Please hand in your solutions after the lectures if you want them to be corrected.

Problem 1. Fix $\alpha \in \mathbb{C}$. Sending an open $U \subset X = \mathbb{C} \setminus \{0\}$ to the complex vector space

$$\mathscr{L}_\alpha(U) \;=\; \big\{ f : U \to \mathbb{C} \text{ holomorphic} \mid z f'(z) = \alpha f(z) \big\}.$$

gives a local system on X. Compute its monodromy around a generator of $\pi_1(X, 1)$.

Problem 2. Show that for the Weyl algebra $\mathscr{D} = \mathbb{C}[z]\langle \partial \rangle$ one has an isomorphism of left $\mathscr{D}$-modules

$$\varphi : \quad \mathscr{D}/\mathscr{D}(z(z+1)\partial + 1) \xrightarrow{\sim} \mathscr{D}/\mathscr{D}(z\partial + 1).$$

Problem 3. Let k be a field. Consider the Weyl algebra $\mathscr{D} = \mathscr{D}_{n,K}$ over $K = k(s)$ and fix a non-constant polynomial $f \in k[x_1, \dots, x_n]$.

(a) Show that $\mathscr{M} = K[x_1, \dots, x_n, 1/f]$ is endowed with a left $\mathscr{D}$-module structure via

$$x_i(g) \;:=\; x_i \cdot g, \quad \partial_i(g) \;:=\; \frac{\partial g}{\partial x_i} + \frac{sg}{f} \cdot \frac{\partial f}{\partial x_i} \quad \text{for} \quad g \in \mathscr{M}.$$

(b) Is this isomorphic to the "usual" $\mathscr{D}$-module structure where $\partial_i(g) := \frac{\partial g}{\partial x_i}$?

Problem 4. Determine the Bernstein-Sato polynomial $b_f(s) \in \mathbb{Q}[s]$ for each of the polynomials

(a) $f(x_1, \dots, x_n) = x_1^{e_1} \cdots x_n^{e_n}$ with $e_1, \dots, e_n \in \mathbb{N}_0$,

(b) $f(x_1, x_2) = x_1^2 - x_2^3$.

Please hand in your solutions after the lectures if you want them to be corrected.

Problem 1. Let $\mathscr{D} = \mathscr{D}_{n,k}$ be the Weyl algebra in n variables over a field k.

(a) Check that both for the order and for the Bernstein filtration the associated graded $gr^F_\bullet(\mathscr{D})$ is a polynomial ring. What is the degree of the variables?

(b) In both cases, find the biggest $\delta \in \mathbb{N}_0$ with $[F_i\mathscr{D}, F_j\mathscr{D}] \subseteq F_{i+j-\delta}\mathscr{D}$ for all i, j.

Problem 2. In the lecture we have used that for a function $h : \mathbb{Z} \to \mathbb{Q}$ the following are equivalent:

(a) There exists $p(t) \in \mathbb{Q}[t]$ with $h(i) = p(i)$ for all large enough $i \in \mathbb{Z}$.

(b) There exists $q(t) \in \mathbb{Q}[t]$ with $h(i) - h(i-1) = q(i)$ for all large enough $i \in \mathbb{Z}$.

Verify this statement and show that in this case $\deg(p) = \deg(q) + 1$.

Problem 3. (a) Show by induction on the degree that

$$R \;=\; \{p \in \mathbb{Q}[t] \mid p(i) \in \mathbb{Z} \text{ for all sufficiently large } i \in \mathbb{Z}\}$$

is a free abelian group on the polynomials

$$p_d(t) \;=\; \frac{t(t-1)\cdots(t-d+1)}{d!} \quad \text{for} \quad d \;\in\; \mathbb{N}_0.$$

(b) Let $\mathscr{D} = \mathscr{D}_{n,k}$ be the Weyl algebra over a field k. Deduce that if we write the Hilbert polynomial of a finitely generated $\mathscr{D}$-module $\mathscr{M}$ with a good filtration $F_\bullet\mathscr{M}$ as

$$p_{\mathscr{M},F_\bullet}(t) \;=\; c \cdot \tfrac{t^d}{d!} + \textit{lower order terms}, \quad \text{then } c \geq 0 \text{ is an integer.}$$

Please hand in your solutions after the lectures if you want them to be corrected.

Problem 1. Let $\mathscr{D} = k[x]\langle\partial\rangle$ for a field k of characteristic zero.

