $\lambda_* = a + b + c - C$ has the smallest absolute value among all eigenvalues of $\mathcal{D}_0$, $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, and $\mathcal{D}_4$.

Proof. This is a summary of the previous lemmas. ♥

This concludes the subproof dealing with the cases $n \leq 4$, leaving us with the more complicated part $n \geq 5$.

By the way: The fortunate coincidence that $\chi_n(a + b + c)$ expands to a polynomial in a, b, and c with coefficients of uniform signs for $n \in \{2, 4\}$ does *not* remain true for greater values of n. (And even if it did, one still would have to come up with something else for the odd values of n.)

Dirac Squares and Centrosymmetry

This part of the paragraph is meant as a brief preparation for the continuation of the proof (cases $n \geq 5$), in which we will apply the Gershgorin-Circle Theorem to $\mathcal{D}_n^2$. In order to do this, we first have to know what the rows of $\mathcal{D}_n^2$ look like.

As $\mathcal{D}_n$ consists of two tridiagonal blocks, $\mathcal{D}_n^2$ will consist of two pentadiagonal blocks. While this might sound a little unpleasant, we must engage in this effort, since otherwise there would be nothing to be gained from the Gershgorin-Circle Theorem. *(Consider it a little blood sacrifice to Gershgorin.)*

Theorem 4.9 *(Squared Dirac Matrices)*

The blocks $\mathcal{A}_n^2 = (\mathcal{A}_n' - C)^2$ and $\mathcal{B}_n^2 = (\mathcal{B}_n' - C)^2$ of the squared Dirac matrix $\mathcal{D}_n^2 = \mathrm{diag}(\mathcal{A}_n^2, \mathcal{B}_n^2)$ consist of the following rows:

If $k \in \{0, 1, \ldots, n\}$ is even:

$$\begin{aligned}
\mathcal{A}_n^2[k, k-2] &= (c-b)(c+b)k(k-1) \\
\mathcal{A}_n^2[k, k-1] &= -2(c-b)(C+a)k \\
\mathcal{A}_n^2[k, k] &= (c-b)^2 k(n-k+1) + (a(n-2k) - C)^2 + (c+b)^2(n-k)(k+1) \\
\mathcal{A}_n^2[k, k+1] &= -2(c+b)(C-a)(n-k) \\
\mathcal{A}_n^2[k, k+2] &= (c+b)(c-b)(n-k)(n-k-1)
\end{aligned}$$

If $k \in \{0, 1, \ldots, n\}$ is odd:

$$\begin{aligned}
\mathcal{A}_n^2[k, k-2] &= (c+b)(c-b)k(k-1) \\
\mathcal{A}_n^2[k, k-1] &= -2(c+b)(C-a)k \\
\mathcal{A}_n^2[k, k] &= (c+b)^2 k(n-k+1) + (a(n-2k) + C)^2 + (c-b)^2(n-k)(k+1) \\
\mathcal{A}_n^2[k, k+1] &= -2(c-b)(C+a)(n-k) \\
\mathcal{A}_n^2[k, k+2] &= (c-b)(c+b)(n-k)(n-k-1)
\end{aligned}$$

The entries of $\mathcal{B}_n^2$ are obtained by swapping "k even" and "k odd".

Proof. This is an elementary computation involving the entries of $\mathcal{A}_n'$ and $\mathcal{B}_n'$ from Theorem 3.12. We know from linear algebra that the entry of M^2 at the position (k, ℓ) is the product of the k-th row and the ℓ-th column of M.

Let $k \in \{0, 1, \ldots, n\}$ be even. Then $k \pm 1$ is odd, and the k-th row of $\mathcal{A}_n = \mathcal{A}_n' - C$ (with all its leading and trailing zeros removed) is

$$\begin{pmatrix} (c-b)k & a(n-2k) - C & (c+b)(n-k) \end{pmatrix}.$$

The relevant parts of the $(k-2)$-th, …, $(k+2)$-th column of $\mathcal{A}_n$ (which have to be multiplied with the k-th row of $\mathcal{A}_n$) are

$$\begin{pmatrix} (c+b)(k-1) & -a(n-2(k-1)) - C & (c-b)(n-k+1) & 0 & 0 \\ 0 & (c-b)k & a(n-2k) - C & (c+b)(n-k) & 0 \\ 0 & 0 & (c+b)(k+1) & -a(n-2(k+1)) - C & (c-b)(n-k-1) \end{pmatrix}.$$

$\vdots$

Similarly, if k is odd, then $k \pm 1$ is even, and the k-th row of $\mathcal{D}_n$ is

$$\begin{pmatrix}(c+b)k & -a(n-2k)-C & (c-b)(n-k)\end{pmatrix}.$$

Analogously, we have the columns

$$\begin{pmatrix}(c-b)(k-1) & a(n-2(k-1))-C & (c+b)(n-k+1) & 0 & 0 \\ 0 & (c+b)k & -a(n-2k)-C & (c-b)(n-k) & 0 \\ 0 & 0 & (c-b)(k+1) & a(n-2(k+1))-C & (c+b)(n-k-1)\end{pmatrix}.$$

The claimed formulas follow immediately. The expressions for $\mathcal{B}_n^2$ are of course the same, since $\mathcal{B}_n'$ was obtained from $\mathcal{A}_n'$ this way.

The formulas also hold at the "borders" of $\mathcal{A}_n^2$ and $\mathcal{B}_n^2$ (that is, for columns at the index $k \in \{0, 1, n-1, n\}$), since the terms in the calculation that correspond to non-existing entries all vanish. ♥

With this, we can now write down explicit expressions for $\Delta_{\mathcal{A}}(n,k)$ and $\Delta_{\mathcal{B}}(n,k)$ (recall Notation 4.2), for which we would like to show that they are non-negative for all $n \geq 5$ and $k \in \{0, 1, \ldots, n\}$. However, before doing so, we should discuss the symmetry of the matrices, which will make our work a little easier.

Definition 4.10 *(Centrosymmetric Matrix)*

An $n \times n$-matrix M is called **centrosymmetric** if it is symmetric about its centre; that is, if $M_{k,\ell} = M_{n-k+1,n-\ell+1}$ holds for all $k, \ell \in \{1, 2, \ldots, n\}$.

Equivalently, M is centrosymmetric if it satisfies the condition $MJ = JM$, where J denotes the exchange matrix (with entries 1 on the antidiagonal and 0 everywhere else). It follows that the square of a centrosymmetric matrix is again centrosymmetric, since one has $M^2J = MJM = JM^2$.

