

Let X be a smooth variety over an algebraically closed field k with $\text{char}(k) = 0$.

Problem 1. Let \mathcal{E} be a locally free \mathcal{O}_X -module with a connection $\nabla : \mathcal{E} \rightarrow \Omega_X^1(\mathcal{E})$.

(a) Show that

$$(\nabla\alpha)(\xi\otimes\eta) = \nabla_\xi(\alpha(\eta)) - \nabla_\eta(\alpha(\xi)) - \alpha([\xi, \eta]) \quad \text{for all } \alpha \in \Omega_X^1(\mathcal{E}), \xi, \eta \in \mathcal{T}_X.$$

(b) Deduce that the connection ∇ is flat iff $[\nabla_\xi, \nabla_\eta] = \nabla_{[\xi, \eta]}$.

Problem 2. Assume that $X = \text{Spec}(A)$ is affine, and put $D = H^0(X, \mathcal{D}_X)$.

(a) Fix $f \in A$. Show that for every $P \in D$ there exist $Q_1, Q_2 \in D$ and $n_1, n_2 \in \mathbb{N}$ with

$$P \cdot f^{n_1} = f \cdot Q_1 \quad \text{and} \quad f^{n_2} \cdot P = Q_2 \cdot f.$$

(b) Deduce that $D \otimes_A A_f \simeq A_f \otimes_A D \simeq H^0(U, \mathcal{D}_X)$ for the open $U = \text{Spec}(A_f)$.

Problem 3. Let V be an n -dimensional k -vector space with dual $V^* = \text{Hom}_k(V, k)$, say

$$\begin{aligned} V &= \text{Spec } k[x_1, \dots, x_n] && \text{with linear coordinates } x_1, \dots, x_n \in V^* \\ V^* &= \text{Spec } k[y_1, \dots, y_n] && \text{with the dual coordinates } y_1, \dots, y_n \in V^{**} \end{aligned}$$

where the dual coordinates satisfy $y_i(x_j) = \delta_{ij}$. Define the *Fourier transform* as the functor

$$\Phi_V : \text{Mod}_{qc}(\mathcal{D}_V) \longrightarrow \text{Mod}_{qc}(\mathcal{D}_{V^*})$$

which is the identity on the underlying vector spaces of global sections but with \mathcal{D}_{V^*} acting on the Fourier transform of $\mathcal{M} \in \text{Mod}_{qc}(\mathcal{D}_V)$ by

$$\left. \begin{aligned} y_i \cdot m &:= \partial_{x_i} \cdot m \\ \partial_{y_i} \cdot m &:= -x_i \cdot m \end{aligned} \right\} \text{ for } m \in H^0(V^*, \Phi_V(\mathcal{M})) = H^0(V, \mathcal{M}).$$

Show that Φ_V is an equivalence of categories and does not depend on the chosen dual coordinate systems. Deduce that for any linear map $f : V \rightarrow W$ of vector spaces the naive pushforward for \mathcal{D} -modules and the pullback under the transpose map $f^t : W^* \rightarrow V^*$ satisfy

$$\Phi_W \circ f_{naive} = (f^t)^* \circ \Phi_V.$$