

Please hand in your solutions at the beginning of the problem class on Friday.

Problem 4. Let X be a Riemann surface and $f : X \setminus \{p\} \rightarrow \mathbb{C}$ a holomorphic function on the complement of a point p . Show that the following three conditions are equivalent:

- (a) f does not extend to a meromorphic function on X ,
- (b) for every open neighborhood $U \subseteq X$ of p , the image $f(U \setminus \{p\})$ is dense in \mathbb{C} ,
- (c) there exists a sequence of points $p_n \in X \setminus \{p\}$ converging to p such that the sequence

$$(f(p_n))_{n \in \mathbb{N}} \text{ does not converge in } \mathbb{P}^1(\mathbb{C}).$$

Problem 5. Determine the group $\text{Aut}(\mathbb{C})$ of biholomorphic automorphisms of the complex plane. Hint: For any such automorphism $f \in \text{Aut}(\mathbb{C})$ consider the Laurent series of $z \mapsto f(1/z)$ on $\mathbb{C} \setminus \{0\}$, and apply the previous exercise.

Problem 6. Let X be a Hausdorff topological space and $\bar{X} = X \cup \{\infty\}$ its disjoint union with another point.

- (a) Show that \bar{X} becomes a compact topological space by declaring $U \subseteq \bar{X}$ to be open iff
 - either $U \subseteq X$ is an open part of the original topological space,
 - or $U = \bar{X} \setminus K$ is the complement of a compact subset $K \subseteq X$.
- (b) Compute \bar{X} for $X = \mathbb{R}^n$ and for $X = (0, 1) \cup (2, 3)$.

Bonus Problem (optional). If you would like to review complex differentiability,

- (a) show that $f : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}, f(z) = \frac{z^5}{|z|^4}$ extends to a partially differentiable function over the origin and satisfies the Cauchy-Riemann equations there. Is the extended function holomorphic at the origin?
- (b) recall that $g : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}, g(z) = z^2 \sin(1/z)$ extends differentiably over the origin but its derivative is not continuous there. What if you replace \mathbb{R} by \mathbb{C} and ask for complex differentiability?