Please bring your written solutions to the first problem session next Thursday, or hand them in earlier in my office (RUD 25, room 1.425).

Problem 1. (Representations of finite abelian groups)

(a) Let $d \in \mathbb{N} = \{1, 2, \dots\}$. Show that every matrix $M \in Gl_d(\mathbb{C})$ of finite order is conjugate to a diagonal matrix.

(b) Deduce that for $n \in \mathbb{N}$, any representation $\rho : \mathbb{Z}/n\mathbb{Z} \to Gl_d(\mathbb{C})$ is a direct sum of 1-dimensional representations.

(c) Generalize this to representations of arbitrary finite abelian groups.

(d) Show that in general, part (c) fails if the word *finite* or *abelian* is omitted.

Problem 2. Let G be a finite group, fix an arbitrary field k, and let $V = k[G]$ denote the vector space of all functions $f : G \to k$. In the first lecture we have defined the right respectively left regular representations $\rho : G \to Gl(V)$ and $\lambda : G \to Gl(V)$ by the formula

$$(\rho(g)(f))(x) \;=\; f(xg) \quad \text{and} \quad (\lambda(g)(f))(x) \;=\; f(g^{-1}x) \quad \text{for} \quad x, g \in G,\ f \in k[G].$$

(a) Verify that ρ and λ, as defined above, are indeed representations.

(b) Express these two representations by matrices in the basis of V given by the characteristic functions

$$e_g(x) \;=\; \begin{cases} 1 & \text{if } x = g \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } g \in G.$$

(c) Find an explicit isomorphism between the representations ρ and λ.

Problem 3. Let $G = \mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}$. Using the basis from problem 2c, describe explicitly how the (right) regular representation $V = \mathbb{C}[G]$ decomposes into a direct sum of 1-dimensional representations.

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Problem 4. Let G be a group and $V \in \mathrm{Rep}_k(G)$ a representation over a field k.

(a) We have defined the symmetric and exterior powers of V as quotients of tensor powers. Show that for $char(k) \nmid n!$ the quotient maps

$$V^{\otimes n} \twoheadrightarrow Sym^n(V) \quad \text{and} \quad V^{\otimes n} \twoheadrightarrow Alt^n(V)$$

can be written as the projection onto a direct summand of $V^{\otimes n} \in \mathrm{Rep}_k(G)$.

(b) Give an example where $char(k) \mid n!$ and where the above property fails.

Problem 5. Let $V \in \mathrm{Rep}_k(G)$. Show that for every *normal* subgroup $H \trianglelefteq G$ we get a subrepresentation

$$V^H \;=\; \{\, v \in V \;\mid\; hv = v \text{ for all } h \in H \,\} \;\in\; \mathrm{Rep}_k(G).$$

In general, does this property hold for non-normal subgroups as well?

Problem 6. Let the group $G = Sl_2(\mathbb{C})$ act on the polynomial ring $\mathbb{C}[x, y]$ in two variables via

$$(g(f))(x, y) = f(ax + cy, bx + dy) \quad \text{for} \quad f \in \mathbb{C}[x, y], \;\; g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl_2(\mathbb{C}).$$

(a) Show that this is a group action and for each $n \in \mathbb{N}_0$, the degree n monomials span a finite-dimensional subrepresentation

$$V_n \;=\; \Big\{ \sum_{\nu=0}^{n} a_\nu \cdot x^\nu y^{d-\nu} \in \mathbb{C}[x, y] \;\Big|\; a_\nu \in \mathbb{C} \Big\} \;\in\; \mathrm{Rep}_{\mathbb{C}}(G).$$

(b) Let $T \subset G$ be the subgroup of diagonal matrices. Show that for each $n \in \mathbb{N}_0$ the restriction $V_n|_T \in \mathrm{Rep}_{\mathbb{C}}(T)$ is semisimple by decomposing it explicitly into a direct sum of irreducible representations.

(c) Deduce that for each $n \in \mathbb{N}_0$, the representation $V_n \in \mathrm{Rep}_{\mathbb{C}}(G)$ is irreducible.

