Problem 1.1. Let $k$ be a field.

(a) Let $R$ be a $k$-algebra which is a domain, not necessarily of finite type over $k$. Show that its Krull dimension is at most the transcendence degree of the quotient field of $R$, i.e. we have
\[ \dim R \leq \text{trdeg}_k(\text{Quot}(R)). \]

(b) Illustrate by an example that in general the above inequality can be strict.

Problem 1.2. Let $k$ be a field. Let $C$ be a normal curve over $k$ that is birational to $\mathbb{P}^1_k$.

(a) Show that $C$ is isomorphic to an open subset of $\mathbb{P}^1_k$.

(b) Deduce that if $C$ is not proper, then it is affine and $\Gamma(C, \mathcal{O}_C)$ is a factorial ring.

Problem 1.3. Let $k$ be an algebraically closed field with $\text{char}(k) \neq 2$.

(a) Show that $A = k[x, y]/(y^2 - x^3 - x)$ is a domain which is integral over $R = k[x]$.

(b) Define an automorphism $\sigma: A \to A$ by $\sigma|_R = \text{id}$ and $\sigma(y) = -y$. Show that we have a multiplicative map
\[ N: A \to R, \quad N(a) := a \cdot \sigma(a). \]

(c) Show that $A^\times = k^\times$ and that $x, y$ are irreducible elements in $A$. Deduce that $A$ is not factorial, and that $\text{Spec}(A)$ is not isomorphic to an open subset of the affine line over $k$.

Problem 1.4. Let $C \subset \mathbb{P}^2_k$ be the projective closure of $C_0 = \text{Spec}(A) \subset \mathbb{A}^2_k$ from problem 1.3.

(a) Show that $C$ is normal and the complement $C \setminus C_0$ consists of a single point $o$.

(b) Let $p, q, r \in C$ be distinct closed points. Show that the following are equivalent:
\begin{itemize}
  \item There is a rational function $f \in k(C)^\times$ with $\text{div}(f) = [p] + [q] + [r] - 3[o]$.
  \item The three points $p, q, r$ are collinear, i.e. they lie on a common line $\ell \subset \mathbb{P}^2_k$.
\end{itemize}

(c) What does this say about the Picard group $\text{Pic}(C)$?
Problem 2.1. Let $k$ be a field.

(a) Show that for any finite set of closed points on a proper normal curve $C$ over $k$, there exists a rational function $f \in k(C)^*$ which has poles only in the points from that set.

(b) Deduce that every non-proper normal curve $C_0$ over $k$ is affine.

Hint: Find a proper normal curve $C \supset C_0$ and $f : C \to \mathbb{P}^1_k$ with $f^{-1}(\mathbb{A}^1_k) = C_0$.

Problem 2.2. Let $f : C' \to C$ be a morphism of normal curves over an alg. closed field $k$.

(a) Check that we have a natural morphism $f^*(\Omega^1_{C/k}) \to \Omega^1_{C'/k}$ of line bundles.

(b) Now assume that $f$ is finite. Show by looking at stalks at the generic point that the above morphism is injective iff the extension of function fields $k(C) \hookrightarrow k(C')$ is separable.

(c) Show that in this case:

- The quotient $\Omega^1_{C'/C} := \Omega^1_{C'/k}/f^*(\Omega^1_{C/k})$ is a skyscraper sheaf, and
  $$\Omega^1_{C'/k} \cong f^*(\Omega^1_{C/k}) \otimes_{\mathcal{O}_{C'}} \mathcal{O}_{C'}(R)$$
  for the ramification divisor $R := \sum_{p \in C'} \text{length}_{\mathcal{O}_{C', p}}(\Omega^1_{C'/C,p}) \cdot [p] \in \text{Div}(C')$.

- The genus $g_{C'}$ of the covering curve is related to the genus $g_C$ of the target curve via
  $$2g_{C'} - 2 = \deg(f) \cdot (2g_C - 2) + \deg(R).$$

- If $\text{char}(k) \nmid e_p(f)$ for all closed points $p \in C'$, then $\deg(R) = \sum_{p \in C'} (e_p(f) - 1)$.

Problem 2.3. Let $k$ be an algebraically closed field with $\text{char}(k) \neq 2$.

(a) Let $C$ be the proper normal compactification of the affine curve $V(y^2 - h(x)) \subset \mathbb{A}^2_k$ for a square-free polynomial

$$h(x) \in k[x] \text{ of degree } \deg(h) = 2g + 2 \geq 4.$$ 

How many 'points at infinity' (i.e. points in the complement $C \setminus C_0$) are there?

