T. Krämer, A. Otwinowska | Due 24 October 2018

Please hand in your solutions at the beginning of the tutorial on Wednesday.

Problem 1. Let R be a principal ideal domain. Describe its prime spectrum $\operatorname{Spec}(R)$, its maximal spectrum $\operatorname{Spm}(R)$, and the ideals of $R/(a)$ for any $a \in R$.

Problem 2. Let $R_1, \ldots, R_n$ be rings and consider $R = R_1 \times \cdots \times R_n$ as a ring with the componentwise addition and multiplication.

(a) How are the ideals of R related to those of the R_i for $i = 1, \ldots, n$?

(b) Show that the spectrum of R is a disjoint union $\operatorname{Spec}(R) = \coprod_{i=1}^n \operatorname{Spec}(R_i)$.

Problem 3. Let R be a ring and $\mathfrak{a}_1, \ldots, \mathfrak{a}_n \trianglelefteq R$.

(a) Find a natural monomorphism

$$R/\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n \hookrightarrow R/\mathfrak{a}_1 \times \cdots \times R/\mathfrak{a}_n.$$

(b) Show that this is an isomorphism if $\mathfrak{a}_i + \mathfrak{a}_j = R$ for all $i \neq j$.

(c) Can the last conditions be replaced by $\mathfrak{a}_1 + \cdots + \mathfrak{a}_n = R$ if $n > 2$?

Problem 4. A ring R is called *Euclidean* if it admits a function $N : R \setminus \{0\} \to \mathbb{N}$ such that for all nonzero ring elements $a, b \in R \setminus \{0\}$,

- $N(a) \leq N(ab)$, and
- there are $q, r \in R$ such that $a = qb + r$ and $N(r) < N(b)$ if $r \neq 0$.

Show that

(a) the Gaussian integers $\mathbb{Z}[i] = \{a + bi \in \mathbb{C} \mid a, b \in \mathbb{Z}\}$ form a Euclidean ring.

(b) every Euclidean ring is a principal ideal domain.

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Problem 5. Let $f : R \to S$ be a ring homomorphism.

(a) Verify that for any ideal $\mathfrak{a} \trianglelefteq S$ the preimage $f^{-1}(\mathfrak{a}) \trianglelefteq R$ is an ideal.

(b) Verify the three implications in the following diagram:

$$\begin{tikzcd} \mathfrak{a} \in \mathrm{Spm}(S) \arrow[d, Rightarrow] & f^{-1}(\mathfrak{a}) \in \mathrm{Spm}(R) \arrow[d, Rightarrow] \\ \mathfrak{a} \in \mathrm{Spec}(S) \arrow[r, Rightarrow] & f^{-1}(\mathfrak{a}) \in \mathrm{Spec}(R) \end{tikzcd}$$

(c) Can you find counterexamples for the remaining five implications?

Problem 6. Show that

(a) any finite integral domain is a field,

(b) in a finite ring every prime ideal is maximal.

Problem 7. Is the ring $\mathbb{Z}[\sqrt{-3}]$ a unique factorization domain? Explain your answer.

Problem 8. Let R be a ring and put $D(f) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid f \notin \mathfrak{p}\}$ for $f \in R$.

(a) Check that $D(f) \subset \mathrm{Spec}(R)$ is an open subset in the Zariski topology.

(b) Show that if elements $f_1, \ldots, f_n \in R$ generate the unit ideal $(f_1, \ldots, f_n) = (1)$, then

$$\mathrm{Spec}(R) \;=\; \bigcup_{i=1}^{n} D(f).$$

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Problem 9. Let R be a ring and $S \subset R$ a multiplicative subset.

(a) Show that localization is compatible with finite intersections of ideals:

$$(\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n)R_S \;=\; \mathfrak{a}_1 R_S \cap \cdots \cap \mathfrak{a}_n R_S \quad \text{for any} \quad \mathfrak{a}_1, \ldots, \mathfrak{a}_n \;\trianglelefteq\; R.$$

(b) Does the analogous statement hold for intersections of infinitely many ideals? Hint: Consider localizations of a polynomial ring over an infinite field.