Problem 7. Using the ideas from problem 6, deduce for all integers $m \geq n \geq 0$ the decomposition

$$V_m \otimes V_n \;\simeq\; V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{m-n} \;\text{ in }\; \mathrm{Rep}_{\mathbb{C}}(Sl_2(\mathbb{C})).$$

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Problem 8. Let G be a group and $V \in \mathrm{Rep}_k(G)$ an irreducible representation over an algebraically closed field k. Show that the following holds:

(a) For $n \in \mathbb{N}$, any subrepresentation of $V^{\oplus n}$ has the form $W \simeq V^{\oplus r}$ with $r \leq n$ and the inclusion
$$V^{\oplus r} \simeq W \hookrightarrow V^{\oplus n}$$
is given by an $r \times n$ matrix with linearly independent rows over k.

(b) If $v_1, \ldots, v_n \in V$ are linearly independent, then the map
$$k[G] \to V^{\oplus n} \quad \text{given by} \quad a \mapsto (av_1, \ldots, av_n) \quad \text{is surjective.}$$

(c) The natural map $k[G] \to \mathrm{End}_k(V)$ is surjective (apply (b) to a basis).

Problem 9. Show that the right or left adjoint of a functor, if it exists, is unique up to isomorphism. Deduce that there exist natural isomorphisms
$$\mathrm{Ind}_H^G(U \otimes \mathrm{Res}_H^G(V)) \simeq \mathrm{Ind}_H^G(U) \otimes V \quad \text{and} \quad \mathrm{Ind}_K^G(\mathrm{Ind}_H^K(U)) \simeq \mathrm{Ind}_H^G(U)$$
for any subgroups $H \leq K \leq G$ of finite index and all $U \in \mathrm{Rep}_k(H)$, $V \in \mathrm{Rep}_k(G)$.

Problem 10. Let $H \trianglelefteq G$ be a normal subgroup of finite index.

(a) Show that for any $U \in \mathrm{Rep}_k(H)$ and $V \in \mathrm{Rep}_k(G)$,
- if V is semisimple, then $\mathrm{Res}_H^G(V)$ is semisimple.
- if $\mathrm{Ind}_H^G(U)$ is semisimple, then U is semisimple.

(b) Show that if $char(k) \nmid n = [G : H]$, the converse implications also hold.

(c) Find counterexamples when $char(k) \mid n$ or when $H \leq G$ is not normal.

Please hand in your written solutions before the lecture next Thursday. Note that there will be **no problem session next week** since we will exceptionally move the lecture to that time: 11:15 - 12:45 in room 1.013 (RUD25).

Problem 11. Suppose that the base field k is algebraically closed, and let $N \trianglelefteq G$ a normal subgroup for which the quotient G/N is abelian. Show that for all irreducible representations $V_1, V_2 \in \mathrm{Rep}_k(G)$,

$$V_1|_N \simeq V_2|_N \iff V_1 \simeq V_2 \otimes \chi \text{ for some 1-dimensional } \chi \in \mathrm{Rep}_k(G/N).$$

Problem 12. In this problem we discuss some character tables over $k = \mathbb{C}$.

(a) Find the character table of $\mathfrak{S}_4$ and describe its irreducible representations.

(b) Determine which of these restricts to an irreducible representation of $\mathfrak{A}_4 \trianglelefteq \mathfrak{S}_4$ and find the character table of this alternating group.

(c) Find the character table of the groups D and Q presented below. Are these two groups isomorphic?

$$D = \langle a, b \mid a^4 = b^2 = abab = 1 \rangle, \quad Q = \langle a, b \mid a^4 = 1,\ bab = a,\ a^2 = b^2 \rangle.$$

Problem 13. Let $m \in \mathbb{N}$. The *elementary symmetric polynomials* in $x = (x_1, \dots, x_m)$ are defined by

$$e_n(x) = \sum_{1 \le i_1 < \cdots < i_n \le m} x_{i_1} \cdots x_{i_n} \quad \text{for} \quad n \in \mathbb{N}_0.$$

(a) Expand $\prod_{i=1}^m (t - x_i)$ in terms of these elementary symmetric polynomials and deduce

$$n e_n(x) = \sum_{i=1}^n (-1)^{i+1} s_i(x)\, e_{n-i}(x) \quad \text{for} \quad s_i(x) = x_1^i + \cdots + x_m^i$$

if $n = m$. Explain why the above formula then remains true also for $n \neq m$.