Usually, this kind of symmetry is not *too* interesting a property that a matrix can have, but it would be helpful for our purposes. Indeed, if one applies the Gershgorin-Circle Theorem to a centrosymmetric matrix, then one only has to look at half of the rows (rounded up), since the rows at the positions k and $n-k+1$ have the same entries in reverse order.

It is a truly pleasant coincidence that our Dirac matrices exhibit exactly this kind of symmetry (as a whole and block-wise, respectively), depending on the parity of n.

Proposition 4.11 *(Centrosymmetry of Dirac Matrices)*

If n is even, then $\mathcal{D}'_n$ is centrosymmetric; that is, $\mathcal{A}'_n[\,\cdot\,,\,\cdot\,] = \mathcal{B}'_n[n-\,\cdot\,, n-\,\cdot\,]$.

If, on the other hand, n is odd, then both $\mathcal{A}'_n$ and $\mathcal{B}'_n$ are centrosymmetric instead; that is, $\mathcal{A}'_n[\,\cdot\,,\,\cdot\,] = \mathcal{A}'_n[n-\,\cdot\,, n-\,\cdot\,]$ and $\mathcal{B}'_n[\,\cdot\,,\,\cdot\,] = \mathcal{B}'_n[n-\,\cdot\,, n-\,\cdot\,]$. *(Recall once more that the indices of $\mathcal{A}'_n$ and $\mathcal{B}'_n$ are counted from zero.)*

Proof. This also follows immediately from Theorem 3.12, using basic arithmetic.

Let $k \in \{0, 1, \ldots, n\}$ be even. Then we have

$$\mathcal{A}'_n[\,\cdot\,, k] = \begin{pmatrix} \vdots \\ 0 \\ (c-b)(n-k+1) \\ a(n-2k) \\ (c+b)(k+1) \\ 0 \\ \vdots \end{pmatrix} \quad \text{and} \quad \mathcal{B}'_n[\,\cdot\,, k] = \begin{pmatrix} \vdots \\ 0 \\ (c+b)(n-k+1) \\ -a(n-2k) \\ (c-b)(k+1) \\ 0 \\ \vdots \end{pmatrix}.$$

for the k-th column of $\mathcal{A}'_n$ and $\mathcal{B}'_n$, respectively. If k is odd instead, the right-hand sides have to be swapped.

Now one easily checks that the following statements hold:

- If n is even and k is even (or odd), then $n-k$ is even (or odd, respectively), and the column $\mathcal{A}'_n[\,\cdot\,, k]$ is the column $\mathcal{B}'_n[\,\cdot\,, n-k]$ in reverse order.
- If n is odd and k is even (or odd), then $n-k$ is odd (or even, respectively), and the columns $\mathcal{A}'_n[\,\cdot\,, k] \,/\, \mathcal{B}'_n[\,\cdot\,, k]$ are the columns $\mathcal{A}'_n[\,\cdot\,, n-k] \,/\, \mathcal{B}'_n[\,\cdot\,, n-k]$ in reverse order.

♥

As mentioned before, $\mathcal{D}_n^2 = (\mathcal{D}'_n - C)^2$ inherits the symmetry (depending on the parity of n) from $\mathcal{D}'_n$. Therefore, if we apply the Gershgorin-Circle Theorem to $\mathcal{D}_n^2$, it suffices to consider $\mathcal{A}_n^2$ (if n is even) and “half” of each $\mathcal{A}_n^2$ and $\mathcal{B}_n^2$ (if n is odd).

We will now advance to the adventurous, exciting, and mind-blowing proof dealing with the cases $n \geq 5$. Together with the already proven cases $n \leq 4$, this will show that λ_* has indeed the smallest absolute value among all eigenvalues of $\mathcal{D}_n$ for *all* $n \in \mathbb{N}_0$.

From now on, assume $1 = a \geq b \geq c > 0$ (recall from page 46 why this is possible) and of course $\mathrm{scal}_{(1,b,c)} > 0$. Taking into account the centrosymmetry of $\mathcal{D}_n^2$ (if n is even) or $\mathcal{A}_n^2$ and $\mathcal{B}_n^2$ (if n is odd), respectively, we would like to prove the statements

- **A0** $\Delta_{\mathcal{A}}(n,k) \geq 0$ for all even $n \geq 6$ and $k \in \{0, 1, \dots, n\}$,
- **A1** $\Delta_{\mathcal{A}}(n,k) \geq 0$ for all odd $n \geq 5$ and $k \in \{0, 1, \dots, \frac{1}{2}(n-1)\}$, and
- **B1** $\Delta_{\mathcal{B}}(n,k) \geq 0$ for all odd $n \geq 5$ and $k \in \{0, 1, \dots, \frac{1}{2}(n-1)\}$

(recall once again the definition of $\Delta_{\mathcal{A}}$ and $\Delta_{\mathcal{B}}$ from Notation 4.2).

Since this is an infinite collection of rather complicated algebraic statements (depending on the four variables n, k, b, and c), we hope to find an inductive argument in (n,k) that holds for all (b,c) in the parameter space. It is almost miraculous that this hope will indeed prevail and that the assumption $\mathrm{scal}_{(1,b,c)} > 0$ will turn out to be *necessary* for our proof to work.

First of all, we have to calculate the expressions $\Delta_{\mathcal{A}}(n,k)$ and $\Delta_{\mathcal{B}}(n,k)$, using the result of the last part of this paragraph, in which we calculated $\mathcal{A}_n^2$ and $\mathcal{B}_n^2$.