Problem 10. Show that if R is a unique factorization domain, then the same holds for its localization at any multiplicative subset $S \subset R$.

Problem 11. Let A be a ring and $S \subset A$ a multiplicative subset.

(a) Let $f : A \to B$ be a ring homomorphism and $f(S) \subset T$ where $T \subset B$ is a multiplicative subset. Show that there is a unique ring homomorphism $f_{S,T}$ which makes the following diagram commutative, where φ_S and φ_T are the localization maps:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varphi_S"'] & B \arrow[d, "\varphi_T"] \\ A_S \arrow[r, dotted, "\exists!\, f_{S,T}"] & B_T \end{tikzcd}$$

(b) Taking $B = A$ and $T = \{a \in A \mid a \text{ divides some } s \in S\}$, show that $A_S \simeq A_T$.

Problem 12. Let k be a field. Show that the inclusion $k[X] \hookrightarrow k[X,Y]$ gives rise to an embedding

$$A \;=\; k[X] \;\hookrightarrow\; B \;=\; k[X,Y]/(XY)$$

which induces an isomorphism $A_S \simeq B_S$ for the localizations at $S = \{X^n \mid n \in \mathbb{N}_0\}$.

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Problem 13. Let k be a field and $R = k[X,Y]/(X - XY^2, Y^3)$. Compute

(a) the nilradical $\mathrm{Rad}(R) \trianglelefteq R$,

(b) the quotient $R_{\mathrm{red}} = R/\mathrm{Rad}(R)$.

Problem 14. Let R be a ring and $\mathfrak{a}, \mathfrak{b} \trianglelefteq R$. Show that

(a) $\sqrt{\mathfrak{a}} = (1)$ iff $\mathfrak{a} = (1)$,

(b) $\sqrt{\mathfrak{a} \cdot \mathfrak{b}} = \sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$,

(c) $\sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt{\mathfrak{c}}$ for the ideal $\mathfrak{c} = \sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}$,

(d) if $\mathfrak{a} = \mathfrak{p}^n$ for a prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ and some $n \geq 1$, then $\sqrt{\mathfrak{a}} = \mathfrak{p}$.

Problem 15. Let R be a ring. Show that the units of the polynomial ring $S = R[X]$ are

$$S^* \ = \ \Big\{ \sum_{n \geq 0} a_n X^n \in S \ \mid \ a_0 \in R^* \text{ and } a_n \in \mathrm{Rad}(R) \text{ for all } n > 0 \Big\}.$$

Deduce that in this case the Jacobson radical is the nilradical: $\mathrm{Jac}(S) = \mathrm{Rad}(S)$.

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Problem 16. Let R be a unique factorization domain and $M = (x, y) \trianglelefteq R$ the ideal generated by two elements $x, y \in R$ with no common factor except units. Consider the epimorphism

$$f : \quad R \oplus R \twoheadrightarrow M, \quad (a, b) \mapsto ax + by$$

of R-modules. Is $\ker(f)$ always a free R-module? Is M always a free R-module?

Problem 17. Let A be a ring.

(a) Is A a reduced ring if its localizations $A_{\mathfrak{p}}$ at all $\mathfrak{p} \in \mathrm{Spec}(A)$ are so?

(b) Is A an integral domain if its localizations $A_{\mathfrak{p}}$ at all $\mathfrak{p} \in \mathrm{Spec}(A)$ are so?

Problem 18. A module $M \in \mathrm{Mod}(A)$ is called *simple* if it has no nonzero submodules other than itself. More generally we say that M is *indecomposable* if it cannot be written as the direct sum of two nonzero submodules.

(a) Show that if M is simple, any $f \in \mathrm{Hom}_A(M, M) \setminus \{0\}$ is an isomorphism.

(b) Can you find a counterexample when M is only indecomposable?

Problem 19. Let $f : M \twoheadrightarrow N$ be an epimorphism of finitely generated A-modules.

(a) If $N \simeq A^n$ is a free module, show that $M \simeq \ker(f) \oplus N$.

(b) Deduce that in this case the kernel $\ker(f)$ is finitely generated.