(b) For finite groups G and $V \in \mathrm{Rep}_{\mathbb{C}}(G)$, show that the characters of the exterior powers satisfy

$$\chi_{\mathrm{Alt}^n(V)}(g) = \frac{1}{n} \sum_{i=1}^n (-1)^{i+1} \chi_V(g^i)\, \chi_{\mathrm{Alt}^{n-i}(V)}(g) \quad \text{for} \quad g \in G,\ n \in \mathbb{N}.$$

Write down a closed formula for these characters for $n = 2, 3$.

Please bring your written solutions to the problem session next Thursday as usual.

Problem 14. Let $G = G_1 \times G_2$ be the product of two groups G_1, G_2, and let k be an algebraically closed field.

(a) Using the result from problem 8, show that for all irreducible $V_i \in \mathrm{Rep}_k(G_i)$ the tensor product $V = V_1 \boxtimes V_2 \in \mathrm{Rep}_k(G)$ is irreducible.

(b) Conversely, if $V \in \mathrm{Rep}_k(G)$ is irreducible and $V_1 \subseteq V|_{G_1}$ is any irreducible constituent of its restriction to $G_1 \times \{1\} \subseteq G$, endow $V_2 = \mathrm{Hom}_{G_1}(V_1, V|_{G_1})$ with a G_2-action and show that

$$V \;\simeq\; V_1 \boxtimes V_2 \quad \text{in} \quad \mathrm{Rep}_k(G).$$

(c) If $|G| < \infty$ and $char(k) = 0$, find a shorter argument via character theory.

Problem 15. Let G be a finite group.

(a) Show that the *lower central series*

$$G \;=\; G_1 \;\trianglerighteq\; G_2 \;\trianglerighteq\; \cdots \quad \text{defined by} \quad G_{i+1} \;=\; [G_i, G]$$

terminates with the trivial group iff the *upper central series*

$$\{1\} \;=\; Z_0 \trianglelefteq Z_1 \;\trianglelefteq\; \cdots \quad \text{defined by} \quad Z_{i+1} \;=\; \{\, z \in G \mid [z,g] \in Z_i \;\forall\, g \in G\}$$

terminates with the whole group. We then say G is *nilpotent*.

(b) Show that subgroups and quotients of nilpotent groups are nilpotent and that in a nilpotent group G every maximal abelian normal subgroup $N \trianglelefteq G$ is equal to its own centralizer: $N = Z_G(N)$.

Problem 16. Recall that a representation of a finite group G is called *primitive* if it is not induced from any proper subgroup.

(a) If G has a primitive faithful irreducible representation, show that any normal abelian subgroup $N \trianglelefteq G$ is contained in the center $Z(G)$.

(b) Using problem 16b, show that any irreducible representation of a nilpotent group is *monomial*, i.e. induced from a 1-dimensional representation.

Please bring your written solutions to the problem session next Thursday.

Problem 17. Let G be a finite group of order n. Show that for $g \in G$ the following are equivalent:

(a) For every $V \in \mathrm{Rep}_{\mathbb{C}}(G)$ we have $\chi_V(g) \in \mathbb{Q}$.

(b) The element g is conjugate to g^m for all $m \in \mathbb{Z}$ with $gcd(m,n) = 1$.

What does this say in the special case of symmetric and alternating groups?

Problem 18. Let G be a finite group.

(a) Show that up to isomorphism, the character table of the group determines the center $Z \subseteq \mathbb{C}[G]$ of the group algebra.

