
Problem 1.1. Let k be a field.

(a) Let R be a k-algebra which is a domain, not necessarily of finite type over k. Show that its Krull dimension is at most the transcendence degree of the quotient field of R, i.e. we have

$$\dim R \ \leq \ \mathrm{trdeg}_k(\mathrm{Quot}(R)).$$

(b) Illustrate by an example that in general the above inequality can be strict.

Problem 1.2. Let k be a field. Let C be a normal curve over k that is birational to $\mathbb{P}^1_k$.

(a) Show that C is isomorphic to an open subset of $\mathbb{P}^1_k$.

(b) Deduce that if C is not proper, then it is affine and $\Gamma(C, \mathcal{O}_C)$ is a factorial ring.

Problem 1.3. Let k be an algebraically closed field with $\mathrm{char}(k) \neq 2$.

(a) Show that $A = k[x,y]/(y^2 - x^3 - x)$ is a domain which is integral over $R = k[x]$.

(b) Define an automorphism $\sigma\colon A \to A$ by $\sigma_{|R} = \mathrm{id}$ and $\sigma(y) = -y$. Show that we have a multiplicative map

$$N\colon \quad A \ \to \ R, \quad N(a) \ := \ a \cdot \sigma(a).$$

(c) Show that $A^\times = k^\times$ and that x, y are irreducible elements in A. Deduce that A is not factorial, and that $\mathrm{Spec}(A)$ is not isomorphic to an open subset of the affine line over k.

Problem 1.4. Let $C \subset \mathbb{P}^2_k$ be the projective closure of $C_0 = \mathrm{Spec}(A) \subset \mathbb{A}^2_k$ from problem 1.3.

(a) Show that C is normal and the complement $C \setminus C_0$ consists of a single point o.

(b) Let $p, q, r \in C$ be distinct closed points. Show that the following are equivalent:

- There is a rational function $f \in k(C)^\times$ with $\mathrm{div}(f) = [p] + [q] + [r] - 3[o]$.
- The three points p, q, r are collinear, i.e. they lie on a common line $\ell \subset \mathbb{P}^2_k$.

(c) What does this say about the Picard group $\mathrm{Pic}(C)$?

Problem 2.1. Let k be a field.

(a) Show that for any finite set of closed points on a proper normal curve C over k, there exists a rational function $f \in k(C)^\times$ which has poles only in the points from that set.

(b) Deduce that every non-proper normal curve C_0 over k is affine.

Hint: Find a proper normal curve $C \supset C_0$ and $f\colon C \to \mathbb{P}^1_k$ with $f^{-1}(\mathbb{A}^1_k) = C_0$.

Problem 2.2. Let $f\colon C' \to C$ be a morphism of normal curves over an alg. closed field k.

(a) Check that we have a natural morphism $f^*(\Omega^1_{C/k}) \to \Omega^1_{C'/k}$ of line bundles.

(b) Now assume that f is finite. Show by looking at stalks at the generic point that the above morphism is injective iff the extension of function fields $k(C) \hookrightarrow k(C')$ is separable.

(c) Show that in this case:

- The quotient $\Omega^1_{C'/C} := \Omega^1_{C'/k}/f^*(\Omega^1_{C/k})$ is a skyscraper sheaf, and

$$\Omega^1_{C'/k} \;\simeq\; f^*(\Omega^1_{C/k}) \otimes_{\mathcal{O}_{C'}} \mathcal{O}_{C'}(R)$$

  for the *ramification divisor* $R := \sum_{p\in C'} \mathrm{length}_{\mathcal{O}_{C',p}}(\Omega^1_{C'/C,p}) \cdot [p] \in \mathrm{Div}(C')$.

- The genus $g_{C'}$ of the covering curve is related to the genus g_C of the target curve via

$$2g_{C'} - 2 \;=\; \deg(f) \cdot (2g_C - 2) + \deg(R).$$

- If $\mathrm{char}(k) \nmid e_p(f)$ for all closed points $p \in C'$, then $\deg(R) \;=\; \sum_{p\in C'}(e_p(f) - 1)$.

Problem 2.3. Let k be an algebraically closed field with $\mathrm{char}(k) \neq 2$.

(a) Let C be the proper normal compactification of the affine curve $V(y^2 - h(x)) \subset \mathbb{A}^2_k$ for a square-free polynomial

$$h(x) \;\in\; k[x] \quad \text{of degree} \quad \deg(h) \;=\; 2g + 2 \;\geq 4.$$

How many 'points at infinity' (i.e. points in the complement $C \setminus C_0$) are there?

