Let X be a smooth variety over an algebraically closed field k with $\mathrm{char}(k) = 0$. In what follows $\otimes = \otimes_{\mathscr{O}_X}$ always denotes the tensor product over $\mathscr{O}_X$.

Problem 1. Show that extending scalars via the homomorphism $\mathscr{O}_X \to \mathscr{D}_X$ gives exact functors

$$\begin{array}{ccc} \mathrm{Coh}(\mathscr{O}_X) & \longrightarrow & \mathrm{Coh}(\mathscr{D}_X), \\ \mathscr{F} & \mapsto & \mathscr{D}_X \otimes \mathscr{F} \end{array} \qquad \begin{array}{ccc} \mathrm{Coh}(\mathscr{O}_X) & \longrightarrow & \mathrm{Coh}(\mathscr{D}_X), \\ \mathscr{G} & \mapsto & \mathscr{G} \otimes \mathscr{D}_X \end{array}$$

and that one has the following isomorphisms of right $\mathscr{D}_X$-modules:

- $\omega_X \otimes (\mathscr{D}_X \otimes \mathscr{F}) \simeq \mathscr{G} \otimes \mathscr{D}_X$ for the coherent sheaf $\mathscr{G} = \omega_X \otimes \mathscr{F}$,
- $\mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{D}_X \otimes \mathscr{F}, \mathscr{D}_X) \simeq \mathscr{G} \otimes \mathscr{D}_X$ for all $j \in \mathbb{N}_0$ and $\mathscr{G} = \mathscr{E}xt^j_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$.

Problem 2. Show that for the Dirac module $\delta_p \in \mathrm{Coh}(\mathscr{D}_X)$ on a point $p \in X$ one has

$$\mathscr{E}xt^j_{\mathscr{D}_X}(\delta_p, \mathscr{D}_X) \simeq \begin{cases} \omega_X \otimes \delta_p & \text{if } j = \dim X, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 3. Show that the left $\mathscr{D}_X$-module $\mathscr{O}_X$ admits a locally free resolution of the form

$$\mathscr{D}_X \otimes \mathrm{Alt}^n_{\mathscr{O}_X}(\mathscr{T}_X) \xrightarrow{d} \mathscr{D}_X \otimes \mathrm{Alt}^{n-1}_{\mathscr{O}_X}(\mathscr{T}_X) \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{D}_X \otimes \mathscr{T}_X \xrightarrow{d} \mathscr{D}_X \xrightarrow{\varphi} \mathscr{O}_X.$$

Here $\varphi(P) := P(1)$ and the differentials are described in local coordinates (x_i, ∂_i) by

$$d\big(P \otimes \partial_{i_1} \wedge \cdots \wedge \partial_{i_r}\big) = \sum_{\nu=1}^{r} (-1)^{\nu+1} (P \cdot \partial_{i_\nu}) \otimes \partial_{i_1} \wedge \cdots \wedge \widehat{\partial}_{i_\nu} \wedge \cdots \wedge \partial_{i_r}$$

where the notation on the right hand side means that the ν-th factor is omitted from the wedge product. Deduce that

$$\mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{D}_X) \simeq \begin{cases} \omega_X & \text{if } j = \dim X, \\ 0 & \text{otherwise.} \end{cases}$$