(b) For any irreducible $V \in \mathrm{Rep}_{\mathbb{C}}(G)$ with central character $\omega : Z \longrightarrow \mathbb{C}$, verify that

$$\omega(e_K) \;=\; \frac{\chi_V(g) \cdot |K|}{\dim V} \quad \text{for} \quad g \in G, \quad K = \mathrm{Cl}(g), \quad e_K = \sum_{h \in K} e_h \in Z.$$

(c) Show that if

$$\sum_{K \in \mathrm{Cl}(G)} e_K \;=\; c \cdot \prod_K e_K$$

for some $c \in \mathbb{C}$, then the character of any non-trivial irreducible $V \in \mathrm{Rep}_{\mathbb{C}}(G)$ vanishes somewhere, and deduce that then $G = [G,G]$.

Problem 19. Let λ be a partition of d, and fix any Young tableau for it.

(a) Find the trace of right multiplication by the Young symmetrizer $c_\lambda = a_\lambda b_\lambda$ on $\mathscr{A} = \mathbb{C}[\mathfrak{S}_d]$ and show

$$c_\lambda^2 \;=\; n_\lambda \cdot c_\lambda \quad \text{for} \quad n_\lambda \;=\; \frac{d!}{\dim_{\mathbb{C}} V_\lambda}.$$

(b) Show $\mathscr{A} \cdot a_\lambda b_\lambda \simeq \mathscr{A} \cdot b_\lambda a_\lambda$ and deduce that for the transpose tableau λ^t one has

$$V_{\lambda^t} \;\simeq\; V_\lambda \otimes sgn.$$

(c) Show that for the partition $\lambda = (d-1, 1)$, the representation $V_\lambda \in \mathrm{Rep}_{\mathbb{C}}(\mathfrak{S}_d)$ is isomorphic to the standard representation of dimension $d-1$.

Please bring your written solutions to the problem session next Thursday.

Problem 20. Let $V = \mathbb{C}^2$ be the standard representation of $G = Gl_2(\mathbb{C})$.

(a) Compute all the Schur functors $S_\lambda(V) \in \mathrm{Rep}_{\mathbb{C}}(G)$.

(b) Determine all Schur polynomials in two variables, and verify directly that the product of any two of them is a sum of Schur polynomials.

(c) What does your result say about the representations of $Sl_2(\mathbb{C})$?

Problem 21. Show by induction on n that the ring $S(x_1, \dots, x_n) = \mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ of symmetric functions is equal to a polynomial ring in the elementary symmetric polynomials:

$$\begin{aligned} \mathbb{Z}[y_1, \dots, y_n] &\xrightarrow{\sim} S(x_1, \dots, x_n), \\ y_\nu &\mapsto e_\nu(x_1, \dots, x_n) = \sum_{1 \le i_1 < \cdots < i_\nu \le n} x_{i_1} \cdots x_{i_\nu}. \end{aligned}$$

Hint: What is the kernel of the restriction map $S(x_1, \dots, x_n) \twoheadrightarrow S(x_1, \dots, x_{n-1})$?

Problem 22. Let $H, K \le G$ be two subgroups of a finite group and $S \subset G$ a set of representatives for the double cosets: $G = \bigsqcup_{s \in S} KsH$.

(a) For $V \in \mathrm{Rep}_{\mathbb{C}}(H)$ with induced representation $\mathrm{Ind}_H^G(V) = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ show that

$$\mathrm{Res}_K^G \, \mathrm{Ind}_H^G(V) = \bigoplus_{s \in S} W^s \quad \text{where} \quad W^s = \mathbb{C}[K] \cdot (s \otimes V) \in \mathrm{Rep}_{\mathbb{C}}(K).$$

(b) Put $V^s = \mathrm{Ad}_s^*(V)$ for the map $\mathrm{Ad}_s : sHs^{-1} \cap K \to H, x \mapsto s^{-1}xs$. Show that one gets a well-defined isomorphism

$$\mathrm{Ind}_{sHs^{-1} \cap K}^K (V^s) \xrightarrow{\sim} W^s \quad \text{via} \quad k \otimes v \mapsto ks \otimes v \quad \text{for } v \in V, \, k \in K.$$

For any partition λ, a Young tableau of shape λ is called a *standard tableau* if its rows and columns are increasing from left to right and top to bottom.