Corollary 4.12 *(Gershgorin Expressions, from Theorem 4.9)*

For all $n \in \mathbb{N}_0$ and $k \in \{0, 1, \dots, n\}$, we have

$$\Delta_{\mathcal{A}}(n,k) = \begin{cases} \Delta_0(n,k) & \textit{(k even)} \\ \Delta_1(n,k) & \textit{(k odd)} \end{cases} \quad \text{and} \quad \Delta_{\mathcal{B}}(n,k) = \begin{cases} \Delta_1(n,k) & \textit{(k even)} \\ \Delta_0(n,k) & \textit{(k odd)} \end{cases},$$

where

$$\begin{aligned} \Delta_0(n,k) = {} & ((n-2k)-C)^2 + (b-c)^2 k(n-k+1) + (b+c)^2 (n-k)(k+1) \\ & - 2(b-c)(C+1)k - 2(b+c)(C-1)(n-k) \\ & - (b-c)(b+c)k(k-1) - (b+c)(b-c)(n-k)(n-k-1) \\ & - (1+b+c-C)^2, \quad \text{and} \end{aligned}$$

$$\begin{aligned} \Delta_1(n,k) = {} & ((n-2k)+C)^2 + (b+c)^2 k(n-k+1) + (b-c)^2 (n-k)(k+1) \\ & - 2(b+c)(C-1)k - 2(b-c)(C+1)(n-k) \\ & - (b+c)(b-c)k(k-1) - (b-c)(b+c)(n-k)(n-k-1) \\ & - (1+b+c-C)^2. \end{aligned}$$

Furthermore, these satisfy $\Delta_0(n,k) = \Delta_1(n, n-k)$ for all (n,k).

Note that in the cases $k \in \{0, n\}$ (where we do not have the columns $k \pm 1$ and $k \pm 2$) and $k \in \{1, n-1\}$ (where we do not have the columns $k \pm 2$), the formulas still hold, since the corresponding terms vanish (namely the left/right term in the second and third line of Δ_0 and Δ_1 if $k \in \{0, n\}$ or just in the third line if $k \in \{1, n-1\}$). Otherwise, we would be subtracting too much in these cases; but, luckily, this is not happening.

> *Proof.* This follows from Theorem 4.9 by plugging $a = 1$ into
>
> $$\Delta_{\mathcal{A}}(n, k) = \mathcal{A}_n^2[k, k] - \sum_{i \in \{\pm 1, \pm 2\}} \left|\mathcal{A}_n^2[k, k+i]\right| - \lambda_*^2.$$
>
> Under the assumption $b \geq c$, we have the signs $(-, +, *, -, -)$ in the k-th row of $\mathcal{A}_n^2$ if k is even, and $(-, -, *, +, -)$ if k is odd (recall $C > a = 1$ from Lemma 4.4 1). Again, $\Delta_{\mathcal{B}}$ is obtained from $\Delta_{\mathcal{A}}$ by swapping "k even" and "k odd".
>
> The identity $\Delta_0(n, k) = \Delta_1(n, n-k)$ is immediate. ♥

The main idea of the proof (which, by the way, was found by accident) will be to write down the statements "$\Delta_i(n, k) \geq 0$" as two triangle formations (depending on the parity of n) for each $i \in \{0, 1\}$ and to derive the following inductive step, which explains the title "Triangle Induction":

If n is even:

$$\begin{array}{ccccccc}
 & & & \Delta_i(0,0) \geq 0 & & & \\
 & & \Delta_i(2,0) \geq 0 & \Delta_i(2,1) \geq 0 & \Delta_i(2,2) \geq 0 & & \\
 & \Delta_i(4,0) \geq 0 & \Delta_i(4,1) \geq 0 & \Delta_i(4,2) \geq 0 & \Delta_i(4,3) \geq 0 & \Delta_i(4,4) \geq 0 & \\
\Delta_i(6,0) \geq 0 & \Delta_i(6,1) \geq 0 & \Delta_i(6,2) \geq 0 & \Delta_i(6,3) \geq 0 & \Delta_i(6,4) \geq 0 & \Delta_i(6,5) \geq 0 & \Delta_i(6,6) \geq 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

If n is odd:

$$\begin{array}{cccccccc}
 & & & \Delta_i(1,0) \geq 0 & \Delta_i(1,1) \geq 0 & & & \\
 & & \Delta_i(3,0) \geq 0 & \Delta_i(3,1) \geq 0 & \Delta_i(3,2) \geq 0 & \Delta_i(3,3) \geq 0 & & \\
 & \Delta_i(5,0) \geq 0 & \Delta_i(5,1) \geq 0 & \Delta_i(5,2) \geq 0 & \Delta_i(5,3) \geq 0 & \Delta_i(5,4) \geq 0 & \Delta_i(5,5) \geq 0 & \\
\Delta_i(7,0) \geq 0 & \Delta_i(7,1) \geq 0 & \Delta_i(7,2) \geq 0 & \Delta_i(7,3) \geq 0 & \Delta_i(7,4) \geq 0 & \Delta_i(7,5) \geq 0 & \Delta_i(7,6) \geq 0 & \Delta_i(7,7) \geq 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Algebraically, this just means $\Delta_i(n+2,k+1) \geq \Delta_i(n,k)$ for all (n,k) and $i \in \{0,1\}$. Now let us assume for a moment that we have already proven this estimate, and recall the "symmetry" $\Delta_0(n,k) = \Delta_1(n,n-k)$ from Corollary 4.12. Then we are still in need of base cases for our proof by induction to be complete.

In order to know which base cases to prove, we need to know who exactly the culprits are, that is, the pairs (n,k) for which $\Delta_{\mathcal{A}}(n,k) \geq 0$ or $\Delta_{\mathcal{B}}(n,k) \geq 0$ fails to hold (recall from Figure 1 that for $n \in \{2,3,4\}$, there do exist such k). This shall be revealed by another numerical experiment, in which $\Delta_{\mathcal{A}}(n,k)$ and $\Delta_{\mathcal{B}}(n,k)$ are plotted as functions of (b,c) for a few (n,k) modulo centrosymmetry (see Figure 2).

We see that only the pairs $(2,0)$, $(3,0)$, and $(4,0)$ contributed by $\Delta_{\mathcal{A}}$ ($= \Delta_0$, since $k=0$ is even) do not show the desired behaviour, while Gershgorin seems to work for all other pairs (modulo centrosymmetry). We will prove $\Delta_{\mathcal{A}}(n,k) \geq 0$ and $\Delta_{\mathcal{B}}(n,k) \geq 0$ for all pairs except those three (modulo centrosymmetry), which then of course implies the weaker statement that this is the case for all $n \geq 5$ and $k \in \{0,1,\ldots,n\}$.

This eventually leads to the following induction strategy:

Lemma 4.13 *(Triangle Induction)*

Assume the inductive steps

1. $\Delta_0(n+2,k+1) \geq \Delta_0(n,k)$ for all (n,k) (yet to be proven), and
2. $\Delta_0(n,k) = \Delta_1(n,n-k)$ for all (n,k) from Corollary 4.12.