(b) Show that there exists a morphism $f : C \to \mathbb{P}^1_k$ of degree two. Determine its ramification divisor, and deduce from the previous problem that the curve $C$ has genus $g$.

(c) Consider the open dense subset $U = D(y) \cap V(y^2 - h(x)) \subset C$. Show that the differential forms

$$\frac{dx}{y}, x \frac{dx}{y}, x^2 \frac{dx}{y}, \ldots, x^{g-1} \frac{dx}{y} \in H^0(U, \Omega^1_{C/k})$$

extend uniquely to global sections of $\Omega^1_{C/k}$ and form a basis of $H^0(C, \Omega^1_{C/k})$ over $k$. 
Problem 3.1. Let $X$ be a scheme over a field $k$, and $\mathcal{L} \in \text{Pic}(X)$. Let $s_0, \ldots, s_m \in \Gamma(X, \mathcal{L})$ and $t_0, \ldots, t_n \in \Gamma(X, \mathcal{L})$ be two sets of sections that span the same basepoint-free linear series
$$V = \langle s_0, \ldots, s_m \rangle_k = \langle t_0, \ldots, t_n \rangle_k \subset \Gamma(X, \mathcal{L}).$$
Show that for $m \geq n$ the morphisms
$$f := [s_0 : \cdots : s_m] : X \to \mathbb{P}^m_k$$
$$g := [t_0 : \cdots : t_n] : X \to \mathbb{P}^n_k$$
satisfy $g = p \circ f$ where $p : \mathbb{P}^m_k \dashrightarrow \mathbb{P}^n_k$ is a linear projection followed by an automorphism.

Problem 3.2. Let $k$ be a field and $X = \mathbb{P}^2_k$ with homogenous coordinates $x, y, z \in \Gamma(X, \mathcal{O}_X(1))$.

(a) Show that $\mathcal{L} = \mathcal{O}_X(2)$ is very ample and $V = \langle x^2, y^2, z^2, y(x-z), (x-y)z \rangle \subset \Gamma(X, \mathcal{L})$ defines a closed immersion
$$\varphi_{\mathcal{L}, V} : X \hookrightarrow \mathbb{P}^4_k.$$
(b) Let $p \in X$ be a closed point. Show that the linear series $W = \{s \in \Gamma(X, \mathcal{L}) \mid s(p) = 0\}$ gives a locally closed immersion $\varphi_{\mathcal{L}, W} : X \setminus p \hookrightarrow \mathbb{P}^4_k$ which after blowing up the point $p$ extends to a closed immersion
$$\tilde{\varphi}_{\mathcal{L}, W} : X = \text{Bl}_p(X) \hookrightarrow \mathbb{P}^4_k.$$
Show moreover that $\tilde{X} \subset \mathbb{P}^4_k$ is a surface which is covered by a family of disjoint lines in $\mathbb{P}^4_k$ which are the proper transforms of the lines through the point $p$ in $X = \mathbb{P}^2_k$.

Problem 3.3. Let $X$ be a scheme of finite type over a Noetherian ring $R$.

(a) Show that if there exists an ample line bundle on $X$, then $X$ is separated.

(b) Now let $X$ be the affine line over a field $k$ with the origin doubled.
   - Compute the group $\text{Pic}(X)$. Which line bundles are globally generated?
   - Verify directly (without using (a)) that there are no ample line bundles on $X$.

Problem 3.4. Let $k$ be a field, and let $X = \text{Proj}(R)$ for a graded $k$-algebra $R$ generated by finitely many elements of degree one. Show that we have a functor
$$\Gamma_* : \text{QCoh}(X) \to \text{GrMod}(R), \quad \mathcal{M} \mapsto \Gamma_*(\mathcal{M}) := \bigoplus_{d \geq 0} \Gamma(X, \mathcal{M} \otimes \mathcal{O}_X(d))$$
and that every $\mathcal{M} \in \text{QCoh}(X)$ arises as $\mathcal{M} \simeq \tilde{M}$ for the graded $R$-module $M := \Gamma_*(\mathcal{M})$. 
Problem 4.1. Verify for the spectrum $X = \text{Spec}(R)$ of the local ring $R = (\mathbb{C}[s,t]/(s^2 - t^3))(s,t)$ that
$$\text{Pic}(X) \cong 0 \quad \text{but} \quad \text{Pic}(X \times \mathbb{A}^1) \not\cong 0.$$ 

Problem 4.2. Let $k$ be a field. Consider the hypersurface $Z = V_+(f) \subset \mathbb{P}^n_k$ for an irreducible homogeneous polynomial $f \in k[x_0, \ldots, x_n]$ of degree $d > 1$, and denote by $H_0, \ldots, H_n \subset \mathbb{P}^n_k$ the coordinate hyperplanes.