(c) Can you find a counterexample when N is not assumed to be free?

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Problem 20. Let $A \to B \to C$ be ring homomorphisms and $M, N \in \mathrm{Mod}(A)$. Show that one has

(a) an isomorphism $(M \otimes_A B) \otimes_B C \simeq M \otimes_A C$ in $\mathrm{Mod}(C)$.

(b) an isomorphism $(M \otimes_A N) \otimes_A B \simeq (M \otimes_A B) \otimes_B (N \otimes_A B)$ in $\mathrm{Mod}(B)$.

Problem 21. Let R be a ring.

(a) Show that for any $M, N \in \mathrm{Mod}(R)$ we may view $\mathrm{Hom}_R(M, N)$ as an R-module via
$$a \cdot \varphi \ := \ (m \mapsto a \cdot \varphi(m)) \quad \text{for} \quad a \in R,\ \varphi \in \mathrm{Hom}_R(M, N).$$

(b) Show that then for all $L, M, N \in \mathrm{Mod}(R)$ there exists a natural isomorphism of R-modules
$$\mathrm{Hom}_R(L \otimes_R M, N) \ \simeq \ \mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N)).$$

Problem 22. If R is a ring and $\mathfrak{a}, \mathfrak{b} \trianglelefteq R$ are two ideals, find an ideal $\mathfrak{c} \trianglelefteq R$ and an isomorphism
$$R/\mathfrak{a} \otimes_R R/\mathfrak{b} \ \simeq \ R/\mathfrak{c} \quad \text{in} \quad \mathrm{Mod}(R).$$

Problem 23. Let R be a ring, $\mathfrak{a} \trianglelefteq R$ and $M \in \mathrm{Mod}(R)$.

(a) Show that for the submodule $\mathfrak{a}M = \{\sum_{i=1}^n a_i m_i \mid a_i \in \mathfrak{a}, m_i \in M\} \subseteq M$ one has
$$M/\mathfrak{a}M \ \simeq \ M \otimes_R R/\mathfrak{a} \quad \text{in } \mathrm{Mod}(R).$$

(b) Is the R-linear map $\mathfrak{a} \otimes_R M \to \mathfrak{a}M, a \otimes m \mapsto am$ always an isomorphism?

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Problem 24. Let R be a ring. Show that in $\mathrm{Mod}(R)$,

(a) a sequence $0 \to M' \to M \to M''$ is exact iff for all $N \in \mathrm{Mod}(R)$ the induced sequence

$$0 \;\to\; \mathrm{Hom}_R(N, M') \;\to\; \mathrm{Hom}_R(N, M) \;\to\; \mathrm{Hom}_R(N, M'') \quad \text{is exact.}$$

(b) a sequence $M' \to M \to M'' \to 0$ is exact iff for all $N \in \mathrm{Mod}(R)$ the induced sequence

$$0 \;\to\; \mathrm{Hom}_R(M'', N) \;\to\; \mathrm{Hom}_R(M, N) \;\to\; \mathrm{Hom}_R(M', N) \quad \text{is exact.}$$

Problem 25. Which of the following rings are Noetherian?

(a) The ring of rational functions $f(z) \in \mathbb{C}(z)$ with no poles on the circle $|z| = 1$.

(b) The ring of all holomorphic functions on the complex plane.

(c) The ring of all formal power series $\mathbb{C}[[z]] = \{\sum_{n \in \mathbb{N}_0} a_i z^i \mid a_i \in \mathbb{C}\}$.

Problem 26. If R is a ring such that for each $\mathfrak{p} \in \mathrm{Spec}(R)$ the localization $R_\mathfrak{p}$ is Noetherian, does it follow that R is Noetherian?

Problem 27. Consider a surjective ring homomorphism $f : R \to R$.

(a) If R is Noetherian, show that f must be an isomorphism.

(b) Find a counterexample if the Noetherian assumption is dropped.

(c) In the lecture we have shown that for finitely generated $M \in \mathrm{Mod}(R)$, every surjective $f \in \mathrm{End}_R(M)$ is an isomorphism. Why doesn't this contradict (b)?