Problem 23. Let $d_\lambda \in \mathbb{N}$ be the number of standard tableaux of shape λ.

(a) Show that $d_\lambda = \sum_{\mu=\lambda\setminus\Box} d_\mu$ where the sum runs over all Young diagrams μ that can be obtained by removing one box from λ.

(b) Show that $(\deg(\mu)+1)\cdot d_\mu = \sum_{\nu=\mu\cup\Box} d_\nu$ where the sum runs over all ν that can be obtained by adding one box to the Young diagram μ.

(c) Deduce inductively that $d! = \sum_\lambda d_\lambda^2$ where the sum is over all λ of degree d.

Problem 24. For any standard tableau T of degree d, let $c_T \in \mathscr{A} = \mathbb{C}[\mathfrak{S}_d]$ denote its Young symmetrizer.

(a) Let $T_1 \neq T_2$ be standard tableaux of the same shape. Suppose that reading row by row from left to right and top to bottom, the first entry where they differ is bigger in T_2 than in T_1. Show that then $c_{T_2}\cdot c_{T_1} = 0$.

(b) Deduce that for any set S of standard tableaux of degree d, the sum of the corresponding left submodules $\sum_{T\in S}\mathscr{A}c_T \subseteq \mathscr{A}$ is direct.

(c) Deduce via problem 23 that the decomposition of $\mathscr{A}$ into isotypic pieces is given by

$$\mathscr{A} \quad = \bigoplus_{\deg(\lambda)=d} \mathscr{A}_\lambda \quad \text{where} \quad \mathscr{A}_\lambda \quad = \bigoplus_{\substack{T \text{ standard} \\ \text{of shape } \lambda}} \mathscr{A}\cdot c_T \quad \simeq \quad V_\lambda^{\oplus d_\lambda}$$

Problem 25. Let $H \leq G$ be finite groups and $V \in \mathrm{Rep}_{\mathbb{C}}(H)$.

(a) Show that $\chi_{\mathrm{Ind}_H^G(V)}(g) = \frac{1}{|H|}\sum_{x\in G, xgx^{-1}\in H} \chi_V(xgx^{-1})$ for all $g\in G$.

(b) Using problem 22 (=Mackey) or otherwise, show that $\mathrm{Ind}_H^G(V)$ is irreducible if and only if

- $V \in \mathrm{Rep}_{\mathbb{C}}(H)$ is irreducible, and
- $\mathrm{Hom}_{sHs^{-1}\cap H}(V^s, V|_{sHs^{-1}\cap H}) = 0$ for all non-trivial $[s]\in H\backslash G/H$.

Please bring your written solutions to the problem session next Thursday.

Problem 27. For $d, e \in \mathbb{N}$ consider the standard embedding $\mathfrak{S}_d \hookrightarrow \mathfrak{S}_{d+e}$.

(a) Show that for any partition μ of degree $d+e$ one has

$$V_\mu|_{\mathfrak{S}_d} \;\simeq\; \bigoplus_\lambda m_{\mu\lambda} \cdot V_\lambda$$

where λ runs over partitions of d whose Young diagram is contained in the diagram of μ, and $m_{\mu\lambda}$ is the number of ways to fill the labels $1, 2, \ldots, e$ into the boxes of the complement $\mu \setminus \lambda$ such that the rows and columns are increasing:

$$\begin{array}{|c|c|c|c|}\hline \bullet & \bullet & 1 & 2 \\ \hline \bullet & 3 & 4 \\ \cline{1-3}\end{array} \quad \begin{array}{|c|c|c|c|}\hline \bullet & \bullet & 1 & 3 \\ \hline \bullet & 2 & 4 \\ \cline{1-3}\end{array} \quad \begin{array}{|c|c|c|c|}\hline \bullet & \bullet & 1 & 4 \\ \hline \bullet & 2 & 3 \\ \cline{1-3}\end{array} \quad \begin{array}{|c|c|c|c|}\hline \bullet & \bullet & 2 & 3 \\ \hline \bullet & 1 & 4 \\ \cline{1-3}\end{array} \quad \begin{array}{|c|c|c|c|}\hline \bullet & \bullet & 2 & 4 \\ \hline \bullet & 1 & 3 \\ \cline{1-3}\end{array} \quad m_{(4,3)(2,1)} = 5$$