Then it suffices to prove the base cases

- $\Delta_0(n,0) \geq 0$ for all $n \in \mathbb{N}_0 \setminus \{2,3,4\}$,
- $\Delta_0(n,n) \geq 0$ for all $n \in \mathbb{N}_0$, and
- $\Delta_0(n,1) \geq 0$ for each $n \in \{4,5,6\}$

in order to show $\Delta_{\mathcal{A}}(n,k) \geq 0$ and $\Delta_{\mathcal{B}}(n,k) \geq 0$ for all (n,k) except $(2,0)$, $(3,0)$, and $(4,0)$ (modulo centrosymmetry).

Proof. This is obvious from the two triangle diagrams above.

Note that for all (n,k), the two statements "$\Delta_0(n+2,k+1) \geq \Delta_0(n,k)$" and "$\Delta_i(n+2,k+1) \geq \Delta_i(n,k)$ for each $i \in \{0,1\}$" are obviously equivalent, since

$$\Delta_1(n+2,k+1) = \Delta_0(n+2,n-k+1) \geq \Delta_0(n,n-k) = \Delta_1(n,k)$$

follows from $\Delta_0(n+2,k+1) \geq \Delta_0(n,k)$ by applying 2 twice. ♥

Plot Collection of $\Delta_{\mathcal{A}}(n,k)$ and $\Delta_{\mathcal{B}}(n,k)$

Figure 2: For each $n \in \{0,1,\dots,7\}$ and each $k \in \{0,1,\dots,n\}$ (if n is even) or $k \in \{0,1,\dots,\frac{1}{2}(n-1)\}$ (if n is odd), the expression $\Delta_{\mathcal{A}}(n,k)$ or $\Delta_{\mathcal{A}}(n,k)/\Delta_{\mathcal{B}}(n,k)$, respectively, is plotted as a function of (b,c) over the parameter space $\{(b,c) : 1 \geq b \geq c > 0,\ \mathrm{scal}_{(1,b,c)} > 0\}$.

The colour map is chosen such that the positive/negative/zero values of $\Delta_{\mathcal{A}}$ and $\Delta_{\mathcal{B}}$ are shown in green/red/yellow (normalised at the maximum and minimum value over all (n,k,b,c)).

Note that for $n \in \{2,3,4\}$, we have $\Delta_0(n,0) < 0$ in a neighbourhood of $(b,c) = (1,\frac{1}{2})$ (which one of course also could check by hand at the boundary point $(b,c) = (1,\frac{1}{2})$). For all other relevant pairs of (n,k), the Gershgorin estimate seems to work.

Additionally, it might be illuminating to consider the picture below, in which the base cases are shown coloured, together with each implied statement "$\Delta_i(n,k) \geq 0$" (using both (1) and (2)) in the same colour, but grouped according to our goals **A0**, **A1**, and **B1** from page 59.

A0 $\Delta_{\mathcal{A}}$, n even

					$\Delta_0(0,0)$					
				$\Delta_0(2,0)$	$\Delta_1(2,1)$	$\Delta_0(2,2)$				
			$\Delta_0(4,0)$	$\Delta_1(4,1)$	$\Delta_0(4,2)$	$\Delta_1(4,3)$	$\Delta_0(4,4)$			
		$\Delta_0(6,0)$	$\Delta_1(6,1)$	$\Delta_0(6,2)$	$\Delta_1(6,3)$	$\Delta_0(6,4)$	$\Delta_1(6,5)$	$\Delta_0(6,6)$		
	$\Delta_0(8,0)$	$\Delta_1(8,1)$	$\Delta_0(8,2)$	$\Delta_1(8,3)$	$\Delta_0(8,4)$	$\Delta_1(8,5)$	$\Delta_0(8,6)$	$\Delta_1(8,7)$	$\Delta_0(8,8)$	
$\Delta_0(10,0)$	$\Delta_1(10,1)$	$\Delta_0(10,2)$	$\Delta_1(10,3)$	$\Delta_0(10,4)$	$\Delta_1(10,5)$	$\Delta_0(10,6)$	$\Delta_1(10,7)$	$\Delta_0(10,8)$	$\Delta_1(10,9)$	$\Delta_0(10,10)$

A1 $\Delta_{\mathcal{A}}$, n odd

					$\Delta_0(1,0)$
				$\Delta_0(3,0)$	$\Delta_1(3,1)$
			$\Delta_0(5,0)$	$\Delta_1(5,1)$	$\Delta_0(5,2)$
		$\Delta_0(7,0)$	$\Delta_1(7,1)$	$\Delta_0(7,2)$	$\Delta_1(7,3)$
	$\Delta_0(9,0)$	$\Delta_1(9,1)$	$\Delta_0(9,2)$	$\Delta_1(9,3)$	$\Delta_0(9,4)$
$\Delta_0(11,0)$	$\Delta_1(11,1)$	$\Delta_0(11,2)$	$\Delta_1(11,3)$	$\Delta_0(11,4)$	$\Delta_1(11,5)$

B1 $\Delta_{\mathcal{B}}$, n odd

					$\Delta_1(1,0)$
				$\Delta_1(3,0)$	$\Delta_0(3,1)$
			$\Delta_1(5,0)$	$\Delta_0(5,1)$	$\Delta_1(5,2)$
		$\Delta_1(7,0)$	$\Delta_0(7,1)$	$\Delta_1(7,2)$	$\Delta_0(7,3)$
	$\Delta_1(9,0)$	$\Delta_0(9,1)$	$\Delta_1(9,2)$	$\Delta_0(9,3)$	$\Delta_1(9,4)$
$\Delta_1(11,0)$	$\Delta_0(11,1)$	$\Delta_1(11,2)$	$\Delta_0(11,3)$	$\Delta_1(11,4)$	$\Delta_0(11,5)$

Figure 3: Each "$\Delta_i(n,k)$" is coloured according to the base case from which $\Delta_i(n,k) \geq 0$ can be concluded (•, •, •) using (1) $\Delta_i(n,k) \rightsquigarrow \Delta_i(n+2,k+1)$ and (2) $\Delta_0(n,k) \rightsquigarrow \Delta_1(n,n-k)$ (with the colour coding from Lemma 4.13). If the colour is purple (•), then $\Delta_i(n,k) \geq 0$ follows both from • and from •. If the colour is gray (•), then $\Delta_i(n,k) \geq 0$ does not hold.

With this being settled, it still remains to prove the base cases (•, •, •) and the inductive step $\Delta_0(n+2,k+1) \geq \Delta_0(n,k)$ as stated in Lemma 4.13.