(a) Show that the morphism $\varphi : \mathscr{D} \oplus \mathscr{D} \to \mathscr{D}, (a, b) \mapsto a\partial + bx$ is onto.

(b) Show that $\ker(\varphi)$ is isomorphic to the left submodule $\mathscr{D} \cdot x^2 + \mathscr{D} \cdot \partial x \subset \mathscr{D}$ and deduce that the latter is projective. Is it free?

Problem 2. Let $\mathscr{D} = k[x]\langle\partial\rangle$ as above.

(a) Compute $\mathrm{Ext}^\bullet_{\mathscr{D}}(\mathscr{M}_2, \mathscr{M}_1)$ for all choices of $\mathscr{M}_i = \mathscr{D}/\mathscr{D}P_i$ with $P_1, P_2 \in \{x, \partial\}$.

(b) In all cases where $\mathrm{Ext}^1_{\mathscr{D}}(\mathscr{M}_2, \mathscr{M}_1) \neq \{0\}$, try to describe explicitly a nonsplit extension

$$0 \to \mathscr{M}_1 \to \mathscr{M} \to \mathscr{M}_2 \to 0 \quad \text{with} \quad \mathscr{M} \in \mathrm{Mod}(\mathscr{D}).$$

Problem 3. Consider $\mathscr{D} = \mathscr{D}_{n,k}$ with the order filtration. For $P \in F_d(\mathscr{D}) \setminus F_{d-1}(\mathscr{D})$ we define its *symbol* to be the image $\sigma(P) = [P] \in \mathrm{Gr}^F_d(\mathscr{D})$. More generally, for a left ideal $\mathscr{I} \trianglelefteq \mathscr{D}$ we put

$$\sigma(\mathscr{I}) \;=\; \langle\, \sigma(P) \mid P \in \mathscr{I} \,\rangle \;\trianglelefteq\; \mathrm{Gr}^F_\bullet(\mathscr{D})$$

Show that

(a) the characteristic variety of $\mathscr{M} = \mathscr{D}/\mathscr{I}$ is the zero locus $\mathrm{Char}(\mathscr{M}) = Z(\sigma(\mathscr{I}))$,

(b) it can happen that $\mathscr{I} = \mathscr{D}\cdot(P_1, \dots, P_m)$ but $Z(\sigma(\mathscr{I})) \neq Z(\sigma(P_1), \dots, \sigma(P_m))$.

Problem 4. We say that in an abelian category a morphism $\varphi : \mathscr{M} \to \mathscr{N}$ of filtered objects is *strict* if $F_i(\mathscr{N}) \cap \varphi(\mathscr{M}) = \varphi(F_i(\mathscr{M}))$ for all $i \in \mathbb{Z}$.

(a) Show that for any exact sequence of strict morphisms of filtered objects the associated sequence of graded objects is exact. Can strictness be dropped?

(b) Show that for $\mathscr{D} = \mathscr{D}_{n,k}$ every $\mathscr{M} \in \mathrm{Mod}(\mathscr{D})$ with a good filtration admits a resolution

$$\cdots \xrightarrow{d_2} \mathscr{M}_1 \xrightarrow{d_1} \mathscr{M}_0 \xrightarrow{d_0} \mathscr{M} \to 0$$

where the $\mathscr{M}_i = \bigoplus_{j=1}^{n_i} \mathscr{D}(a_{ij})$ are free filtered modules and all d_i are strict.

Let X be a smooth variety over an algebraically closed field k. Unless otherwise stated we always assume that $\mathrm{char}(k) = 0$.

Problem 1. Let $\mathscr{T}$ be a Lie algebroid on X and $U(\mathscr{T}/\mathscr{O}_X)$ its universal enveloping algebra as defined in the lecture. Show that sheaves of left modules for $U(\mathscr{T}/\mathscr{O}_X)$ correspond bijectively to $\mathscr{O}_X$-modules $\mathscr{M}$ endowed with a homomorphism of sheaves of Lie algebras

$$\mathscr{T} \oplus \mathscr{O}_X \longrightarrow \mathscr{E}nd_k(\mathscr{M})$$

such that

$$\begin{aligned} &f \cdot (\xi \cdot m) = (f \cdot \xi) \cdot m, \\ &\xi \cdot (f \cdot m) = \xi(f) \cdot m + f \cdot (\xi \cdot m) \quad \forall\, f \in \mathscr{O}_X,\, \xi \in \mathscr{T},\, m \in \mathscr{M}. \end{aligned}$$

Can you formulate corresponding statements for right modules?