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Problem 28. Let p be a prime number.

(a) Determine all Artin rings with precisely p^2 elements.

(b) Which of them are $\mathbb{F}_p$-algebras, where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$?

Problem 29. Let R be a local ring with maximal ideal $\mathfrak{m}$.

(a) What do the simple R-modules look like?

(b) Assume that there is a subfield $k \subseteq R$ for which the composite $k \hookrightarrow R \twoheadrightarrow R/\mathfrak{m}$ is an isomorphism. Show that then a module $M \in \mathrm{Mod}(R)$ is of finite length iff $\dim_k(M) < \infty$, in which case

$$\ell(M) \;=\; \dim_k(M).$$

Problem 30. Show that for Noetherian modules $M \in \mathrm{Mod}(R)$ over any ring R the following are equivalent:

(a) M has finite length.

(b) There exist $\mathfrak{m}_1, \dots, \mathfrak{m}_n \in \mathrm{Spm}(R)$ with $\mathfrak{m}_1 \cdots \mathfrak{m}_n \subseteq \mathrm{Ann}(M)$.

(c) Every prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{Ann}(M) \subseteq \mathfrak{p}$ is maximal.

(d) The quotient $R/\mathrm{Ann}(M)$ is an Artinian ring.

Problem 31. Let R be a ring. Show by direct computation that for any ideal $\mathfrak{a} \trianglelefteq R$ of nilpotent elements, the quotient map $R \to R/\mathfrak{a}$ induces a bijection between the sets of idempotents:

$$\Big\{ e \in R : \; e^2 = e \Big\} \;\xrightarrow{\sim}\; \Big\{ f \in R/\mathfrak{a} : \; f^2 = f \Big\}.$$

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Problem 32. Which of the following topological spaces X are connected?

(a) $X = \operatorname{Spec}(k[x,y]/(xy-1))$ with the Zariski topology, where k is any field.

(b) $X = \{(x,y) \in \mathbb{R}^2 \mid xy = 1\}$ where $\mathbb{R}^2$ is given the usual Euclidean topology.

(c) $X = \{(x,y) \in \mathbb{C}^2 \mid xy = 1\}$ where $\mathbb{C}^2$ is given the usual Euclidean topology.

Problem 33. Find all implications between the following statements for $f \in \mathbb{R}[x,y]$, with a proof or counterexample in each case:

(a) $\operatorname{Spec}(\mathbb{R}[x,y]/(f))$ is irreducible. (c) $\operatorname{Spec}(\mathbb{R}[x,y]/(f))$ is connected.

(b) $\operatorname{Spec}(\mathbb{C}[x,y]/(f))$ is irreducible. (d) $\operatorname{Spec}(\mathbb{C}[x,y]/(f))$ is connected.

Problem 34. Let k be a field. Verify that

(a) the ideal $\mathfrak{a} = (X, Y^2) \trianglelefteq k[X,Y]$ is primary but not a power of a prime ideal.

(b) the ideal $\mathfrak{p} = (x,z) \trianglelefteq R = k[X,Y,Z]/(XY - Z^2)$ is prime but $\mathfrak{p}^2$ is not primary.

Here as usual we denote by x, y, z the images of X, Y, Z in the quotient ring R.

Problem 35. An ideal is called *irreducible* if it cannot be written as an intersection of two strictly bigger ideals. Show that in the polynomial ring $k[x,y]$ over a field k the ideals

$$\mathfrak{q}_1 = (x, y^2), \quad \mathfrak{q}_2 = (y, x^2), \quad \mathfrak{q}_3 = (x+y, x^2)$$

are irreducible and pairwise distinct, but for their intersections $\mathfrak{q}_1 \cap \mathfrak{q}_2 = \mathfrak{q}_1 \cap \mathfrak{q}_3$.

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This is a bonus sheet to review some topics we have seen so far. Try to solve it without your notes at first. Don't worry about the number of problems, the exam will be shorter. The solutions will be discussed in the next problem session.

Problem 1. Let R be a ring.

(a) Show that every maximal proper ideal $\mathfrak{m}$ of R is prime.