(b) What does this say in the special cases where $d = 1$ or $e = 1$?

Problem 28. Let $W \in \mathrm{Rep}(\mathfrak{S}_d)$ be the standard representation of dimension $d-1$.

(a) Using the previous result, show by induction on $d \in \mathbb{N}$ that for every $\nu < d$ one has

$$V_{(d-\nu,1,\ldots,1)} \;\simeq\; \mathrm{Alt}^\nu(W).$$

(b) Show similarly that

$$\mathrm{Sym}^2(W) \;\simeq\; \mathbf{1} \oplus W \oplus V_{(d-2,2)} \quad \text{for} \quad d \;\geq\; 4.$$

Problem 29. Show by induction on $d \in \mathbb{N}$ that the only irreducible $V \in \mathrm{Rep}(\mathfrak{S}_d)$ with $1 < \dim(V) < d$ are the standard representation and its tensor product with the sign character, plus the following exceptions:

- $d = 4$ and $V \simeq V_{(2,2)}$,
- $d = 6$ and $V \simeq V_{(3,3)}$ or $V \simeq V_{(2,2,2)}$.

What are the dimensions $\dim(V)$ in these exceptional cases?

Please bring your written solutions to the problem session next Thursday.

Problem 30. Let $\mathsf{M} = M_{n\times n}(\mathbb{R})$ be the vector space of real $n\times n$ matrices and $\mathsf{S} \subset \mathsf{M}$ the subspace of all symmetric matrices.

(a) Identifying these spaces with their respective tangent spaces, show that the derivative of $f : \mathsf{M} \to \mathsf{S}, A \mapsto A^t A$ at any point $A \in O(n) = f^{-1}(\mathbf{1})$ is the linear map

$$(Df)(A) : \quad T_A(\mathsf{M}) \longrightarrow T_{\mathbf{1}}(\mathsf{S}), \quad X \mapsto A^t X + X^t A.$$

(b) Show that this linear map is surjective, and deduce that the subset $O(n) \subset \mathsf{M}$ is a compact Lie group. What is its dimension?

(c) Prove the corresponding statements for $U(n)$ and $Sp(n)$.

Problem 31. Viewing the Hamiltonian quaternions $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ as a $\mathbb{C}$-algebra, find an embedding $M_{n\times n}(\mathbb{H}) \hookrightarrow M_{2n\times 2n}(\mathbb{C})$ of matrix rings that gives rise to an isomorphism

$$Sp(n) \simeq U(2n) \cap Sp_{2n}(\mathbb{C}).$$

Problem 32. Let G be a Lie group.

(a) Show that there is a unique homomorphism $c : G \to \mathbb{R}^\times$ such that for any left invariant volume form ω on G the pull-back under right translations is given by

$$\rho_h^*(\omega) = c(h) \cdot \omega \quad \text{for} \quad h \in G.$$

(b) Compute c for the Lie group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; b \in \mathbb{R},\, a \in \mathbb{R}^\times \right\} \subset Gl_2(\mathbb{R}).$$

(c) If G is compact, show that $c(h) \in \{\pm 1\}$ for all $h \in G$.

(d) Looking at orientations, show that c is non-trivial for $G = O(2)$.

Please bring your written solutions to the problem session next Thursday.

Problem 33. Let $V = \mathbb{R}^n$ for some $n \in \mathbb{N}$.

(a) Show that every continuous map $f : V \to \mathbb{R}$ with $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ must be $\mathbb{R}$-linear.

