Please hand in your solutions at the beginning of the problem class on Friday.

Problem 10. Let X be a topological space.

(a) Show that if $U_1, U_2 \subseteq X$ are simply connected open subsets with $U_1 \cap U_2$ pathconnected and nonempty, then $U_1 \cup U_2$ is simply connected.

(b) Deduce that the sphere $X = S^n$ is simply connected for $n \geq 2$.

(c) For $n = 2$, find a continuous curve $\gamma : [0,1] \twoheadrightarrow S^2$ covering the sphere.

Problem 11. Show that for $X \subseteq \mathbb{C}$ open, a holomorphic function $f : X \to \mathbb{C}$ is a local homeomorphism near a point $x_0 \in X$ iff $f'(x_0) \neq 0$. Apply this criterion to see that

$$\sin : \quad X \;=\; \mathbb{C} \setminus \left\{ \tfrac{\pi}{2} + k\pi \mid k \in \mathbb{Z} \right\} \;\to\; Y \;=\; \mathbb{C} \setminus \{\pm 1\}$$

is a covering map. Let $u_\pm : [0,1] \to Y$ be the curves given by $t \mapsto \pm(e^{2\pi i t} - 1)$, and consider the composite

$$\gamma_+ \;=\; u_+ \cdot u_- \quad \text{and} \quad \gamma_- \;=\; u_- \cdot u_+.$$

Find $\tilde{\gamma}_\pm(1)$ for the unique continuous lifts $\tilde{\gamma}_\pm : [0,1] \to X$ of $\gamma_\pm$ with $\tilde{\gamma}_\pm(0) = 0$.

Problem 12. Let $p : X \longrightarrow Y$ be a covering map of finite degree between Riemann surfaces. Show that for any $h \in \mathscr{O}(X)$,

$$P_h(t) \quad := \prod_{x \in p^{-1}(y)} \big(t - h(x)\big) \;\in\; \mathscr{O}(Y)[t]$$

is a polynomial in t whose coefficients depend holomorphically on y. Deduce that the ring extension $\mathscr{O}(Y) \hookrightarrow \mathscr{O}(X), f \mapsto f \circ h$ is *integral*, i.e. any element of the bigger ring is a zero of a monic polynomial over the smaller ring.

Bonus problem. If $U \subseteq \mathbb{C}$ is a connected open subset and $\{f_n : U \to \mathbb{C}\}_{n \in \mathbb{N}}$ is a sequence of injective holomorphic functions converging uniformly on compact subsets to a function $f : U \to \mathbb{C}$, show that f is either injective or constant.