(b) Show that there exists a morphism $f\colon C \to \mathbb{P}^1_k$ of degree two. Determine its ramification divisor, and deduce from the previous problem that the curve C has genus g.

(c) Consider the open dense subset $U = D(y) \cap V(y^2 - h(x)) \subset C$. Show that the differential forms

$$\tfrac{dx}{y},\; x\tfrac{dx}{y},\; x^2\tfrac{dx}{y},\; \dots,\; x^{g-1}\tfrac{dx}{y} \;\in\; H^0(U, \Omega^1_{C/k})$$

extend uniquely to global sections of $\Omega^1_{C/k}$ and form a basis of $H^0(C, \Omega^1_{C/k})$ over k.

Problem 3.1. Let X be a scheme over a field k, and $\mathscr{L} \in \mathrm{Pic}(X)$. Let $s_0, \dots, s_m \in \Gamma(X, \mathscr{L})$ and $t_0, \dots, t_n \in \Gamma(X, \mathscr{L})$ be two sets of sections that span the same basepoint-free linear series

$$V = \langle s_0, \dots, s_m \rangle_k = \langle t_0, \dots, t_n \rangle_k \subset \Gamma(X, \mathscr{L}).$$

Show that for $m \geq n$ the morphisms

$$f := [s_0 : \cdots : s_m] \colon X \longrightarrow \mathbb{P}^m_k$$
$$g := [t_0 : \cdots : t_n] \colon X \longrightarrow \mathbb{P}^n_k$$

satisfy $g = p \circ f$ where $p \colon \mathbb{P}^m_k \dashrightarrow \mathbb{P}^n_k$ is a linear projection followed by an automorphism.

Problem 3.2. Let k be a field and $X = \mathbb{P}^2_k$ with homogenous coordinates $x, y, z \in \Gamma(X, \mathscr{O}_X(1))$.

(a) Show that $\mathscr{L} = \mathscr{O}_X(2)$ is very ample and $V = \langle x^2, y^2, z^2, y(x-z), (x-y)z \rangle \subset \Gamma(X, \mathscr{L})$ defines a closed immersion

$$\varphi_{\mathscr{L},V} \colon \quad X \hookrightarrow \mathbb{P}^4_k.$$

(b) Let $p \in X$ be a closed point. Show that the linear series $W = \{s \in \Gamma(X, \mathscr{L}) \mid s(p) = 0\}$ gives a locally closed immersion $\varphi_{\mathscr{L},W} \colon X \setminus p \hookrightarrow \mathbb{P}^4_k$ which after blowing up the point p extends to a closed immersion

$$\tilde{\varphi}_{\mathscr{L},W} \colon \quad \tilde{X} = \mathrm{Bl}_p(X) \hookrightarrow \mathbb{P}^4_k.$$

Show moreover that $\tilde{X} \subset \mathbb{P}^4_k$ is a surface which is covered by a family of disjoint lines in $\mathbb{P}^4_k$ which are the proper transforms of the lines through the point p in $X = \mathbb{P}^2_k$.

Problem 3.3. Let X be a scheme of finite type over a Noetherian ring R.

(a) Show that if there exists an ample line bundle on X, then X is separated.

(b) Now let X be the affine line over a field k with the origin doubled.

- Compute the group $\mathrm{Pic}(X)$. Which line bundles are globally generated?
- Verify directly (without using (a)) that there are no ample line bundles on X.

Problem 3.4. Let k be a field, and let $X = \mathrm{Proj}(R)$ for a graded k-algebra R generated by finitely many elements of degree one. Show that we have a functor

$$\Gamma_* \colon \quad \mathrm{QCoh}(X) \to \mathrm{GrMod}(R), \quad \mathscr{M} \mapsto \Gamma_*(\mathscr{M}) := \bigoplus_{d \geq 0} \Gamma(X, \mathscr{M} \otimes \mathscr{O}_X(d))$$

and that every $\mathscr{M} \in \mathrm{QCoh}(X)$ arises as $\mathscr{M} \simeq \tilde{M}$ for the graded R-module $M := \Gamma_*(\mathscr{M})$.