(b) Deduce from this that any continuous homomorphism $\rho : V \to U(1)$ has the form $\rho(x) = e^{if(x)}$ for some $f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$.

(c) Find all irreducible continuous representations of tori $U(1) \times \cdots \times U(1)$.

Problem 34. Let G be a compact Lie group. If $V_1, V_2 \in \mathrm{Rep}_{\mathbb{C}}(G)$ are isomorphic representations, show that for any G-invariant Hermitian inner products $\langle \cdot, \cdot \rangle_i$ on V_i there is an isomorphism

$$f \in \mathrm{Hom}_G(V_1, V_2) \quad \text{with} \quad \big\langle f(u), f(v) \big\rangle_2 = \langle u, v \rangle_1 \quad \text{for all } u, v \in V_1,\ g \in G.$$

Problem 35. For a compact Lie group G and $V \in \mathrm{Irr}_{\mathbb{C}}(G)$, a bilinear form on V is called *G-invariant* if it is preserved under the action of the group. Show that there is a nondegenerate such bilinear form iff

$$\mathrm{Hom}_G(V, \overline{V}) \neq \{0\}.$$

Deduce that in this case there exists

$$J \in \mathrm{Hom}_G(V, \overline{V}) \quad \text{with} \quad J \circ J = \pm \mathrm{id}_V$$

and that the bilinear form is

(a) symmetric iff $V \simeq W \otimes_{\mathbb{R}} \mathbb{C}$ for some real representation $W \in \mathrm{Rep}_{\mathbb{R}}(G)$,

(b) alternating iff V is a vector space over the quaternions $\mathbb{H}$ and $V \in \mathrm{Rep}_{\mathbb{H}}(G)$.

Looking at the character of $Sym^2(V)$ and $Alt^2(V)$, show

$$\int_G \chi_V(g^2)\, dg = \begin{cases} +1 & \text{in case (a)}, \\ -1 & \text{in case (b)}, \\ 0 & \text{otherwise}. \end{cases}$$

Please bring your written solutions to the problem session next Thursday.

Problem 36. Let $n \in \mathbb{N}$. The action of $G = O(n)$ on $\mathbb{R}^n$ induces for each $d \in \mathbb{N}$ an action on

$$V_d \;=\; \big\{ \text{homogenous polynomials } P : \mathbb{R}^n \longrightarrow \mathbb{C} \text{ of degree } d \big\} \;\subset\; \mathbb{C}[x_1, \dots, x_n],$$

given by

$$(g \cdot P)(x_1, \dots, x_n) \;:=\; P(y_1, \dots, y_n) \quad \text{for} \quad y_j \;:=\; \sum_{i=1}^{n} g_{ij} \cdot x_i$$

where g_{ij} are the matrix entries of $g \in G$. Show that

$$\langle \cdot, \cdot \rangle : \quad V_d \times V_d \;\longrightarrow\; \mathbb{C}, \quad \langle P, Q \rangle \;:=\; \overline{P}\big(\tfrac{\partial}{\partial x_1}, \dots, \tfrac{\partial}{\partial x_n}\big)\big(Q(x_1, \dots, x_n)\big)$$

defines a G-invariant Hermitian inner product, and that for this inner product one has an orthogonal splitting

$$V_d \;=\; \ker(\Delta) \oplus \big(x_1^2 + \cdots + x_n^2\big)\, V_{d-2} \quad \text{where} \quad \Delta \;:=\; \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \;\in\; \mathrm{Hom}_G(V_d, V_{d-2}).$$

Problem 37. Compute the Weyl groups of $SO(2n)$, $SO(2n+1)$ and $Sp(n)$.

Problem 38. Consider the *left* action of $Sp(n)$ on $\mathbb{H}^n$ (as a *right* $\mathbb{H}$-module).

