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Please hand in your solutions after the lectures if you want them to be corrected.

Problem 1. Let $\mathscr{D} = \mathscr{D}_{n,k}$ be the Weyl algebra in n variables over a field k.

(a) Check that both for the order and for the Bernstein filtration the associated graded $gr^F_\bullet(\mathscr{D})$ is a polynomial ring. What is the degree of the variables?

(b) In both cases, find the biggest $\delta \in \mathbb{N}_0$ with $[F_i\mathscr{D}, F_j\mathscr{D}] \subseteq F_{i+j-\delta}\mathscr{D}$ for all i, j.

Problem 2. In the lecture we have used that for a function $h : \mathbb{Z} \to \mathbb{Q}$ the following are equivalent:

(a) There exists $p(t) \in \mathbb{Q}[t]$ with $h(i) = p(i)$ for all large enough $i \in \mathbb{Z}$.

(b) There exists $q(t) \in \mathbb{Q}[t]$ with $h(i) - h(i-1) = q(i)$ for all large enough $i \in \mathbb{Z}$.

Verify this statement and show that in this case $\deg(p) = \deg(q) + 1$.

Problem 3. (a) Show by induction on the degree that

$$R \;=\; \{p \in \mathbb{Q}[t] \mid p(i) \in \mathbb{Z} \text{ for all sufficiently large } i \in \mathbb{Z}\}$$

is a free abelian group on the polynomials

$$p_d(t) \;=\; \frac{t(t-1)\cdots(t-d+1)}{d!} \quad \text{for} \quad d \in \mathbb{N}_0.$$

(b) Let $\mathscr{D} = \mathscr{D}_{n,k}$ be the Weyl algebra over a field k. Deduce that if we write the Hilbert polynomial of a finitely generated $\mathscr{D}$-module $\mathscr{M}$ with a good filtration $F_\bullet\mathscr{M}$ as

$$p_{\mathscr{M},F_\bullet}(t) \;=\; c \cdot \tfrac{t^d}{d!} + \textit{lower order terms}, \quad \text{then } c \geq 0 \text{ is an integer.}$$