Problem 4.1. Verify for the spectrum $X = \operatorname{Spec}(R)$ of the local ring $R = (\mathbb{C}[s,t]/(s^2 - t^3))_{(s,t)}$ that

$$\operatorname{Pic}(X) \simeq 0 \quad \text{but} \quad \operatorname{Pic}(X \times \mathbb{A}^1) \not\simeq 0.$$

Problem 4.2. Let k be a field. Consider the hypersurface $Z = V_+(f) \subset \mathbb{P}^n_k$ for an irreducible homogenous polynomial $f \in k[x_0, \dots, x_n]$ of degree $d > 1$, and denote by $H_0, \dots, H_n \subset \mathbb{P}^n_k$ the coordinate hyperplanes.

(a) Show that $U := \mathbb{P}^n_k \setminus Z$ is affine and covered by affine open subsets $U_i := U \setminus H_i$.

(b) Show that $\operatorname{Pic}(U) \not\simeq 0$ but $\operatorname{Pic}(U_i) \simeq 0$ for $i = 0, 1, \dots, n$.

Hence an affine scheme with a cover by spectra of UFD's need not be the spectrum of a UFD.

Problem 4.3. Consider the normalization $\pi \colon \widetilde{C} \to C$ of an integral proper curve C over an algebraically closed field k. For closed points $p \in C$, let $\widetilde{\mathscr{O}}_{C,p}$ be the normalization of the corresponding local ring. Denote by $\mathscr{K}_C$ the sheaf of total quotient rings of the curve. Deduce from the short exact sequence of sheaves

$$0 \longrightarrow \pi_*(\mathscr{O}^\times_{\widetilde{C}})/\mathscr{O}^\times_C \longrightarrow \mathscr{K}^\times_C/\mathscr{O}^\times_C \longrightarrow \mathscr{K}^\times_C/\pi_*(\mathscr{O}^\times_{\widetilde{C}}) \longrightarrow 0$$

that we have a short exact sequence of abelian groups

$$0 \longrightarrow \bigoplus_{p \in C} \widetilde{\mathscr{O}}^\times_{C,p}/\mathscr{O}^\times_{C,p} \longrightarrow \operatorname{Pic}(C) \longrightarrow \operatorname{Pic}(\widetilde{C}) \longrightarrow 0.$$

Problem 4.4. Let k be an algebraically closed field. Deduce from the previous exercise:

(a) The Picard group of the cuspidal cubic $C = V(y^2 z - x^3) \subset \mathbb{P}^2_k = \operatorname{Proj} k[x,y,z]$ fits in an exact sequence

$$0 \to \mathbb{G}_a \longrightarrow \operatorname{Pic}(C) \longrightarrow \operatorname{Pic}(\widetilde{C}) \to 0 \quad \text{for the additive group } \mathbb{G}_a = (k, +).$$

(b) The Picard group of the nodal cubic $C = V(y^2 z - x^2(x+z)) \subset \mathbb{P}^2_k$ fits in an exact sequence

$$0 \to \mathbb{G}_m \longrightarrow \operatorname{Pic}(C) \longrightarrow \operatorname{Pic}(\widetilde{C}) \to 0 \quad \text{for the multiplicative group } \mathbb{G}_m = (k^\times, \cdot).$$

Problem 5.1. Describe $\Omega^1_{X/S}$ for each of the following morphisms:

(a) $X = \mathrm{Spec}(\mathbb{Z}[i]) \to S = \mathrm{Spec}(\mathbb{Z})$,

(b) $X = \mathrm{Spec}(\mathbb{C}[x,y]/(xy)) \to S = \mathrm{Spec}(\mathbb{C})$,

(c) $X = \mathrm{Bl}_p(S) \to S = \mathbb{A}^2_{\mathbb{C}}$, the blowup of the plane in the point $p = (0,0) \in \mathbb{A}^2(\mathbb{C})$.

Is $\Omega^1_{X/S} \in \mathrm{Mod}(\mathscr{O}_X)$ locally free? At which points $p \in X$ is the morphism $X \to S$ smooth?

Problem 5.2. Verify the claim from the lecture that the morphism $\mathrm{Spec}(\mathbb{F}_p(t)) \to \mathrm{Spec}(\mathbb{F}_p(t^p))$ is neither smooth nor unramified. More generally, let X be a scheme over a field k. Show that the following properties are equivalent:

(a) $X \to \mathrm{Spec}(k)$ is étale.

(b) $X \to \mathrm{Spec}(k)$ is unramified.

(c) $X \simeq \bigsqcup_{i \in I} \mathrm{Spec}(k_i)$ for finite separable field extensions $k_i \supseteq k$.

Problem 5.3. Let X, Y be two locally Noetherian integral schemes and $f\colon Y \to X$ a dominant morphism of finite type such that the extension $k(X) \subset k(Y)$ of the corresponding function fields is separable (resp. finite and separable).

