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Please hand in your solutions after the lectures if you want them to be corrected.

Problem 1. Fix $\alpha \in \mathbb{C}$. Sending an open $U \subset X = \mathbb{C} \setminus \{0\}$ to the complex vector space

$$\mathscr{L}_\alpha(U) \;=\; \bigl\{ f : U \to \mathbb{C} \text{ holomorphic} \mid zf'(z) = \alpha f(z) \bigr\}.$$

gives a local system on X. Compute its monodromy around a generator of $\pi_1(X, 1)$.

Problem 2. Show that for the Weyl algebra $\mathscr{D} = \mathbb{C}[z]\langle\partial\rangle$ one has an isomorphism of left $\mathscr{D}$-modules

$$\varphi : \quad \mathscr{D}/\mathscr{D}(z(z+1)\partial + 1) \;\xrightarrow{\sim}\; \mathscr{D}/\mathscr{D}(z\partial + 1).$$

Problem 3. Let k be a field. Consider the Weyl algebra $\mathscr{D} = \mathscr{D}_{n,K}$ over $K = k(s)$ and fix a non-constant polynomial $f \in k[x_1, \dots, x_n]$.

(a) Show that $\mathscr{M} = K[x_1, \dots, x_n, 1/f]$ is endowed with a left $\mathscr{D}$-module structure via

$$x_i(g) \;:=\; x_i \cdot g, \quad \partial_i(g) \;:=\; \frac{\partial g}{\partial x_i} + \frac{sg}{f} \cdot \frac{\partial f}{\partial x_i} \quad \text{for} \quad g \in \mathscr{M}.$$

(b) Is this isomorphic to the "usual" $\mathscr{D}$-module structure where $\partial_i(g) := \frac{\partial g}{\partial x_i}$?

Problem 4. Determine the Bernstein-Sato polynomial $b_f(s) \in \mathbb{Q}[s]$ for each of the polynomials

(a) $f(x_1, \dots, x_n) = x_1^{e_1} \cdots x_n^{e_n}$ with $e_1, \dots, e_n \in \mathbb{N}_0$,

(b) $f(x_1, x_2) = x_1^2 - x_2^3$.