(a) Show that $U := \mathbb{P}^n_k \setminus Z$ is affine and covered by affine open subsets $U_i := U \setminus H_i$.

(b) Show that $\text{Pic}(U) \not\cong 0$ but $\text{Pic}(U_i) \cong 0$ for $i = 0, 1, \ldots, n$.

Hence an affine scheme with a cover by spectra of UFD’s need not be the spectrum of a UFD.

Problem 4.3. Consider the normalization $\pi: \tilde{C} \rightarrow C$ of an integral proper curve $C$ over an algebraically closed field $k$. For closed points $p \in C$, let $\tilde{O}_{C,p}$ be the normalization of the corresponding local ring. Denote by $\mathcal{K}_C$ the sheaf of total quotient rings of the curve. Deduce from the short exact sequence of sheaves
$$0 \rightarrow \pi_*(\mathcal{O}_{\tilde{C}}^\times)/\mathcal{O}_{C}^\times \rightarrow \mathcal{K}_C^\times/\mathcal{O}_{C}^\times \rightarrow \mathcal{K}_C^\times/\pi_*(\mathcal{O}_{\tilde{C}}^\times) \rightarrow 0$$
that we have a short exact sequence of abelian groups
$$0 \rightarrow \bigoplus_{p \in C} \tilde{O}_{C,p}^\times/\mathcal{O}_{C,p}^\times \rightarrow \text{Pic}(C) \rightarrow \text{Pic}(\tilde{C}) \rightarrow 0.$$ 

Problem 4.4. Let $k$ be an algebraically closed field. Deduce from the previous exercise:

(a) The Picard group of the cuspidal cubic $C = V(y^2z - x^3) \subset \mathbb{P}^2_k = \text{Proj} k[x,y,z]$ fits in an exact sequence
$$0 \rightarrow \mathbb{G}_a \rightarrow \text{Pic}(C) \rightarrow \text{Pic}(\tilde{C}) \rightarrow 0 \quad \text{for the additive group } \mathbb{G}_a = (k, +).$$

(b) The Picard group of the nodal cubic $C = V(y^2z - x^2(x + z)) \subset \mathbb{P}^2_k$ fits in an exact sequence
$$0 \rightarrow \mathbb{G}_m \rightarrow \text{Pic}(C) \rightarrow \text{Pic}(\tilde{C}) \rightarrow 0 \quad \text{for the multiplicative group } \mathbb{G}_m = (k^\times, \cdot).$$
Problem 5.1. Describe $\Omega^1_{X/S}$ for each of the following morphisms:

(a) $X = \text{Spec}(\mathbb{Z}[i]) \to S = \text{Spec}(\mathbb{Z})$,
(b) $X = \text{Spec}(\mathbb{C}[x, y]/(xy)) \to S = \text{Spec}(\mathbb{C})$,
(c) $X = \text{Bl}_p(S) \to S = \mathbb{A}^2_{\mathbb{C}}$, the blowup of the plane in the point $p = (0, 0) \in \mathbb{A}^2(\mathbb{C})$.

Is $\Omega^1_{X/S} \in \text{Mod}(\mathcal{O}_X)$ locally free? At which points $p \in X$ is the morphism $X \to S$ smooth?

Problem 5.2. Verify the claim from the lecture that the morphism $\text{Spec}(\mathbb{F}_p(t)) \to \text{Spec}(\mathbb{F}_p(t^p))$ is neither smooth nor unramified. More generally, let $X$ be a scheme over a field $k$. Show that the following properties are equivalent:

(a) $X \to \text{Spec}(k)$ is étale.
(b) $X \to \text{Spec}(k)$ is unramified.
(c) $X \simeq \bigsqcup_{i \in I} \text{Spec}(k_i)$ for finite separable field extensions $k_i \supseteq k$.