(a) Show that there is an open dense $V \subset Y$ such that $f|_V\colon V \to X$ is smooth (resp. étale).

(b) Show that if moreover the morphism f is proper and the fiber $f^{-1}(\eta) \to \mathrm{Spec}(k(\eta))$ over the generic point $\eta \in X$ is smooth, then one may find an open dense subset $U \subset X$ such that the restriction $f|_{f^{-1}(U)}\colon f^{-1}(U) \to U$ is smooth (resp. étale).

Problem 5.4. Let $f\colon X \to S$ be a morphism of schemes, and let Sch_S be the category of schemes over S. Consider the functor

$$T_{X/S}\colon \quad \mathrm{Sch}_S^{\mathrm{op}} \longrightarrow \mathrm{Sets}, \quad Z \mapsto \mathrm{Hom}_S(Z[\varepsilon], X)$$

where $Z[\varepsilon] = Z \times \mathrm{Spec}\,\mathbb{Z}[\varepsilon]$ denotes the trivial first order thickening of Z (with $\epsilon^2 = 0$).

(a) If $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ are affine, show that the functor $T_{X/S}$ is represented by

$$T_{X/S} \;=\; \mathrm{Spec}(\mathrm{Sym}^\bullet_B(\Omega^1_{B/A})) \;\in\; \mathrm{Sch}_S.$$

(b) Generalize your result to morphisms between arbitrary (not necessarily affine) schemes.

Problem 6.1. Let X, Y be schemes of finite type over a field k.

(a) Show that if k is algebraically closed and X and Y are regular, then $X \times_k Y$ is regular.

(b) Illustrate by an example that this fails in general if k is not algebraically closed.

Problem 6.2. Let $p > 0$ be a prime.

(a) Let k be a field of characteristic p, and let $a \in k$ be an element which is not a p-th power in k. Show that the curve

$$X \;=\; V(x^p + y^p - a) \;\subset\; \mathbb{A}^2_k$$

is not smooth over k, but normal, hence regular. Do the same for $Y = V(x^p + y^2 - a)$.

(b) Is the scheme $Z = \operatorname{Spec} \mathbb{Z}[x, y]/(xy - p)$ regular? Is is smooth over $\operatorname{Spec} \mathbb{Z}$?

Problem 6.3. Let $f\colon X \to Y$ be a morphism of schemes. Fix a point $x \in X$, and let $y = f(x)$.

(a) Show that for the fiber $X_y = X \times_Y \operatorname{Spec}(\kappa(y))$ we have an exact sequence of $\kappa(x)$-vector spaces:

$$0 \longrightarrow T_x X_y \longrightarrow T_x X \longrightarrow \kappa(x) \otimes_{\kappa(y)} T_y Y$$

(b) Suppose now that f has a section $g\colon Y \to X$ with $x = g(y)$. Show:

- We have $\kappa(x) = \kappa(y)$ and $T_x X \simeq T_x X_y \oplus T_y Y$.
- If X, Y are regular and locally Noetherian and f is of finite type, then f is smooth at x and

$$\widehat{\mathscr{O}}_{X,x} \;\simeq\; \widehat{\mathscr{O}}_{Y,y}[[t_1, \ldots, t_d]] \quad \text{where} \quad d = \dim_x X_y.$$

Problem 6.4. Let k be a field. By *smooth* we mean *smooth over k*.

(a) Let $f_1, \ldots, f_m \in k[x_0, \ldots, x_n]$ be homogenous. Show that the closed subscheme

$$X = V_+(f_1, \ldots, f_m) \subset \mathbb{P}^n_k = \operatorname{Proj} k[x_0, \ldots, x_n]$$

is smooth at a point $p \in X(k)$ if and only if $\operatorname{rk}(\partial f_i/\partial x_j)(p) = n - \dim_p X$.

(b) Let $X_0 = V(y^2 - g(x)) \subset \mathbb{A}^2_k$. For which $g \in k[x]$ is the closure $X = \overline{X}_0 \subset \mathbb{P}^2_k$ smooth?

Problem 7.1.

(a) Is the $\mathbb{Z}$-module $M = \mathbb{Q}$ flat? Is it projective? Is it free?

(b) Let $A = \prod_{i\in\mathbb{N}} \mathbb{F}_2$. Show that the A-module

$$M \;=\; \prod_{i\in\mathbb{N}} \mathbb{F}_2 \,/\, \bigoplus_{i\in\mathbb{N}} \mathbb{F}_2$$

is finitely generated and flat over A, but not projective over A.