Problem 2. In the lecture we formally defined the *symbol* of $P \in F_d\mathscr{D}_X$ in local coordinates $x_1, \dots, x_n$ as the d-fold commutator

$$\sigma_d(P) \;=\; \tfrac{1}{d!}[\cdots[[P,f],f],\dots,f] \;\in\; \mathscr{O}_X[\xi_1,\dots,\xi_n] \quad \text{with} \quad f \;=\; \sum_{i=1}^n \xi_i\,\partial_i.$$

Verify that

$$\sigma_d\Big(\textstyle\sum_{|I|\le d} f_I\,\partial^I\Big) \;=\; \textstyle\sum_{|I|=d} f_I\,\xi^I \quad \text{for all} \quad f_I \;\in\; \mathscr{O}_X.$$

Problem 3. In the lecture we have seen three possible definitions for the sheaf of differential operators on X: As the universal enveloping algebra of the tangent Lie algebroid, its image inside $\mathscr{E}nd_k(\mathscr{O}_X)$, or via Grothendieck's definition. Are the two maps

$$U(\mathscr{T}_X/\mathscr{O}_X) \;\twoheadrightarrow\; \mathscr{D}_X \;\hookrightarrow\; \mathscr{D}i\!f\!f_X$$

isomorphisms for the affine line $X = \mathbb{A}^1_k$ in characteristic $\mathrm{char}(k) = p > 0$?

Problem 4. Show directly from the definition of the sheaf $\mathscr{D}i\!f\!f_X \subset \mathscr{E}nd_k(\mathscr{O}_X)$ of Grothendieck differential operators that if $x_1, \dots, x_n$ is a local coordinate system on X, then

$$\big\{P \in \mathscr{D}i\!f\!f_X \mid [P, x_i] \;=\; 0 \text{ for } i = 1,\dots,n\big\} \;=\; \mathscr{O}_X.$$

Let X be a smooth variety over an algebraically closed field k with $\mathrm{char}(k) = 0$.

Problem 1. Let $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module with a connection $\nabla : \mathscr{E} \to \Omega^1_X(\mathscr{E})$.

(a) Show that

$$(\nabla\alpha)(\xi\otimes\eta) \;=\; \nabla_\xi(\alpha(\eta)) - \nabla_\eta(\alpha(\xi)) - \alpha([\xi,\eta]) \quad \text{for all } \alpha \in \Omega^1_X(\mathscr{E}),\ \xi,\eta \in \mathscr{T}_X.$$

(b) Deduce that the connection ∇ is flat iff $[\nabla_\xi, \nabla_\eta] = \nabla_{[\xi,\eta]}$.

Problem 2. Assume that $X = \mathrm{Spec}(A)$ is affine, and put $D = H^0(X, \mathscr{D}_X)$.

(a) Fix $f \in A$. Show that for every $P \in D$ there exist $Q_1, Q_2 \in D$ and $n_1, n_2 \in \mathbb{N}$ with

$$P \cdot f^{n_1} \;=\; f \cdot Q_1 \quad \text{and} \quad f^{n_2} \cdot P \;=\; Q_2 \cdot f.$$

(b) Deduce that $D \otimes_A A_f \simeq A_f \otimes_A D \simeq H^0(U, \mathscr{D}_X)$ for the open $U = \mathrm{Spec}(A_f)$.

Problem 3. Let V be an n-dimensional k-vector space with dual $V^* = \mathrm{Hom}_k(V,k)$, say

$$\begin{aligned} V \;&=\; \mathrm{Spec}\, k[x_1,\dots,x_n] \qquad &&\text{with linear coordinates} \quad x_1,\dots,x_n \;\in\; V^* \\ V^* \;&=\; \mathrm{Spec}\, k[y_1,\dots,y_n] \qquad &&\text{with the dual coordinates} \quad y_1,\dots,y_n \;\in\; V^{**} \end{aligned}$$

where the dual coordinates satisfy $y_i(x_j) = \delta_{ij}$. Define the *Fourier transform* as the functor

$$\Phi_V : \quad \mathrm{Mod}_{qc}(\mathscr{D}_V) \;\longrightarrow\; \mathrm{Mod}_{qc}(\mathscr{D}_{V^*})$$

