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Let X be a smooth variety over an algebraically closed field k. Unless otherwise stated we always assume that $\mathrm{char}(k) = 0$.

Problem 1. Let $\mathscr{T}$ be a Lie algebroid on X and $U(\mathscr{T}/\mathscr{O}_X)$ its universal enveloping algebra as defined in the lecture. Show that sheaves of left modules for $U(\mathscr{T}/\mathscr{O}_X)$ correspond bijectively to $\mathscr{O}_X$-modules $\mathscr{M}$ endowed with a homomorphism of sheaves of Lie algebras

$$\mathscr{T} \oplus \mathscr{O}_X \longrightarrow \mathscr{E}nd_k(\mathscr{M})$$

such that

$$\begin{aligned} &f \cdot (\xi \cdot m) = (f \cdot \xi) \cdot m, \\ &\xi \cdot (f \cdot m) = \xi(f) \cdot m + f \cdot (\xi \cdot m) \quad \forall\, f \in \mathscr{O}_X,\, \xi \in \mathscr{T},\, m \in \mathscr{M}. \end{aligned}$$

Can you formulate corresponding statements for right modules?

Problem 2. In the lecture we formally defined the *symbol* of $P \in F_d\mathscr{D}_X$ in local coordinates $x_1, \dots, x_n$ as the d-fold commutator

$$\sigma_d(P) \;=\; \tfrac{1}{d!}[\cdots[[P,f],f],\dots,f] \;\in\; \mathscr{O}_X[\xi_1,\dots,\xi_n] \quad \text{with} \quad f \;=\; \sum_{i=1}^{n} \xi_i\,\partial_i.$$

Verify that

$$\sigma_d\Big(\textstyle\sum_{|I|\le d} f_I\,\partial^I\Big) \;=\; \textstyle\sum_{|I|=d} f_I\,\xi^I \quad \text{for all} \quad f_I \;\in\; \mathscr{O}_X.$$

Problem 3. In the lecture we have seen three possible definitions for the sheaf of differential operators on X: As the universal enveloping algebra of the tangent Lie algebroid, its image inside $\mathscr{E}nd_k(\mathscr{O}_X)$, or via Grothendieck's definition. Are the two maps

$$U(\mathscr{T}_X/\mathscr{O}_X) \;\twoheadrightarrow\; \mathscr{D}_X \;\hookrightarrow\; \mathscr{D}i\!f\!f_X$$

isomorphisms for the affine line $X = \mathbb{A}^1_k$ in characteristic $\mathrm{char}(k) = p > 0$?

Problem 4. Show directly from the definition of the sheaf $\mathscr{D}i\!f\!f_X \subset \mathscr{E}nd_k(\mathscr{O}_X)$ of Grothendieck differential operators that if $x_1,\dots,x_n$ is a local coordinate system on X, then

$$\big\{P \in \mathscr{D}i\!f\!f_X \mid [P,x_i] \;=\; 0 \text{ for } i = 1,\dots,n\big\} \;=\; \mathscr{O}_X.$$