(c) Is the morphism $\operatorname{Spec}(\mathbb{C}[x,y,z,w]/(z,w)\cap(x+z,y+w)) \to \operatorname{Spec}(\mathbb{C}[x,y])$ flat? Describe its fibers and the irreducible components of its source.

Problem 7.2. Let $f\colon X \to Y$ be a finite morphism, with Y Noetherian.

(a) Show that f is flat if and only if $f_*(\mathscr{O}_X)$ is locally free.

(b) Show that if Y is integral, this is equivalent to $\dim_{\kappa(y)}(f_*(\mathscr{O}_X)\otimes_{\mathscr{O}_{Y,y}}\kappa(y)) = \text{constant}$.

(c) Deduce that the normalization of an integral Noetherian non-normal scheme is not flat.

Problem 7.3. Let $f\colon Y \to X$ be a morphism of schemes, with X integral. Let $Z_1, Z_2 \subset Y$ be two closed subschemes which coincide over an open dense subscheme $U \subset X$ in the sense that $Z_1 \cap f^{-1}(U) = Z_1 \cap f^{-1}(U)$ as closed subsechemes of $f^{-1}(U)$.

(a) Show that if Z_1, Z_2 are both flat over X, then $Z_1 = Z_2$.

(b) Illustrate by an example that this may fail if Z_1, Z_2 are not both flat over X.

Problem 7.4. Let $f\colon Y \to X$ be a flat morphism of finite type, where X and Y are Noetherian schemes. Show in the following steps that f is open:

(a) Show that $f(Y)$ contains a non-empty open subset $U \subset X$.

(b) Show that $f(Y) \setminus U$ contains a non-empty open subset of $Z = X \setminus U$.

(c) Show that $X \setminus f(Y) \subset X$ is closed, using that $|X|$ is Noetherian.

Problem 8.1. Show that

(a) for any commutative ring R, the category $\mathrm{Mod}(R)$ has enough injectives.

(b) for any ringed space $(X, \mathscr{O}_X)$, the category $\mathrm{Mod}(\mathscr{O}_X)$ has enough injectives.

Problem 8.2. Let $(X, \mathscr{O}_X)$ be a ringed space and $V \subset X$ a non-empty open subset.

(a) Show that $j^* \colon \mathrm{Mod}(\mathscr{O}_X) \to \mathrm{Mod}(\mathscr{O}_V)$ sends injective objects to injective objects.

(b) Does the same hold for the left adjoint functor $j_! \colon \mathrm{Mod}(\mathscr{O}_V) \to \mathrm{Mod}(\mathscr{O}_X)$?

Problem 8.3. Let $\mathscr{A}$ be an abelian category.

(a) Deduce from the universal property of the kernel and cokernel the snake lemma: Every commutative diagram

$$\begin{array}{ccccccc} & & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & & \end{array}$$

in $\mathscr{A}$ with exact rows induces an exact sequence

$$\ker(f') \to \ker(f) \to \ker(f'') \to \mathrm{coker}(f') \to \mathrm{coker}(f) \to \mathrm{coker}(f'').$$

(b) Deduce that any short exact sequence $0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$ of complexes in $\mathscr{A}$ gives rise to a long exact sequence

$$\cdots \to H^{i-1}(M^\bullet) \to H^i(K^\bullet) \to H^i(L^\bullet) \to H^i(M^\bullet) \to H^{i+1}(K^\bullet) \to \cdots$$

Problem 8.4. A morphism of complexes in an abelian category is called a *quasi-isomorphism* if it induces an isomorphism between their cohomology groups in each degree.

(a) Let $\mathscr{A} = \mathrm{AbGps}$ be the abelian category of abelian groups. Show that for every bounded above complex $K^\bullet \in \mathscr{C}^-(\mathscr{A})$ of *free* abelian groups K^i there exists a quasi-isomorphism between the complex and the direct sum of its cohomology groups:

$$K^\bullet \xrightarrow{\sim} \Big[\cdots \xrightarrow{0} H^i(K^\bullet) \xrightarrow{0} H^{i+1}(K^\bullet) \xrightarrow{0} \cdots\Big]$$

(b) Find an abelian category $\mathscr{A}$ and complexes $K^\bullet, L^\bullet \in \mathscr{C}^-(\mathscr{A})$ which have isomorphic cohomology groups but no quasi-isomorphism between them in either direction.

