

Please hand in your solutions at the beginning of the problem class on Friday.

**Problem 1.** (a) Show that the Cauchy-Riemann equations for  $f = u + iv : U \rightarrow \mathbb{C}$  can be written as

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \phi}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \phi}$$

in polar coordinates  $(r, \phi)$  on any sufficiently small open subset  $U \subset \mathbb{C} \setminus \{0\}$ .

(b) Deduce that on any such open there exists a holomorphic function  $f : U \rightarrow \mathbb{C}$  such that  $\exp(f(z)) = z$ . Fixing any such function near  $z = 1$ , compute its analytic continuation along the two paths

$$\gamma_{\pm} : [0, \pi] \rightarrow \mathbb{C} \setminus \{0\}, \quad t \mapsto \exp(\pm it).$$

**Problem 2.** Show that every 1-dimensional complex torus  $\mathbb{C}/\Lambda$  is isomorphic to one of the form

$$X_{\tau} = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau \quad \text{for some } \tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}.$$

Show furthermore that

$$X_{\sigma} \simeq X_{\tau} \quad \text{if } \sigma = \frac{a\tau + b}{c\tau + d} \quad \text{for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

**Problem 3.** In the lecture we have defined a topological manifold as a topological space which is

- (a) Hausdorff,
- (b) second countable, and
- (c) covered by charts homeomorphic to open subsets in  $\mathbb{R}^n$  for some  $n \in \mathbb{N}$ .

Show by examples that none of these three conditions follows from the other two.