Problem 5.3. Let $X, Y$ be two locally Noetherian integral schemes and $f: Y \to X$ a dominant morphism of finite type such that the extension $k(X) \subseteq k(Y)$ of the corresponding function fields is separable (resp. finite and separable).

(a) Show that there is an open dense $V \subset Y$ such that $f|_V: V \to X$ is smooth (resp. étale).
(b) Show that if moreover the morphism $f$ is proper and the fiber $f^{-1}(\eta) \to \text{Spec}(k(\eta))$ over the generic point $\eta \in X$ is smooth, then one may find an open dense subset $U \subset X$ such that the restriction $f|_{f^{-1}(U)}: f^{-1}(U) \to U$ is smooth (resp. étale).

Problem 5.4. Let $f: X \to S$ be a morphism of schemes, and let $\text{Sch}_S$ be the category of schemes over $S$. Consider the functor

$$T_{X/S}: \text{Sch}_S^{\text{op}} \to \text{Sets}, \quad Z \mapsto \text{Hom}_S(Z[\varepsilon], X)$$

where $Z[\varepsilon] = Z \times \text{Spec} \mathbb{Z}[\varepsilon]$ denotes the trivial first order thickening of $Z$ (with $\varepsilon^2 = 0$).

(a) If $X = \text{Spec}(B)$ and $S = \text{Spec}(A)$ are affine, show that the functor $T_{X/S}$ is represented by

$$T_{X/S} = \text{Spec}(\text{Sym}^\bullet_B(\Omega^1_{B/A})) \in \text{Sch}_S.$$

(b) Generalize your result to morphisms between arbitrary (not necessarily affine) schemes.
Problem 6.1. Let $X, Y$ be schemes of finite type over a field $k$.

(a) Show that if $k$ is algebraically closed and $X$ and $Y$ are regular, then $X \times_k Y$ is regular.

(b) Illustrate by an example that this fails in general if $k$ is not algebraically closed.

Problem 6.2. Let $p > 0$ be a prime.

(a) Let $k$ be a field of characteristic $p$, and let $a \in k$ be an element which is not a $p$-th power in $k$. Show that the curve $X = V(x^p + y^p - a) \subset \mathbb{A}^2_k$ is not smooth over $k$, but normal, hence regular. Do the same for $Y = V(x^p + y^2 - a)$.

(b) Is the scheme $Z = \text{Spec} \mathbb{Z}[x, y]/(xy - p)$ regular? Is it smooth over $\text{Spec} \mathbb{Z}$?

Problem 6.3. Let $f: X \to Y$ be a morphism of schemes. Fix a point $x \in X$, and let $y = f(x)$.

(a) Show that for the fiber $X_y = X \times_Y \text{Spec}(\kappa(y))$ we have an exact sequence of $\kappa(x)$-vector spaces:
$$0 \to T_{x,X} \to T_x X \to \kappa(x) \otimes_{\kappa(y)} T_y Y$$

(b) Suppose now that $f$ has a section $g: Y \to X$ with $x = g(y)$. Show:
- We have $\kappa(x) = \kappa(y)$ and $T_x X \simeq T_x X_y \oplus T_y Y$.
- If $X, Y$ are regular and locally Noetherian and $f$ is of finite type, then $f$ is smooth at $x$ and
$$\hat{\partial}_{X,x} \simeq \hat{\partial}_{Y,y}[t_1, \ldots, t_d] \quad \text{where} \quad d = \dim_x X_y.$$

Problem 6.4. Let $k$ be a field. By smooth we mean smooth over $k$.

(a) Let $f_1, \ldots, f_m \in k[x_0, \ldots, x_n]$ be homogenous. Show that the closed subscheme
$$X = V_1(f_1, \ldots, f_m) \subset \mathbb{P}^n_k = \text{Proj} k[x_0, \ldots, x_n]$$
is smooth at a point $p \in X(k)$ if and only if $\text{rk}(\partial f_i/\partial x_j)(p) = n - \dim_p X$.

(b) Let $X_0 = V(y^2 - g(x)) \subset \mathbb{A}^2_k$. For which $g \in k[x]$ is the closure $X = \overline{X}_0 \subset \mathbb{P}^2_k$ smooth?
Problem 7.1.

(a) Is the \( \mathbb{Z} \)-module \( M = \mathbb{Q} \) flat? Is it projective? Is it free?

(b) Let \( A = \prod_{i \in \mathbb{N}} \mathbb{F}_2 \). Show that the \( A \)-module

\[
M = \prod_{i \in \mathbb{N}} \mathbb{F}_2 / \bigoplus_{i \in \mathbb{N}} \mathbb{F}_2
\]

is finitely generated and flat over \( A \), but not projective over \( A \).