Algebraic Geometry II — Problem Set 09

HU Berlin, Prof. Dr. T. Krämer — Due: January 12, 2023

Problem 9.1. Show that for Noetherian schemes X, the following are equivalent:

(a) The scheme X is affine.

(b) The underlying reduced closed subscheme X^{red} is affine.

(c) Every irreducible component of X is affine.

Problem 9.2. Let k be a field and $X = \mathbb{A}^n_k \setminus \{0\}$ for $n \in \mathbb{N}$. Compute $H^\bullet(X, \mathscr{O}_X)$.

Problem 9.3. Let $\mathscr{A}$ be an abelian category with enough injectives, and let $A, B \in \mathscr{A}$.

(a) Put $\mathrm{Ext}^i_{\mathscr{A}}(B, -) := R^i\mathrm{Hom}_{\mathscr{A}}(B, -)$. Show that $\mathrm{Ext}^1_{\mathscr{A}}(B, A)$ is in natural bijection with the set of isomorphism classes of short exact sequences $0 \to A \to E \to B \to 0$ in $\mathscr{A}$, where two such sequences with middle terms E, E' are called *isomorphic* if they fit in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & E' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

Hint: Embed A into an injective $I \in \mathscr{A}$ and consider fiber products $E = I \times_Q B$.

(b) Describe the group structure on $\mathrm{Ext}^1_{\mathscr{A}}(B, A) = R^1\mathrm{Hom}_{\mathscr{A}}(B, A)$ in these terms.

Problem 9.4. Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{G} \in \mathrm{Mod}(\mathscr{O}_X)$.

(a) A *$\mathscr{G}$-torsor* is a sheaf $\mathscr{F}$ of sets on X with an action $\mathscr{G} \times \mathscr{F} \to \mathscr{F}$ such that

- the open sets $U \subset X$ with $\mathscr{F}(U) \neq \varnothing$ form a basis for the topology of X, and
- for all such open sets, the action of the group $\mathscr{G}(U)$ on $\mathscr{F}(U)$ is simply transitive.

Show that $\mathscr{G}$-torsors form a category $T(\mathscr{G})$ in which every morphism is an isomorphism.

(b) Let $E(\mathscr{G})$ be the category of exact sequences $0 \to \mathscr{G} \to \mathscr{E} \to \mathscr{O}_X \to 0$ in $\mathrm{Mod}(\mathscr{O}_X)$, with morphisms the isomorphisms of short exact sequences as in problem 9.3. Show that we have an equivalence of categories

$$\tau: \quad E(\mathscr{G}) \xrightarrow{\sim} T(\mathscr{G}), \qquad \tau(\, 0 \to \mathscr{G} \xrightarrow{f} \mathscr{E} \xrightarrow{g} \mathscr{O}_X \to 0 \,) \; := \; g^{-1}(1).$$

Hint: First show full faithfulness (locally), and deduce essential surjectivity via gluing.

(c) Show that the set $\mathrm{Tors}(\mathscr{G})$ of isomorphism classes of $\mathscr{G}$-torsors forms a group and we have isomorphisms

$$H^1(X, \mathscr{G}) \;\simeq\; \mathrm{Ext}^1_{\mathrm{Mod}(\mathscr{O}_X)}(\mathscr{O}_X, \mathscr{G}) \;\simeq\; \mathrm{Tors}(\mathscr{G})$$

Problem 10.1. Let $X = \mathbb{P}^n_k$ for a field k. Show via the Euler sequence that

$$\dim_k H^i(X, \wedge^j \Omega^1_{X/k}) \;=\; \begin{cases} 1 & \text{if } 0 \le i = j \le n, \\ 0 & \text{else.} \end{cases}$$

Deduce that for $n > 1$ the coherent sheaf $\Omega^1_{X/k}$ is *not* a successive extension of line bundles.

Problem 10.2. Let k be an algebraically closed field. Consider a plane curve $X = V_+(f) \subset \mathbb{P}^2_k$ which is cut out by a homogenous polynomial $f \in H^0(\mathbb{P}^2_k, \mathscr{O}(d))$ of degree $d > 0$. Show that we have

$$\dim_k H^1(X, \mathscr{O}_X) \;=\; \frac{(d-1)(d-2)}{2}.$$

Problem 10.3. Let k be an algebraically closed field, and $X \subset Y = \mathbb{P}^3_k$ a proper integral curve which is a complete intersection of two homogenous polynomials, i.e. $X = V_+(f_1, f_2) \subset Y$ for suitable $f_i \in H^0(Y, \mathscr{O}_Y(d_i))$ such that the multiplication map

$$\mathscr{O}_{V_+(f_1)} \xrightarrow{f_2} \mathscr{O}_{V_+(f_1)} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y(d_2)$$

is injective. Show that we have an exact sequence

$$0 \longrightarrow \mathscr{O}_Y(-d_1 - d_2) \longrightarrow \mathscr{O}_Y(-d_1) \oplus \mathscr{O}_Y(-d_2) \longrightarrow \mathscr{O}_Y \longrightarrow \mathscr{O}_X \longrightarrow 0.$$