(b) Show that if $\mathfrak{p} \trianglelefteq R$ is a prime ideal and $R/\mathfrak{p}$ is finite, then $\mathfrak{p}$ is maximal.

(c) Find an example of a ring with a strictly ascending infinite chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \trianglelefteq R.$$

Problem 2. Let R be a local ring with maximal ideal $\mathfrak{m} \trianglelefteq R$.

(a) What does Nakayama's lemma say?

(b) If R is Noetherian, show that $\mathfrak{m}/\mathfrak{m}^2 = \{0\}$ iff R is a field.

(c) Can you find a counterexample to part b when R is not Noetherian?

Problem 3. For an exact sequence $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ of modules over a ring, answer the following questions with a sketch of proof or counterexample:

(a) Is the module M Artinian if both M' and M'' are so?

(b) Are two submodules $M_1, M_2 \subseteq M$ equal if

$$f^{-1}(M_1) = f^{-1}(M_2) \quad \text{and} \quad g(M_1) = g(M_2)?$$

Problem 4. Let R be a ring.

(a) Show that $R/\mathfrak{a} \otimes_R R/\mathfrak{b} \simeq R/(\mathfrak{a} + \mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b} \trianglelefteq R$.

(b) In the lecture we have seen that any finitely generated nonzero $M \in \mathrm{Mod}(R)$ has a cyclic quotient. Deduce that then $M \otimes_R M \neq 0$.

(c) Can you find a counterexample to part b with $M \neq 0$ not finitely generated?

Problem 5. Let R be a ring.

(a) Show that
$$\sqrt{\mathfrak{a}} \;=\; \bigcap_{\substack{\mathfrak{p}\in\mathrm{Spec}(R)\\ \mathfrak{p}\supseteq\mathfrak{a}}} \mathfrak{p} \quad \text{for all ideals} \quad \mathfrak{a} \trianglelefteq R.$$

(b) Deduce that if $\mathfrak{a}$ is a radical ideal, then there is a multiplicative subset $S \subset R$ such that
$$\mathfrak{a}\cdot R_S \;=\; Jac(R_S).$$

Problem 6. Let R be a ring.

(a) When is an element $e \in R$ called idempotent?

(b) Show that then $f = 1 - e$ is also idempotent with $ef = 0$ and one has a ring isomorphism
$$R\cdot e \;\simeq\; R\big[\tfrac{1}{e}\big].$$

Problem 7. Let $\mathfrak{a} = (xy, xz, yz) \trianglelefteq R = k[x, y, z]$ for a field k.

(a) What are the irreducible components of $\mathrm{Spec}(R/\mathfrak{a})$?

(b) Find a minimal primary decomposition of the ideal $\mathfrak{a}$.

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Problem 36. If you haven't seen this before,

(a) check whether $\sqrt{2+\sqrt{2}}+\sqrt[3]{3}/2$ is an algebraic integer.

(b) verify that every unique factorization domain is normal.

(c) show that for an integral extension $k \subset K$ of domains, K is a field iff k is.

Problem 37. Show that the extension $A = \mathbb{R}[t^2-1] \subset B = \mathbb{R}[t]$ is integral, but that the extension

$$A_{\mathfrak{p}} \;\subset\; B_{\mathfrak{q}} \quad \text{obtained by localizing at the primes} \quad \begin{cases} \mathfrak{q} = (t-1) \trianglelefteq B \\ \mathfrak{p} = \mathfrak{q} \cap A \;\; \trianglelefteq A \end{cases}$$

is *not* integral. Didn't we show in class that integrality is stable under localization?

Problem 38. Let $A \subset B$ be an extension of domains, and $\overline{A}$ the integral closure of the former domain in the latter. Show that for any monic polynomials $f, g \in B[t]$ one has:

$$f \cdot g \;\in\; \overline{A}[t] \quad \Longleftrightarrow \quad f, g \;\in\; \overline{A}[t]$$

Problem 39. Show that in an arbitrary ring extension $A \subseteq B$,

(a) the conclusion of the going up theorem holds iff for all primes $\mathfrak{q} \in \operatorname{Spec}(B)$ and $\mathfrak{p} = \mathfrak{q} \cap A$, the natural map $\operatorname{Spec}(B/\mathfrak{q}) \to \operatorname{Spec}(A/\mathfrak{p})$ is surjective.