(c) Is the morphism \( \text{Spec}(\mathbb{C}[x,y,z,w]/(z,w) \cap (x+z,y+w)) \to \text{Spec}(\mathbb{C}[x,y]) \) flat? Describe its fibers and the irreducible components of its source.

Problem 7.2. Let \( f: X \to Y \) be a finite morphism, with \( Y \) Noetherian.

(a) Show that \( f \) is flat if and only if \( f^*(\mathcal{O}_X) \) is locally free.

(b) Show that if \( Y \) is integral, this is equivalent to \( \dim_{\kappa(y)}(f^*(\mathcal{O}_X) \otimes_{\mathcal{O}_{\text{Spec} \mathcal{O}_X}} \kappa(y)) = \text{constant} \).

(c) Deduce that the normalization of an integral Noetherian non-normal scheme is not flat.

Problem 7.3. Let \( f: Y \to X \) be a morphism of schemes, with \( X \) integral. Let \( Z_1, Z_2 \subset Y \) be two closed subschemes which coincide over an open dense subscheme \( U \subset X \) in the sense that \( Z_1 \cap f^{-1}(U) = Z_2 \cap f^{-1}(U) \) as closed subschemes of \( f^{-1}(U) \).

(a) Show that if \( Z_1, Z_2 \) are both flat over \( X \), then \( Z_1 = Z_2 \).

(b) Illustrate by an example that this may fail if \( Z_1, Z_2 \) are not both flat over \( X \).

Problem 7.4. Let \( f: Y \to X \) be a flat morphism of finite type, where \( X \) and \( Y \) are Noetherian schemes. Show in the following steps that \( f \) is open:

(a) Show that \( f(Y) \) contains a non-empty open subset \( U \subset X \).

(b) Show that \( f(Y) \setminus U \) contains a non-empty open subset of \( Z = X \setminus U \).

(c) Show that \( X \setminus f(Y) \subset X \) is closed, using that \( |X| \) is Noetherian.
Problem 8.1. Show that

(a) for any commutative ring \( R \), the category \( \text{Mod}(R) \) has enough injectives.

(b) for any ringed space \( (X, \mathcal{O}_X) \), the category \( \text{Mod}(\mathcal{O}_X) \) has enough injectives.

Problem 8.2. Let \( (X, \mathcal{O}_X) \) be a ringed space and \( V \subset X \) a non-empty open subset.

(a) Show that \( j^* : \text{Mod}(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_V) \) sends injective objects to injective objects.

(b) Does the same hold for the left adjoint functor \( j_! : \text{Mod}(\mathcal{O}_V) \to \text{Mod}(\mathcal{O}_X) \)?

Problem 8.3. Let \( \mathcal{A} \) be an abelian category.

(a) Deduce from the universal property of the kernel and cokernel the snake lemma: Every commutative diagram

\[
\begin{array}{ccc}
A' & \rightarrow & A & \rightarrow & A'' & \rightarrow & 0 \\
\downarrow f & & \downarrow f & & \downarrow f'' & & \\
0 & \rightarrow & B' & \rightarrow & B & \rightarrow & B''
\end{array}
\]

in \( \mathcal{A} \) with exact rows induces an exact sequence

\[
\text{ker}(f') \rightarrow \text{ker}(f) \rightarrow \text{ker}(f'') \rightarrow \text{coker}(f') \rightarrow \text{coker}(f) \rightarrow \text{coker}(f'').
\]

(b) Deduce that any short exact sequence \( 0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0 \) of complexes in \( \mathcal{A} \) gives rise to a long exact sequence

\[
\cdots \rightarrow H^{i-1}(M^\bullet) \rightarrow H^i(K^\bullet) \rightarrow H^i(L^\bullet) \rightarrow H^i(M^\bullet) \rightarrow H^{i+1}(K^\bullet) \rightarrow \cdots
\]

Problem 8.4. A morphism of complexes in an abelian category is called a quasi-isomorphism if it induces an isomorphism between their cohomology groups in each degree.