Deduce that

$$\dim_k H^1(X, \mathscr{O}_X) \;=\; \binom{d_1 + d_2 - 1}{3} - \binom{d_1 - 1}{3} - \binom{d_2 - 1}{3}$$

and conclude that there exist projective curves that cannot be embedded in the plane.

Problem 10.4. Let X be a projective scheme over a Noetherian ring and $\mathscr{F}^1 \to \mathscr{F}^2 \to \cdots \to \mathscr{F}^r$ an exact sequence of coherent sheaves on X. Show that there exists an integer n_0 such that for all $n \ge n_0$ the following sequence remains exact:

$$\Gamma(X, \mathscr{F}^1(n)) \to \Gamma(X, \mathscr{F}^2(n)) \to \cdots \to \Gamma(X, \mathscr{F}^r(n)).$$

Problem 11.1. Let k be a field, and let $X = V_+(f_1, \dots, f_r) \subset \mathbb{P}^n_k$ be a complete intersection of dimension $\dim(X) = n - r > 0$ cut out by a sequence of hypersurfaces $f_i \in H^0(\mathbb{P}^n_k, \mathscr{O}(d_i))$ of degrees $d_1, \dots, d_r \geq 1$. Show:

(a) The restriction maps $H^0(\mathbb{P}^n_k, \mathscr{O}(m)) \to H^0(X, \mathscr{O}(m)|_X)$ are surjective for all m.

(b) The scheme X is geometrically connected.

(c) If X is smooth over k, then $\omega_X := \wedge^{n-r}\Omega^1_X \simeq \mathscr{O}(d_1 + \cdots + d_r - n - 1)|_X$.

Problem 11.2. Let X be a projective 1-dimensional scheme over an algebraically closed field which contains a cycle of curves in the sense that for some integer $n \geq 2$ there are pairwise distinct 1-dimensional integral closed subschemes $C_1, \dots, C_n \subset X$ and pairwise distinct closed points $p_1, \dots, p_n \in X$ with

$$p_i \ \in \ C_i \cap C_{i+1 \bmod n}$$

for all $i \in \{1, \dots, n\}$. Show that $H^1(X, \mathscr{F}) \neq 0$ for the image $\mathscr{F} = \mathrm{im}(\mathscr{O}_X \to \bigoplus_{i=1}^n \mathscr{O}_{C_i})$, and deduce that

$$H^1(X, \mathscr{O}_X) \ \neq \ 0.$$

Problem 11.3. Let $X = \mathbb{P}^1_k$ for an infinite field k.

(a) Show that $\mathscr{O}_X$ cannot be written as a quotient of a projective object in $\mathrm{Mod}(\mathscr{O}_X)$.
Hint: Consider $j_!(\mathscr{O}_U) \twoheadrightarrow i_*(\kappa(p))$ for $j \colon U \hookrightarrow X$ open and a closed point $i \colon \{p\} \hookrightarrow X$.

(b) Show that $\mathscr{O}_X$ cannot be written as a quotient of a projective object in $\mathrm{QCoh}(X)$.
Hint: Consider $\mathscr{L} \twoheadrightarrow \mathscr{L} \otimes \kappa(p)$ for a line bundle $\mathscr{L} \in \mathrm{Pic}(X)$ and a closed point $p \in X$.

Problem 11.4. Let X be a noetherian scheme, and assume that $\mathrm{Coh}(X)$ has enough locally free sheaves. For any coherent sheaf $\mathscr{F} \in \mathrm{Coh}(X)$, define its homological dimension $\mathrm{hd}(\mathscr{F})$ by

$$\mathrm{hd}(\mathscr{F}) \ := \ \min \{n \in \mathbb{N}_0 \mid \exists \text{locally free resolution } 0 \to \mathscr{L}_n \to \cdots \to \mathscr{L}_0 \to \mathscr{F} \to 0\} \cup \{\infty\}.$$

Show:

(a) $\mathscr{F}$ is locally free iff $\mathscr{E}xt^1_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) = 0$ for all $\mathscr{G} \in \mathrm{Mod}(X)$.