However, before we immerse ourselves into further calculations, let us gather one more estimate, which required a bit more creativity to be revealed. It will play a key role in proving the induction step $\Delta_0(n,k) \rightsquigarrow \Delta_0(n+2,k+1)$, and it is absolutely reliant on the assumption $\mathrm{scal}_{(1,b,c)} > 0$.

Lemma 4.14 *(Schüth's Lemma)*

For all (b,c) with $1 \geq b \geq c > 0$ and $\mathrm{scal}_{(1,b,c)} > 0$, we have $bC - b^2 - c < 0$, and the assumption $\mathrm{scal}_{(1,b,c)} > 0$ is *necessary* for this to hold.

It also follows that $C < b+1$, which is an even better upper bound on C than the one from Lemma 4.4(3).

Proof. Recall from the proof of Lemma 4.3 the factorised expression

$$-\tfrac{1}{2}b^2c^2\,\mathrm{scal}_{(1,b,c)} = \left(b^2(1-c)^2 - c^2\right)\left(b^2(1+c)^2 - c^2\right),$$

and that $\mathrm{scal}_{(1,b,c)} > 0$ is equivalent to $b^2(1-c)^2 - c^2 < 0$. Since

$$\begin{aligned} 2c(bC - b^2 - c) &= bc\left(\frac{b}{c} + bc + \frac{c}{b}\right) - 2b^2c - 2c^2 \\ &= b^2 + b^2c^2 + c^2 - 2b^2c - 2c^2 \\ &= b^2(1-c)^2 - c^2 \end{aligned}$$

has the same sign as $bC - b^2 - c$, we conclude that $\mathrm{scal}_{(1,b,c)} > 0$ is also equivalent to $bC - b^2 - c < 0$, just as desired.

The second estimate is then obtained from $bC < b^2 + c \le b^2 + b$ (divide by b). ♥

We can now start proving the base cases.

Lemma 4.15 *(Triangle Induction — Base Cases)*

The base cases from Lemma 4.13 are in fact true; that is, we have

1 • $\Delta_0(n,0) \ge 0$ for all $n \in \mathbb{N}_0 \setminus \{2,3,4\}$,

2 • $\Delta_0(n,n) \ge 0$ for all $n \in \mathbb{N}_0$, and

3 • $\Delta_0(n,1) \ge 0$ for each $n \in \{4,5,6\}$

on the set $\{(b,c) : 1 \ge b \ge c > 0,\ \mathrm{scal}_{(1,b,c)} > 0\}$.

Proof. The proof is quite lengthy due to its algebraic nature. Beware!

1 The expression

$$\begin{aligned} \Delta_0(n,0) &= (n-C)^2 + (b+c)^2 n - 2(b+c)(C-1)n \\ &\qquad - (b+c)(b-c)n(n-1) - \lambda_*^2 \\ &= \left(1 - b^2 + c^2\right)n^2 + 2\big((b+c)(1+b-C) - C\big)\,n + C^2 - \lambda_*^2 \end{aligned}$$

is quadratic in n with leading coefficient $1 - b^2 + c^2 > 1 - \frac{3}{4}b^2 > 0$ (due to $c > \frac{1}{2}b$ from Lemma 4.3).

⋮

We will prove $\Delta_0(n,0) \geq 0$ for $n \in \{0,1\}$ and $\Delta_0(n,0) \geq 0$ for all $n \geq 5$. The first two cases can be calculated explicitly. Indeed, one easily computes

- $\Delta_0(0,0) = C^2 - \lambda_*^2 \geq 0$ (the equality being true if and only if $1 = b = c$, see Lemma 4.4(2)) and
- $\Delta_0(1,0) \equiv 0$ (the constant zero map).

The latter is quite interesting and on top of that also very useful: We know that the quadratic function $n \mapsto \Delta_0(n,0)$ has a root at $n = 1$ *independent (!)* of b and c. By Vieta's Theorem, the roots x_1 and x_2 of the quadratic function $x \mapsto x^2 + px + q$ satisfy $x_1 x_2 = q$, from which we conclude that the other root of $n \mapsto \Delta_0(n,0)$ lies at

$$q(b,c) = \frac{C^2 - (1+b+c-C)^2}{1-b^2+c^2} \geq 0.$$

In particular, $\Delta_0(n,0) \geq 0$ for all $n \geq \max\{1,q\}$. We would like to show $q \leq 5$, which is equivalent to $f(b,c) \coloneqq 5(1-b^2+c^2) - C^2 + (1+b+c-C)^2 \geq 0$.

We will prove $f(b,c) > 0$ even on the triangle $\{(b,c) : 1 \geq b \geq c > \frac{1}{2}b > 0\}$ (which is a larger set by Lemma 4.3), using the following trick: Consider the triangle as the union of all segments $b \mapsto c = \xi b$, where $\frac{1}{2} < \xi \leq 1$. We may reparametrise this by setting $\xi \eqqcolon \frac{1}{2}(\eta+1)$, so that $(b,\eta) \in (0,1] \times (0,1]$. Then one can check that $\tilde{f}(b,\eta) \coloneqq f(b,c)$ is a rational function of the form

$$\tilde{f}(b,\eta) = \frac{\beta_3\eta^3 + \beta_2\eta^2 + \beta_1\eta + \beta_0}{4(\eta+1)},$$

whose coefficients $\beta_3, \beta_2, \beta_1, \beta_0$ (which in turn are polynomials in b) can be estimated from below (quite barbarically, using only $0 < b \leq 1$) by

$$\begin{aligned}
\beta_3 &= -b^3 + 6b^2 - b &&\geq -2, \\
\beta_2 &= -5b^3 + 20b^2 - b - 2 &&\geq -8, \\
\beta_1 &= -7b^3 + 6b^2 + 5b + 20 &&\geq 13, \text{ and} \\
\beta_0 &= -3b^3 - 8b^2 - 3b + 14 &&\geq 0.
\end{aligned}$$

Since we have $\eta^3 \leq \eta^2 \leq \eta$ on the interval $(0,1]$, it follows that

$$\beta_3\eta^3 + \beta_2\eta^2 + \beta_1\eta + \beta_0 \geq (-2-8+13)\eta = 3\eta > 0.$$

This implies $\tilde{f}(b,\eta) = f(b,c) > 0$, from which we conclude $q < 5$.