(a) Let \( \mathcal{A} = \text{AbGps} \) be the abelian category of abelian groups. Show that for every bounded above complex \( K^\bullet \in \text{C}^- \mathcal{A} \) of free abelian groups \( K^i \) there exists a quasi-isomorphism between the complex and the direct sum of its cohomology groups:

\[
K^\bullet \sim \rightarrow [\cdots \to H^i(K^\bullet) \to H^{i+1}(K^\bullet) \to \cdots]
\]

(b) Find an abelian category \( \mathcal{A} \) and complexes \( K^\bullet, L^\bullet \in \text{C}^- \mathcal{A} \) which have isomorphic cohomology groups but no quasi-isomorphism between them in either direction.
Problem 9.1. Show that for Noetherian schemes $X$, the following are equivalent:

(a) The scheme $X$ is affine.

(b) The underlying reduced closed subscheme $X^{\text{red}}$ is affine.

(c) Every irreducible component of $X$ is affine.

Problem 9.2. Let $k$ be a field and $X = \mathbb{A}^n_k \setminus \{0\}$ for $n \in \mathbb{N}$. Compute $H^\bullet(X, \mathcal{O}_X)$.

Problem 9.3. Let $\mathcal{A}$ be an abelian category with enough injectives, and let $A, B \in \mathcal{A}$.

(a) Put $\text{Ext}^1_{\mathcal{A}}(B, -) := R^1\text{Hom}_{\mathcal{A}}(B, -)$. Show that $\text{Ext}^1_{\mathcal{A}}(B, A)$ is in natural bijection with the set of isomorphism classes of short exact sequences $0 \to A \to E \to B \to 0$ in $\mathcal{A}$, where two such sequences with middle terms $E, E'$ are called isomorphic if they fit in a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & A & \to & E & \to & B & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & A & \to & E' & \to & B & \to & 0
\end{array}
\]

Hint: Embed $A$ into an injective $I \in \mathcal{A}$ and consider fiber products $E = I \times_Q B$.

(b) Describe the group structure on $\text{Ext}^1_{\mathcal{A}}(B, A) = R^1\text{Hom}_{\mathcal{A}}(B, A)$ in these terms.

Problem 9.4. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{G} \in \text{Mod}(\mathcal{O}_X)$.

(a) A $\mathcal{G}$-torsor is a sheaf $\mathcal{F}$ of sets on $X$ with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that

- the open sets $U \subset X$ with $\mathcal{F}(U) \neq \emptyset$ form a basis for the topology of $X$, and
- for all such open sets, the action of the group $\mathcal{G}(U)$ on $\mathcal{F}(U)$ is simply transitive.

Show that $\mathcal{G}$-torsors form a category $T(\mathcal{G})$ in which every morphism is an isomorphism.

(b) Let $E(\mathcal{G})$ be the category of exact sequences $0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{O}_X \to 0$ in $\text{Mod}(\mathcal{O}_X)$, with morphisms the isomorphisms of short exact sequences as in problem 9.3. Show that we have an equivalence of categories

\[
\tau: E(\mathcal{G}) \xrightarrow{\sim} T(\mathcal{G}), \quad \tau(0 \to \mathcal{G} \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{O}_X \to 0) := g^{-1}(1).
\]

Hint: First show full faithfulness (locally), and deduce essential surjectivity via gluing.

(c) Show that the set $\text{Tors}(\mathcal{G})$ of isomorphism classes of $\mathcal{G}$-torsors forms a group and we have isomorphisms

\[
H^1(X, \mathcal{G}) \cong \text{Ext}^1_{\text{Mod}(\mathcal{O}_X)}(\mathcal{O}_X, \mathcal{G}) \cong \text{Tors}(\mathcal{G})
\]
Problem 10.1. Let $X = \mathbb{P}^n_k$ for a field $k$. Show via the Euler sequence that

$$\dim_k H^i(X, \wedge^j \Omega^1_{X/k}) = \begin{cases} 1 & \text{if } 0 \leq i = j \leq n, \\ 0 & \text{else}. \end{cases}$$

Deduce that for $n > 1$ the coherent sheaf $\Omega^1_{X/k}$ is not a successive extension of line bundles.