(b) the conclusion of the going down theorem holds iff for all primes $\mathfrak{q} \in \operatorname{Spec}(B)$ and $\mathfrak{p} = \mathfrak{q} \cap A$, the natural map $\operatorname{Spec}(B_{\mathfrak{q}}) \to \operatorname{Spec}(A_{\mathfrak{p}})$ is surjective.

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Problem 40. Which of the following ring extensions $A = \mathbb{C}[x] \subset B$ are integral?

(a) $B = \mathbb{C}[x, y, z]/(z^2 - xy)$,

(b) $B = \mathbb{C}[x, y, z]/(z^2 - xy, y^3 - x^2)$,

(c) $B = \mathbb{C}[x, y, z]/(z^2 - xy, x^3 - yz)$.

Problem 41. Let k be an arbitrary field. Find a Noether normalization $A \hookrightarrow B$ for the k-algebras

(a) $B = k[x, y]/(xy(x + y) - 1)$,

(b) $B = k[x, y, z]/(y - z^2, xz - y^2)$.

In each case, can you achieve your result by a linear coordinate change?

Problem 42. Let $f_1, \ldots, f_m \in \mathbb{Z}[x_1, \ldots, x_n]$ be polynomials which have no common complex zeroes:

$$V \;:=\; \big\{a \in \mathbb{C}^n \mid f_1(a) = \cdots = f_m(a) = 0\big\} \;=\; \varnothing.$$

Show that there are polynomials with integer coefficients $g_1, \ldots, g_m \in \mathbb{Z}[x_1, \ldots, x_n]$ such that $g_1 f_1 + \cdots + g_m f_m \;\in\; \mathbb{Z} \setminus \{0\}$. What if we only assume that $V \cap \mathbb{R}^n = \varnothing$?

Problem 43. Which of the following rings R are Jacobson?

(a) $R = \mathbb{Z}$,

(b) $R = \mathbb{R}[x]$,

(c) the formal power series ring $R = \mathbb{C}[[x]]$.

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Problem 44. Let R be a Noetherian ring. Recall that among the associated primes for a primary decomposition of a proper ideal $\mathfrak{a} \subsetneq R$, the minimal ones are called isolated. Show that

$$\dim(R/\mathfrak{a}) \;=\; \max\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \text{ is an isolated prime of } \mathfrak{a}\}.$$

Problem 45. Let B_1, B_2 be finitely generated algebras over a field k and $B_1 \otimes_k B_2$ their tensor product, viewed as a k-algebra in the natural way with multiplication defined by

$$(b_1 \otimes b_2) \cdot (c_1 \otimes c_2) \;=\; b_1c_1 \otimes b_2c_2 \quad \text{for} \quad b_i, c_i \;\in\; B_i.$$

Use Noether normalizations to show that the Krull dimension of the tensor product is

$$\dim(B_1 \otimes_k B_2) \;=\; \dim(B_1) + \dim(B_2).$$

Problem 46. Let $A \hookrightarrow B$ be an integral extension where A is a finitely generated algebras over a field. Suppose that we have an inclusion of prime ideals $\mathfrak{q} \hookrightarrow \mathfrak{q}'' \trianglelefteq B$ and let $\mathfrak{p} = \mathfrak{q} \cap A \hookrightarrow \mathfrak{p}'' = \mathfrak{q}'' \cap A$.

(a) Use Noether normalization and going down to show that if $\mathfrak{p} \subset \mathfrak{p}' \subset \mathfrak{p}''$ for some $\mathfrak{p}' \in \operatorname{Spec}(A)$, then $\mathfrak{q} \subset \mathfrak{q}' \subset \mathfrak{q}''$ for some $\mathfrak{q} \in \operatorname{Spec}(B)$.

(b) Can you always find such a prime $\mathfrak{q}'$ which furthermore lies above $\mathfrak{p}'$?