⋮

It turns out, by the way, that this estimate is *sharp*. Indeed, one can check $\Delta_0(5,0) = 0$ at the boundary point $(b,c) = (1,\frac{1}{2})$ (satisfying $\mathrm{scal}_{(1,b,c)} = 0$), which corresponds to $(b,\eta) = (1,0)$. That is, the condition $n \geq 5$ is *necessary* for Gershgorin to work (while one could have expected an arbitrary number somewhere between 4 and 5 as the purely analytic result).

2 Again, the function $n \mapsto \Delta_0(n,n)$ is quadratic and has the same leading coefficient $1 - b^2 + c^2 > 0$ (see below).

Therefore, in order to prove $\Delta_0(n,n) \geq 0$ for all $n \in \mathbb{N}_0$, it suffices to prove $\frac{\mathrm{d}}{\mathrm{d}n}\big|_{n=0} \Delta_0(n,n) \geq 0$, since we already know $\Delta_0(0,0) \geq 0$ from 1.
This, however, is easy to see. Indeed,

$$\begin{aligned}\Delta_0(n,n) &= (n+C)^2 + (b-c)^2 n - 2(b-c)(C+1)n \\ &\qquad - (b-c)(b+c)n(n-1) - \lambda_*^2 \\ &= (1-b^2+c^2)\,n^2 + 2\big(C - (b-c)(1-b+C)\big)\,n + C^2 - \lambda_*^2\end{aligned}$$

implies that (rewriting the middle coefficient)

$$\frac{\mathrm{d}}{\mathrm{d}n}\Big|_{n=0} \Delta_0(n,n) = 2\big(C(1-b+c) - (b-c)(1-b)\big).$$

Since $1 - b + c > 1 - \frac{b}{2} > 0$, we may use the lower bound from Lemma 4.4 2 to obtain

$$\begin{aligned}C(1-b+c) - (b-c)(1-b) &= C(1-b+c) + b^2 - bc - b + c \\ &\geq \tfrac{1}{2}(1+b+c)(1-b+c) + b^2 - bc - b + c \\ &= \tfrac{1}{2}b^2 - bc + \tfrac{1}{2}c^2 - b + 2c + \tfrac{1}{2} \\ &= \tfrac{1}{2}(b-c)^2 + (-b+2c) + \tfrac{1}{2} \\ &> 0 + 0 + \tfrac{1}{2} = \tfrac{1}{2} > 0,\end{aligned}$$

which proves the claim.

3 We will prove $\Delta_0(n,1) \geq 0$ separately for each $n \in \{4,5,6\}$.

Case $n = 4$: Recall the upper bounds $bC < b^2 + c$ and $C < b+1$ from Lemma 4.14. Then we have

$$\begin{aligned}\Delta_0(4,1) &= -6bC - 2(c+1)C + 3b^2 + 2bc + 2b + 15c^2 + 6c + 3 \\ &> -6(b^2+c) - 2(c+1)(b+1) + 3b^2 + 2bc + 2b + 15c^2 + 6c + 3 \\ &\quad \vdots\end{aligned}$$

$$
\begin{aligned}
&= -3b^2 + 15c^2 - 2c + 1\\
&> -12c^2 + 15c^2 - 2c + 1\\
&= 3c^2 - 2c + 1.
\end{aligned}
$$

Note that we used $c > \frac{1}{2}b$ in the last estimate (see Lemma 4.3).

One can easily verify that this quadratic function attains its minimum $\frac{2}{3} > 0$ at the point $c = \frac{1}{3}$. Thus, $\Delta_0(4,1) > \frac{2}{3} > 0$.

For the cases $n \in \{5, 6\}$, this kind of estimate will not work. But, luckily, the trick from 1 succeeds (while it does not for $n = 4$, by the way): Substitute $c = \frac{1}{2}(\eta + 1)b$ to obtain rational functions in $(b, \eta) \in (0, 1] \times (0, 1]$.

Case $n = 5$: One can check that $\Delta_0(5,1) = \dfrac{\beta_3\eta^3 + \beta_2\eta^2 + \beta_1\eta + \beta_0}{2(\eta+1)}$, where

$$
\begin{aligned}
\beta_3 &= -b^3 + 12b^2 - b &&\geq -b &&\geq -1,\\
\beta_2 &= -7b^3 + 38b^2 + b - 2 &&\geq b - 2 &&\geq -2,\\
\beta_1 &= -11b^3 + 40b^2 + 9b + 12 &&\geq 9b + 12 &&\geq 12, \text{ and}\\
\beta_0 &= -5b^3 + 14b^2 - 9b + 6.
\end{aligned}
$$

The estimates for β_3, β_2, and β_1 follow from $b^2 \geq b^3$ for $0 < b \leq 1$; that is, the second term always kills the first one. This is too rough for β_0, but one can check $\min\{\beta_0(b)\}_{0<b\leq 1} > 4$ manually (using elementary analysis). Together, this yields

$$
\beta_3\eta^3 + \beta_2\eta^2 + \beta_1\eta + \beta_0 \geq (-1 - 2 + 12)\eta + 4 = 9\eta + 4 > 0
$$

for all $(b, \eta) \in (0, 1] \times (0, 1]$, which implies $\Delta_0(5, 1) > 0$ even on the triangle $\{(b, c) : 1 \geq b \geq c > \frac{1}{2}b > 0\}$.

(Just by the way: In the case $n = 4$, we would have $\beta_0 < 0$ for some b, which destroys the argument. In fact, one can check numerically that $\Delta_0(4, 1) \geq 0$ does not hold on the larger triangular set $\{(b, c) : 1 \geq b \geq c > \frac{1}{2}b > 0\}$.)

Case $n = 6$: Similarly, one has $\Delta_0(6,1) = \dfrac{\beta_3\eta^3 + \beta_2\eta^2 + \beta_1\eta + \beta_0}{4(\eta+1)}$, where

$$
\begin{aligned}
\beta_3 &= -3b^3 + 35b^2 - 3b &&\geq -3b &&\geq -3,\\
\beta_2 &= -19b^3 + 111b^2 + b - 6 &&\geq b - 6 &&\geq -6,\\
\beta_1 &= -29b^3 + 97b^2 + 23b + 48 &&\geq 23b + 48 &&\geq 48, \text{ and}\\
\beta_0 &= -13b^3 + 21b^2 - 21b + 30.
\end{aligned}
$$

$\vdots$

Again, one can check by using elementary analysis that $\min\{\beta_0(b)\}_{0<b\leq 1} = 17$. It follows that

$$\beta_3\eta^3 + \beta_2\eta^2 + \beta_1\eta + \beta_0 \geq (-3-6+48)\eta + 17 = 39\eta + 17 > 0$$

holds for all $(b,\eta) \in (0,1] \times (0,1]$, from which we conclude (just as above) that $\Delta_0(6,1) > 0$ holds even on the triangle $\{(b,c) : 1 \geq b \geq c > \frac{1}{2}b > 0\}$.