(b) $\mathrm{hd}(\mathscr{F}) \leq n$ iff $\mathscr{E}xt^i_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) = 0$ for all $i > n$ and all $\mathscr{G} \in \mathrm{Mod}(X)$.

(c) $\mathrm{hd}(\mathscr{F}) = \sup_{x \in X} \mathrm{pd}(\mathscr{F}_x)$ (where $\mathrm{pd} \colon \mathrm{Mod}(\mathscr{O}_{X,x}) \to \mathbb{N}_0 \cup \{\infty\}$ denotes the projective dimension, i.e. the smallest length of a resolution by projective modules over $\mathscr{O}_{X,x}$).

Problem 12.1. Let X be a Cohen-Macaulay scheme, and denote by $\pi\colon \tilde{X} = \mathrm{Bl}_Z(X) \to X$ its blowup along a complete intersection $Z \subset X$.

(a) Show that $E = \pi^{-1}(Z)$ is Cohen-Macaulay, and deduce that $\tilde{X}$ is Cohen-Macaulay.

(b) Let $f\colon X \to S$ be a morphism to a regular scheme S such that all fibers of f are equidimensional of the same dimension. Fix a closed point $\tilde{x} \in E$ with images $x = \pi(\tilde{x})$ and $s = f(x)$. Show that the following two properties are equivalent:

- $f \circ \pi\colon \tilde{X} \to S$ is flat at $\tilde{x}$.
- $\dim_x(Z) - \dim_x(Z \cap X_s) \geq \dim_s(S) - 1$.

Problem 12.2. Let X be a smooth variety of dimension n over a field k. Let $\pi\colon \tilde{X} \to X$ be its blowup in a point $p \in X(k)$, and let $E = \pi^{-1}(p) \simeq \mathbb{P}_k^{n-1}$ be the exceptional divisor.

(a) Show that the normal bundle of $E \subset \tilde{X}$ is isomorphic to $\mathscr{O}_E(-1)$.

(b) Let $\mathscr{M} := \omega_{\tilde{X}} \otimes \pi^*(\omega_X^\vee) \in \mathrm{Pic}(\tilde{X})$. Show that the restriction $\mathscr{M}|_{\tilde{X}\setminus E}$ is trivial, and deduce that

$$\omega_{\tilde{X}} \;\simeq\; \pi^*(\omega_X) \otimes \mathscr{O}_{\tilde{X}}((n-1)E).$$

Problem 12.3. Let $f\colon X \to Y$ be a finite morphism of Noetherian schemes.

(a) Let $\mathscr{G} \in \mathrm{Coh}(X)$. Show that up to isomorphism there exists a unique $f^!(\mathscr{G}) \in \mathrm{Coh}(X)$ with

$$f_*(f^!(\mathscr{G})) \;\simeq\; \mathscr{H}om_Y(f_*(\mathscr{O}_X), \mathscr{G}) \quad \text{in} \quad \mathrm{Mod}(f_*\mathscr{O}_X),$$

(b) Show that we have $f_*\mathscr{H}om(\mathscr{F}, f^!(\mathscr{G})) \simeq \mathscr{H}om_Y(f_*(\mathscr{F}), \mathscr{G})$ for all $\mathscr{F} \in \mathrm{Coh}(X)$.

Problem 12.4. Let X be a projective scheme of pure dimension n over a field k.

(a) Let $f\colon X \to Y$ be a finite flat morphism to another projective scheme over k. Show that for any dualizing sheaf $(\omega_Y^\circ, \mathrm{tr}_Y)$ on Y we get a dualizing sheaf $(\omega_X^\circ, \mathrm{tr}_X)$ on X by taking $\omega_X^\circ := f^!(\omega_Y^\circ)$ (with notations as in the previous problem), endowed with the trace map

$$\mathrm{tr}_X\colon \quad H^n(X, f^!\omega_Y^\circ) \;\simeq\; H^n(Y, f_*f^!\omega_Y^\circ) \;\longrightarrow\; H^n(Y, \omega_Y^\circ) \xrightarrow{\mathrm{tr}_Y} k.$$

(b) Use this to construct on any projective Cohen-Macaulay scheme X of pure dimension n over k a dualizing sheaf, by taking $f\colon X \to Y = \mathbb{P}^n$ to be a suitable projection.