(a) Show that every $A \in Sp(n)$ has an eigenvector with a complex eigenvalue in the sense that

$$A \cdot v \;=\; v \cdot \lambda \quad \text{for some} \quad v \;\in\; \mathbb{H}^n \setminus \{0\} \quad \text{and} \quad \lambda \;\in\; \mathbb{C} \;\subset\; \mathbb{H}.$$

(b) Deduce that the right $\mathbb{H}$-module $\mathbb{H}^n$ has an orthonormal basis of eigenvectors for A with complex eigenvalues, and that there exists a matrix $B \in Sp(n)$ such that

$$B^{-1}AB \;=\; \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad \text{with} \quad \lambda_1, \dots, \lambda_n \;\in\; U(1).$$

Please bring your written solutions to the problem session next Thursday.

Problem 39. Show that $\mathbb{R}^3$ forms a Lie algebra with respect to the bracket defined by the cross-product

$$\times : \quad \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \vec{v} \times \vec{w} := \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$$

and find an isomorphism of Lie algebras

$$f : \quad \mathbb{R}^3 \xrightarrow{\sim} \mathfrak{so}(3) := \mathrm{Lie}(SO(3))$$

such that for the standard action of $\mathfrak{so}(3)$ on $\mathbb{R}^3$ one has $\vec{x} \times \vec{v} = (f(\vec{x}))(\vec{v})$.

Problem 40. For $G = Sp(n), SO(2n)$ or $SO(2n+1)$, determine the weights of the adjoint representation

$$\big(\,\mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C},\, Ad\big)|_T \;\in\; \mathrm{Rep}_{\mathbb{C}}(T)$$

with respect to the maximal torus $T \subset G$ that we considered in the lecture.

Problem 41. Show that $Sp(1) \simeq SU(2)$ and this group is simply connected. Look at its adjoint representation to find a homomorphism $SU(2) \to SO(3)$ with finite kernel, so $SO(3)$ is not simply connected. What is its fundamental group?

Let G be an arbitrary Lie group with Lie algebra $\mathfrak{g} = Lie(G)$.

Problem 42. Let $X_1, X_2, \dots \in \mathfrak{g} \setminus \{0\}$ be a sequence of nonzero elements converging to zero. Passing to a subsequence we may assume that for suitable $\lambda_1, \lambda_2, \dots \in \mathbb{R}_{>0}$ and $Y_n := \lambda_n \cdot X_n$ the limit

$$Y := \lim_{n \to \infty} Y_n$$

exists and is nonzero. If $H \subset G$ is a closed topological subgroup with $exp(X_n) \in H$ for all $n \in \mathbb{N}$, show that

$$exp(sY) \in H \quad \text{for all} \quad s \in \mathbb{R}.$$

Problem 43. Verify that for any closed topological subgroup $H \subset G$ the following subset is a real vector subspace:

$$\mathfrak{h} := \left\{ \tfrac{d\alpha}{dt}|_{t=0} \;\middle|\; \alpha : \mathbb{R} \to G \text{ smooth with } \alpha(\mathbb{R}) \subseteq H \text{ and } \alpha(0) = e \right\} \subseteq \mathfrak{g}.$$

Locally writing $\alpha = exp \circ \beta$, observe that $\frac{d\alpha}{dt}|_{t=0} = \lim_{n\to\infty} Y_n$ for $Y_n = n \cdot \beta(\frac{1}{n})$ and deduce

$$\mathfrak{h} = \left\{ Y \in \mathfrak{g} \;\middle|\; exp(sY) \in H \text{ for all } s \in \mathbb{R} \right\}.$$

Problem 44. In the above setting, show that for any subspace $\mathfrak{h}' \subseteq \mathfrak{g}$ with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$ there are open neighborhoods

$$0 \in U \subseteq \mathfrak{h} \quad \text{and} \quad 0 \in U' \subseteq \mathfrak{h}'$$

such that the map

$$\varphi : \quad U \times U' \longrightarrow G, \quad (u, u') \mapsto exp(u) \cdot exp(u')$$

is an open embedding with $\varphi^{-1}(H) = U \times \{0\}$. Deduce that H is a Lie subgroup.