This conclues the proof dealing with the base cases. ♥

The good news is that we are almost done now. The even better news is that the key argument, which will prove the induction step, is truly magical (and not as boring as the cases 2 and 3 in the previous proof).

Lemma 4.16 *(Magical Lemma)*

For each $\delta \in \mathbb{R}$, the sequence

$$n \mapsto \Delta_0\Big(n, \frac{n}{2} + \delta\Big)$$

is increasing in each point of $\{(b,c) : 1 \geq b \geq c > 0,\ \mathrm{scal}_{(1,b,c)} > 0\}$.

Proof. One easily calculates

$$\Delta_0\Big(n, \frac{n}{2} + \delta\Big) = c^2\, n^2 + 2(-bC + b^2 + c)\, n + \text{constant terms in } n.$$

Note that the coefficients at n^2 and, more surprisingly, at n do *not* depend on δ.

The statement now follows from $c^2 > 0$ and $-bC + b^2 + c > 0$, see Lemma 4.14. ♥

Just in case that the reader is not yet aware of the impact of this statement: This is indeed **HUGE**, and discovering this step of the proof only by random intuition felt deeply rewarding (from a mathematician's point of view).

The claimed induction step now follows immediately.

Corollary 4.17 *(Triangle Induction — Induction Step)*

For all $n \in \mathbb{N}_0$ and $k \in \{0, 1, \ldots, n\}$, we have

$$\Delta_0(n+2, k+1) \geq \Delta_0(n,k).$$

Proof. Let $n \in \mathbb{N}_0$ and $k \in \{0, 1, \dots, n\}$. Then there exists a fixed $\delta \in \mathbb{Z}$ or $\delta \in \mathbb{Z}+\frac{1}{2}$ (if n is even or odd, respectively) with $-\frac{n}{2} \le \delta \le \frac{n}{2}$ such that $k = \frac{n}{2} + \delta$, namely $\delta = k - \frac{n}{2}$. Applying Lemma 4.16 yields

$$\Delta_0(n, k) = \Delta_0\Big(n, \frac{n}{2} + \delta\Big) \le \Delta_0\Big(n+2, \frac{n+2}{2} + \delta\Big) = \Delta_0(n+2, k+1),$$

just as desired. ♥

This *finally* concludes the proof dealing with the infinitely many cases $n \ge 5$.

Fragment 2 / 3 of Main Theorem *(Completion of Subproof for $n \ge 5$)*

The number $\lambda_*^2 = (a+b+c-C)^2$ is less than or equal to each eigenvalue of $\mathcal{D}_n^2$ for all $n \ge 5$.

It follows that $\lambda_* \le |\lambda|$ for all eigenvalues λ of $\mathcal{D}_n$, where $n \ge 5$.

Proof. Recall from page 46 that it suffices to consider the case $1 = a \ge b \ge c > 0$. In that case, follow the procedure described in Lemma 4.13.

- **Base Cases:** See Lemma 4.15.
- **Induction Step:** See Corollary 4.17 and Corollary 4.12.

By Lemma 4.13, this completes the proof by triangle induction of the fact that $\Delta_0(n, k) \ge 0$ and $\Delta_1(n, k) \ge 0$ holds for all $n \ge 5$ and $k \in \{0, 1, \dots, n\}$. ♥

Combining the Fragments 1 and 2, we now know $\lambda_* \le |\lambda|$ for all eigenvalues λ of all $\mathcal{D}_n$, where $n \in \mathbb{N}_0$. That is, we finally know that if $\mathrm{scal}_{(a,b,c)} > 0$, then

$$\lambda_{*(a,b,c)} := \min|\operatorname{spec}(\mathscr{D})| = \lambda_* = a + b + c - \frac{1}{2}\Big(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}\Big) > 0 \tag{3}$$

for the Dirac operator associated with $\mathbb{S}^3_{(a,b,c)}$ endowed with either orientation (see the remark at the beginning of page 45 on reversing the orientation of $\mathbb{S}^3$).

Now it just remains to prove spectral invariance, but this is merely a cosy calculation to end the day (or rather, the icing on the cake that the reader absolutely deserves for enduring the proof of $\lambda_{*(a,b,c)} = \lambda_*$).

In this very last part of the paragraph, we will complete the proof of the Main Theorem, earning our well-deserved retirement from homogeneous 3-spheres. Just as Lauret did with the Laplace operator, we will show that the Dirac spectrum of $\mathbb{S}^3_{(a,b,c)}$ determines the isometry class of $\langle \cdot\,, \cdot \rangle_{(a,b,c)}$ if $\mathrm{scal}_{(a,b,c)} > 0$.

We know that $\lambda_{*(a,b,c)}$ is a spectral invariant by (3). The result cited below states that $\mathrm{vol}_{(a,b,c)}$ and $\mathrm{scal}_{(a,b,c)}$ are also spectral invariants, on whose aid we will rely.

Theorem 4.18 *(Volume and Total Scalar Curvature are Spectral Invariants)*

Let $\mathcal{M}^n$ be a closed spin manifold of dimension $n \geq 3$. Then the Dirac spectrum determines the volume $\mathrm{vol}_{\mathcal{M}}$ and the total scalar curvature $\int_{\mathcal{M}} \mathrm{scal}_{\mathcal{M}}$ of $\mathcal{M}$.

Proof. This was proven by Baum (formerly Dlubek) and Friedrich by calculating the asymptotic expansion of the Zeta function of $\mathscr{D}^2$ (which is constructed from the spectrum), whose first two coefficients are expressions in $\mathrm{vol}_{\mathcal{M}}$ and $\int_{\mathcal{M}} \mathrm{scal}_{\mathcal{M}}$, respectively. See [8] for reference. ♥

Since $\int_{\mathbb{S}^3} \mathrm{scal}_{(a,b,c)} = \mathrm{scal}_{(a,b,c)} \, \mathrm{vol}_{(a,b,c)}$ in our case, we know that $\mathrm{scal}_{(a,b,c)}$ is a spectral invariant. This already leads us to the final proof.