Problem 10.2. Let $k$ be an algebraically closed field. Consider a plane curve $X = V_+(f) \subset \mathbb{P}^2_k$ which is cut out by a homogenous polynomial $f \in H^0(\mathbb{P}^2_k, \mathcal{O}(d))$ of degree $d > 0$. Show that we have

$$\dim_k H^1(X, \mathcal{O}_X) = \frac{(d-1)(d-2)}{2}.$$

Problem 10.3. Let $k$ be an algebraically closed field, and $X \subset Y = \mathbb{P}^3_k$ a proper integral curve which is a complete intersection of two homogenous polynomials, i.e. $X = V_+(f_1, f_2) \subset Y$ for suitable $f_i \in H^0(Y, \mathcal{O}_Y(d))$ such that the multiplication map

$$\mathcal{O}_{V_+(f_1)} \xrightarrow{f_2} \mathcal{O}_{V_+(f_1)} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(d_2)$$

is injective. Show that we have an exact sequence

$$0 \rightarrow \mathcal{O}_Y(-d_1 - d_2) \rightarrow \mathcal{O}_Y(-d_1) \oplus \mathcal{O}_Y(-d_2) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_X \rightarrow 0.$$

Deduce that

$$\dim_k H^1(X, \mathcal{O}_X) = \binom{d_1 + d_2 - 1}{3} - \binom{d_1 - 1}{3} - \binom{d_2 - 1}{3}$$

and conclude that there exist projective curves that cannot be embedded in the plane.

Problem 10.4. Let $X$ be a projective scheme over a Noetherian ring and $\mathcal{F}^1 \rightarrow \mathcal{F}^2 \rightarrow \cdots \rightarrow \mathcal{F}^r$ an exact sequence of coherent sheaves on $X$. Show that there exists an integer $n_0$ such that for all $n \geq n_0$ the following sequence remains exact:

$$\Gamma(X, \mathcal{F}^1(n)) \rightarrow \Gamma(X, \mathcal{F}^2(n)) \rightarrow \cdots \rightarrow \Gamma(X, \mathcal{F}^r(n)).$$
Problem 11.1. Let $k$ be a field, and let $X = V_+(f_1, \ldots, f_r) \subset \mathbb{P}^n_k$ be a complete intersection of dimension $\dim(X) = n - r > 0$ cut out by a sequence of hypersurfaces $f_i \in H^0(\mathbb{P}^n_k, \mathcal{O}(d_i))$ of degrees $d_1, \ldots, d_r \geq 1$. Show:

(a) The restriction maps $H^0(\mathbb{P}^n_k, \mathcal{O}(m)) \to H^0(X, \mathcal{O}(m)|_X)$ are surjective for all $m$.

(b) The scheme $X$ is geometrically connected.

(c) If $X$ is smooth over $k$, then $\omega_X := \bigwedge^{n-r} \Omega^1_X \simeq \mathcal{O}(d_1 + \cdots + d_r - n - 1)|_X$.

Problem 11.2. Let $X$ be a projective 1-dimensional scheme over an algebraically closed field which contains a cycle of curves in the sense that for some integer $n \geq 2$ there are pairwise distinct 1-dimensional integral closed subschemes $C_1, \ldots, C_n \subset X$ and pairwise distinct closed points $p_1, \ldots, p_n \in X$ with $p_i \in C_i \cap C_{i+1 \mod n}$ for all $i \in \{1, \ldots, n\}$. Show that $H^1(X, \mathcal{F}) \neq 0$ for the image $\mathcal{F} = \text{im}(\mathcal{O}_X \to \bigoplus_{i=1}^n \mathcal{O}_{C_i})$, and deduce that $H^1(X, \mathcal{O}_X) \neq 0$.

Problem 11.3. Let $X = \mathbb{P}^1_k$ for an infinite field $k$.

(a) Show that $\mathcal{O}_X$ cannot be written as a quotient of a projective object in $\text{Mod}(\mathcal{O}_X)$.

Hint: Consider $j_! (\mathcal{O}_U) \to i_* (\kappa(p))$ for $j: U \hookrightarrow X$ open and a closed point $i: \{p\} \hookrightarrow X$.

(b) Show that $\mathcal{O}_X$ cannot be written as a quotient of a projective object in $\text{QCoh}(X)$.

Hint: Consider $L \to L \otimes \kappa(p)$ for a line bundle $L \in \text{Pic}(X)$ and a closed point $p \in X$.

Problem 11.4. Let $X$ be a noetherian scheme, and assume that $\text{Coh}(X)$ has enough locally free sheaves. For any coherent sheaf $\mathcal{F} \in \text{Coh}(X)$, define its homological dimension $\text{hd}(\mathcal{F})$ by

$$\text{hd}(\mathcal{F}) := \min \{ n \in \mathbb{N}_0 \mid \exists \text{ locally free resolution } 0 \to \mathcal{L}_n \to \cdots \to \mathcal{L}_0 \to \mathcal{F} \to 0 \} \cup \{\infty\}.$$ 

Show:

(a) $\mathcal{F}$ is locally free iff $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = 0$ for all $\mathcal{G} \in \text{Mod}(\mathcal{O}_X)$.