which is the identity on the underlying vector spaces of global sections but with $\mathscr{D}_{V^*}$ acting on the Fourier transform of $\mathscr{M} \in \mathrm{Mod}_{qc}(\mathscr{D}_V)$ by

$$\left.\begin{aligned} y_i \cdot m \;&:=\; \partial_{x_i} \cdot m \\ \partial_{y_i} \cdot m \;&:=\; -x_i \cdot m \end{aligned}\right\} \quad \text{for} \quad m \;\in\; H^0(V^*, \Phi_V(\mathscr{M})) \;=\; H^0(V, \mathscr{M}).$$

Show that Φ_V is an equivalence of categories and does not depend on the chosen dual coordinate systems. Deduce that for any linear map $f : V \to W$ of vector spaces the naive pushforward for $\mathscr{D}$-modules and the pullback under the transpose map $f^t : W^* \to V^*$ satisfy

$$\Phi_W \circ f_{naive} \;=\; (f^t)^* \circ \Phi_V.$$

Let X be a smooth variety over an algebraically closed field k with $\mathrm{char}(k) = 0$. In what follows $\otimes = \otimes_{\mathscr{O}_X}$ always denotes the tensor product over $\mathscr{O}_X$.

Problem 1. Show that extending scalars via the homomorphism $\mathscr{O}_X \to \mathscr{D}_X$ gives exact functors

$$\begin{aligned} \mathrm{Coh}(\mathscr{O}_X) &\longrightarrow \mathrm{Coh}(\mathscr{D}_X), \\ \mathscr{F} &\mapsto \mathscr{D}_X \otimes \mathscr{F} \end{aligned} \qquad \begin{aligned} \mathrm{Coh}(\mathscr{O}_X) &\longrightarrow \mathrm{Coh}(\mathscr{D}_X), \\ \mathscr{G} &\mapsto \mathscr{G} \otimes \mathscr{D}_X \end{aligned}$$

and that one has the following isomorphisms of right $\mathscr{D}_X$-modules:

- $\omega_X \otimes (\mathscr{D}_X \otimes \mathscr{F}) \simeq \mathscr{G} \otimes \mathscr{D}_X$ for the coherent sheaf $\mathscr{G} = \omega_X \otimes \mathscr{F}$,
- $\mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{D}_X \otimes \mathscr{F}, \mathscr{D}_X) \simeq \mathscr{G} \otimes \mathscr{D}_X$ for all $j \in \mathbb{N}_0$ and $\mathscr{G} = \mathscr{E}xt^j_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$.

Problem 2. Show that for the Dirac module $\delta_p \in \mathrm{Coh}(\mathscr{D}_X)$ on a point $p \in X$ one has

$$\mathscr{E}xt^j_{\mathscr{D}_X}(\delta_p, \mathscr{D}_X) \;\simeq\; \begin{cases} \omega_X \otimes \delta_p & \text{if } j = \dim X, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 3. Show that the left $\mathscr{D}_X$-module $\mathscr{O}_X$ admits a locally free resolution of the form

$$\mathscr{D}_X \otimes \mathrm{Alt}^n_{\mathscr{O}_X}(\mathscr{T}_X) \xrightarrow{d} \mathscr{D}_X \otimes \mathrm{Alt}^{n-1}_{\mathscr{O}_X}(\mathscr{T}_X) \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{D}_X \otimes \mathscr{T}_X \xrightarrow{d} \mathscr{D}_X \xrightarrow{\varphi} \mathscr{O}_X.$$

Here $\varphi(P) := P(1)$ and the differentials are described in local coordinates (x_i, ∂_i) by

$$d\big(P \otimes \partial_{i_1} \wedge \cdots \wedge \partial_{i_r}\big) \;=\; \sum_{\nu=1}^{r} (-1)^{\nu+1} (P \cdot \partial_{i_\nu}) \otimes \partial_{i_1} \wedge \cdots \wedge \widehat{\partial}_{i_\nu} \wedge \cdots \wedge \partial_{i_r}$$

where the notation on the right hand side means that the ν-th factor is omitted from the wedge product. Deduce that

$$\mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{D}_X) \;\simeq\; \begin{cases} \omega_X & \text{if } j = \dim X, \\ 0 & \text{otherwise.} \end{cases}$$