Fragment 3 / 3 of Main Theorem *(Spectral Invariance of $\langle \cdot\,, \cdot \rangle_{(a,b,c)}$)*

The triple $\{\mathrm{vol}_{(a,b,c)}, \mathrm{scal}_{(a,b,c)}, \lambda_{*(a,b,c)}\}$ determines (a,b,c) up to order.

In other words, two arbitrary homogeneous 3-spheres of positive scalar curvature are Dirac isospectral if and only if they are isometric.

Proof. We know the isometry class of $\langle \cdot\,, \cdot \rangle_{(a,b,c)}$ if we know the triple (a,b,c) up to order. This is the case if we know the symmetric polynomials

$$\sigma_1 = a + b + c, \quad \sigma_2 = ab + bc + ca, \quad \text{and} \quad \sigma_3 = abc.$$

This is because we then know the polynomial $x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3$, which is obtained by expanding $(x-a)(x-b)(x-c)$. In particular, we know its roots a, b, and c.

Since $\mathrm{vol}_{(a,b,c)}$ depends only on $\sqrt{|\det\langle \cdot\,, \cdot \rangle_{(a,b,c)}|} = (abc)^{-1}$, we know that σ_3 is determined by the Dirac spectrum. We may then assume $\sigma_3 = 1$ without loss of

⋮

generality. This just serves to simplify the further calculation, in which we want to obtain σ_1 and σ_2 from $\lambda_{*(a,b,c)}$ and $\mathrm{scal}_{(a,b,c)}$.

Indeed, under the assumption $\sigma_3 = abc = 1$, one can derive

$$\begin{aligned}
2\lambda_{*(a,b,c)} &= 2(a+b+c) - \Big(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}\Big) \\
&= 2(a+b+c) - (a^2b^2 + b^2c^2 + c^2a^2) \\
&= 2(a+b+c) - \big((ab+bc+ca)^2 - 2(ab\cdot bc + bc\cdot ca + ca\cdot ab)\big) \\
&= 2\sigma_1 - ({\sigma_2}^2 - 2\sigma_1\sigma_3) \\
&= 4\sigma_1 - {\sigma_2}^2
\end{aligned}$$

and, reusing the identity $a^2b^2 + b^2c^2 + c^2a^2 = {\sigma_2}^2 - 2\sigma_1\sigma_3$,

$$\begin{aligned}
\mathrm{scal}_{(a,b,c)} &= 4(a^2+b^2+c^2) - 2\Big(\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2}\Big) \quad \textit{(recall Proposition 1.5)} \\
&= 4(a^2+b^2+c^2) - 2(a^4b^4 + b^4c^4 + c^4a^4) \\
&= 4\big((a+b+c)^2 - 2(ab+bc+ca)\big) \\
&\quad - 2\big((a^2b^2+b^2c^2+c^2a^2)^2 - 2(a^2b^2\cdot b^2c^2 + b^2c^2\cdot c^2a^2 + c^2a^2\cdot a^2b^2)\big) \\
&= 4({\sigma_1}^2 - 2\sigma_2) - 2\big(({\sigma_2}^2 - 2\sigma_1\sigma_3)^2 - 2{\sigma_3}^2({\sigma_1}^2 - 2\sigma_2)\big) \\
&= 8({\sigma_1}^2 - 2\sigma_2) - 2({\sigma_2}^2 - 2\sigma_1)^2 \\
&= 8{\sigma_1}^2 - 16\sigma_2 - 2{\sigma_2}^4 + 8\sigma_1{\sigma_2}^2 - 8{\sigma_1}^2 \\
&= 4\lambda_{*(a,b,c)}{\sigma_2}^2 - 16\sigma_2.
\end{aligned}$$

The latter implies

$$\sigma_2 = \frac{2}{\lambda_{*(a,b,c)}} \pm \sqrt{\Big(\frac{2}{\lambda_{*(a,b,c)}}\Big)^2 + \frac{\mathrm{scal}_{(a,b,c)}}{4\lambda_{*(a,b,c)}}}.$$

Since $\mathrm{scal}_{(a,b,c)} > 0$, this yields exactly one *positive* solution for σ_2, which we can plug into the first identity to obtain a unique solution for σ_1. We then know that all the values σ_1, σ_2, and σ_3 are determined by the Dirac spectrum. ♥

Congratulations!

We can now — **finally** — combine the three fragments of the Main Theorem...

Main Theorem *(Dirac Eigenvalues of Homogeneous 3-Spheres)*

Let $\mathbb{S}^3_{(a,b,c)}$ be a homogeneous 3-sphere of *positive scalar curvature*, endowed with either orientation.

Then the smallest absolute value of its Dirac eigenvalues is given by

$$\lambda_{*(a,b,c)} = a + b + c - \frac{1}{2}\Big(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}\Big) > 0,$$

and its Dirac spectrum determines the underlying metric up to isometry (within the class of homogeneous 3-spheres).

...and die in peace.

(∘Θ‿‿Θ∘)...zZZ♥

Conclusion

In this thesis, we dealt with the spin geometry of the Lie group $\mathbb{S}^3$ endowed with the naturally-arising left-invariant metrics. We derived a formula for the associated Dirac operator (which was already known before in a much more general setting) only by elementary means. We eventually succeeded in calculating the smallest Dirac eigenvalue of each homogeneous 3-sphere of positive scalar curvature (which, to the knowledge of the author, was *not* known before) and proved that two such spheres are Dirac isospectral if and only if they are isometric.

That's it — there is nothing more to conclude about the mathematical theory. Really. It's not much, but it's honest work. However, as for the thesis per se, there are a few more words to say:

The author would like to express his gratitude to his supervisor for suggesting him a topic that allowed the creation of new knowledge (be it only to a negligible extent) instead of something that only involves researching literature, which probably would have been very boring. Nevertheless, it was also an educational experience to review many mathematical concepts from the lectures Differential Geometry I–IV, which the author now understands a little better.

The most satisfying experience, however, was the continuous process of being totally intimidated and clueless at the beginning, gradually obtaining hypotheses and gaining knowledge through numerical experiments (or even by accident), and being able to write things down rigorously. It has been truly fulfilling that everything worked out in the end, which is the highest reward a mathematician can hope for.

The End

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