Please hand in your solutions at the beginning of the problem class on Friday.

Problem 7. In class we have seen that any meromorphic function $f \in \mathscr{M}(\mathbb{P}^1(\mathbb{C}))$ is a rational function

$$f(z) \;=\; \frac{p(z)}{q(z)} \quad \text{for some} \quad p, q \;\in\; \mathbb{C}[z], \quad q \neq 0.$$

Use this to determine the group of biholomorphic automorphisms $\mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$.

Problem 8. Which of the following Riemann surfaces are isomorphic? In each case write down a biholomorphic map or explain why such a map cannot exist:

$$\begin{aligned} D &= \{z \in \mathbb{C} \mid |z| < 1\}, & A &= \{z \in \mathbb{C} \mid \tfrac{1}{2} < |z| < 1\}, & &\mathbb{C}, \\ D^* &= D \setminus \{0\}, & \mathbb{H} &= \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}, & &\mathbb{P}^1(\mathbb{C}). \end{aligned}$$

Problem 9. Let $f_s : (X, x_0) \to (Y, y_0)$ for $s = 0, 1$ be continuous maps of pointed topological spaces. We say that the maps are *homotopic* if there is a continuous map

$$H: \quad [0,1] \times X \;\to\; Y \quad \text{with} \quad \begin{cases} H(s, -) \;=\; f_s & \text{for } s = 0, 1, \\ H(s, x_0) \;=\; y_0 & \text{for all } s \in [0,1]. \end{cases}$$

Show that this gives an equivalence relation on maps. If $HTop_*$ is the category whose objects are pointed topological spaces and whose morphisms are homotopy classes of maps, verify that $HTop_* \to Groups$, $(X, x_0) \mapsto \pi_1(X, x_0)$ is a well-defined functor. What is the relation between

(a) homeomorphisms,

(b) isomorphisms in the category $HTop_*$,

(c) continuous maps inducing an isomorphism on fundamental groups?

Bonus Problem (optional). If you feel uneasy about line integrals, reprove Liouville's theorem for bounded holomorphic $f : \mathbb{C} \to \mathbb{C}$ by looking at

$$\oint_{|\zeta|=R} \frac{f(\zeta)}{(\zeta - a)(\zeta - b)}\, d\zeta \qquad \text{for} \quad a, b \in \mathbb{C} \quad \text{and} \quad R \to \infty.$$