Lectures on $\mathscr{D}$-Modules | Problem Set 3

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Please hand in your solutions after the lectures if you want them to be corrected.

Problem 1. Let $\mathscr{D} = k[x]\langle\partial\rangle$ for a field k of characteristic zero.

(a) Show that the morphism $\varphi : \mathscr{D} \oplus \mathscr{D} \to \mathscr{D}, (a, b) \mapsto a\partial + bx$ is onto.

(b) Show that $\ker(\varphi)$ is isomorphic to the left submodule $\mathscr{D} \cdot x^2 + \mathscr{D} \cdot \partial x \subset \mathscr{D}$ and deduce that the latter is projective. Is it free?

Problem 2. Let $\mathscr{D} = k[x]\langle\partial\rangle$ as above.

(a) Compute $\mathrm{Ext}^\bullet_{\mathscr{D}}(\mathscr{M}_2, \mathscr{M}_1)$ for all choices of $\mathscr{M}_i = \mathscr{D}/\mathscr{D}P_i$ with $P_1, P_2 \in \{x, \partial\}$.

(b) In all cases where $\mathrm{Ext}^1_{\mathscr{D}}(\mathscr{M}_2, \mathscr{M}_1) \neq \{0\}$, try to describe explicitly a nonsplit extension

$$0 \to \mathscr{M}_1 \to \mathscr{M} \to \mathscr{M}_2 \to 0 \quad \text{with} \quad \mathscr{M} \in \mathrm{Mod}(\mathscr{D}).$$

Problem 3. Consider $\mathscr{D} = \mathscr{D}_{n,k}$ with the order filtration. For $P \in F_d(\mathscr{D}) \setminus F_{d-1}(\mathscr{D})$ we define its *symbol* to be the image $\sigma(P) = [P] \in \mathrm{Gr}^F_d(\mathscr{D})$. More generally, for a left ideal $\mathscr{I} \trianglelefteq \mathscr{D}$ we put

$$\sigma(\mathscr{I}) \;=\; \langle\, \sigma(P) \mid P \in \mathscr{I} \,\rangle \;\trianglelefteq\; \mathrm{Gr}^F_\bullet(\mathscr{D})$$

Show that

(a) the characteristic variety of $\mathscr{M} = \mathscr{D}/\mathscr{I}$ is the zero locus $\mathrm{Char}(\mathscr{M}) = Z(\sigma(\mathscr{I}))$,

(b) it can happen that $\mathscr{I} = \mathscr{D}\cdot(P_1, \ldots, P_m)$ but $Z(\sigma(\mathscr{I})) \neq Z(\sigma(P_1), \ldots, \sigma(P_m))$.

Problem 4. We say that in an abelian category a morphism $\varphi : \mathscr{M} \to \mathscr{N}$ of filtered objects is *strict* if $F_i(\mathscr{N}) \cap \varphi(\mathscr{M}) = \varphi(F_i(\mathscr{M}))$ for all $i \in \mathbb{Z}$.

(a) Show that for any exact sequence of strict morphisms of filtered objects the associated sequence of graded objects is exact. Can strictness be dropped?

(b) Show that for $\mathscr{D} = \mathscr{D}_{n,k}$ every $\mathscr{M} \in \mathrm{Mod}(\mathscr{D})$ with a good filtration admits a resolution

$$\cdots \xrightarrow{d_2} \mathscr{M}_1 \xrightarrow{d_1} \mathscr{M}_0 \xrightarrow{d_0} \mathscr{M} \to 0$$

where the $\mathscr{M}_i = \bigoplus_{j=1}^{n_i} \mathscr{D}(a_{ij})$ are free filtered modules and all d_i are strict.