(b) $\text{hd}(\mathcal{F}) \leq n$ if $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = 0$ for all $i > n$ and all $\mathcal{G} \in \text{Mod}(\mathcal{O}_X)$.

(c) $\text{hd}(\mathcal{F}) = \sup_{x \in X} \text{pd}(\mathcal{F}_x)$ (where $\text{pd}: \text{Mod}(\mathcal{O}_{X,x}) \to \mathbb{N}_0 \cup \{\infty\}$ denotes the projective dimension, i.e. the smallest length of a resolution by projective modules over $\mathcal{O}_{X,x}$).
Problem 12.1. Let $X$ be a Cohen-Macaulay scheme, and denote by $\pi : \tilde{X} = \text{Bl}_Z(X) \to X$ its blowup along a complete intersection $Z \subset X$.

(a) Show that $E = \pi^{-1}(Z)$ is Cohen-Macaulay, and deduce that $\tilde{X}$ is Cohen-Macaulay.

(b) Let $f : X \to S$ be a morphism to a regular scheme $S$ such that all fibers of $f$ are equidimensional of the same dimension. Fix a closed point $\tilde{x} \in E$ with images $x = \pi(\tilde{x})$ and $s = f(x)$. Show that the following two properties are equivalent:

\begin{itemize}
  \item $f \circ \pi : \tilde{X} \to S$ is flat at $\tilde{x}$.
  \item $\dim_x(Z) - \dim_x(Z \cap X_s) \geq \dim_s(S) - 1$.
\end{itemize}

Problem 12.2. Let $X$ be a smooth variety of dimension $n$ over a field $k$. Let $\pi : \tilde{X} \to X$ be its blowup in a point $p \in X(k)$, and let $E = \pi^{-1}(p) \cong \mathbb{P}^{n-1}_k$ be the exceptional divisor.

(a) Show that the normal bundle of $E \subset \tilde{X}$ is isomorphic to $\mathcal{O}_E(-1)$.

(b) Let $\mathcal{M} := \omega_{\tilde{X}} \otimes \pi^*(\omega_Y^\vee) \in \text{Pic}(\tilde{X})$. Show that the restriction $\mathcal{M}|_{\tilde{X} \setminus E}$ is trivial, and deduce that $\omega_{\tilde{X}} \cong \pi^*(\omega_X) \otimes \mathcal{O}_{\tilde{X}}((n-1)E)$.

Problem 12.3. Let $f : X \to Y$ be a finite morphism of Noetherian schemes.

(a) Let $\mathcal{G} \in \text{Coh}(X)$. Show that up to isomorphism there exists a unique $f^!(\mathcal{G}) \in \text{Coh}(X)$ with $f_*(f^!(\mathcal{G})) \cong \mathcal{H}om_Y(f_*(\mathcal{O}_X), \mathcal{G})$ in $\text{Mod}(f_*(\mathcal{O}_X))$.

(b) Show that we have $f_*\mathcal{H}om(\mathcal{F}, f^!(\mathcal{G})) \cong \mathcal{H}om_Y(f_*(\mathcal{F}), \mathcal{G})$ for all $\mathcal{F} \in \text{Coh}(X)$.

Problem 12.4. Let $X$ be a projective scheme of pure dimension $n$ over a field $k$.

(a) Let $f : X \to Y$ be a finite flat morphism to another projective scheme over $k$. Show that for any dualizing sheaf $(\omega_Y^\vee, \text{tr}_Y)$ on $Y$, we get a dualizing sheaf $(\omega_X^\vee, \text{tr}_X)$ on $X$ by taking $\omega_X^\vee := f^!(\omega_Y^\vee)$ (with notations as in the previous problem), endowed with the trace map

$$\text{tr}_X : H^n(X, f^!(\omega_Y^\vee)) \cong H^n(Y, f_*f^!(\omega_Y^\vee)) \to H^n(Y, \omega_Y^\vee) \cong k.$$

(b) Use this to construct on any projective Cohen-Macaulay scheme $X$ of pure dimension $n$ over $k$ a dualizing sheaf, by taking $f : X \to Y = \mathbb{P}^n$ to be a suitable projection.