Please hand in your solutions at the beginning of the problem class on Friday.

Problem 13. Let $\Lambda, \Lambda' \subset \mathbb{C}$ be lattices.

(a) Show that any holomorphic map $f : \mathbb{C}/\Lambda' \to \mathbb{C}/\Lambda$ with $f(0) = 0$ comes from a linear map

$$\tilde{f} : \mathbb{C} \to \mathbb{C}, \; z \mapsto az \quad \text{for a unique } a \in \mathbb{C} \text{ with } a \cdot \Lambda' \subseteq \Lambda.$$

(b) If $\Lambda' = \Lambda$ and if $f \neq \pm\mathrm{id}$ is a non-trivial automorphism, show that a must be a root of unity with the property that $\Lambda = \mathbb{Z}\lambda \oplus \mathbb{Z}a\lambda$ for any non-zero lattice vector $\lambda \in \Lambda \setminus \{0\}$ of minimal length.

(c) Find all automorphisms of the complex tori

$$\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}) \quad \text{and} \quad \mathbb{C}/(\mathbb{Z} \oplus \rho\mathbb{Z}) \quad \text{for} \quad \rho = \exp\left(\tfrac{2\pi i}{3}\right),$$

and show that any torus as in part b is isomorphic to one of these two.

Problem 14. For $n \in \mathbb{N}$, show that $f_n : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}), z \mapsto z^n + z^{-n}$ is a branched Galois cover whose group of deck transformations is the dihedral group

$$D_{2n} = \left\langle \sigma, \rho \;\middle|\; \rho^n = \sigma^2 = \mathrm{id}, \;\; \sigma\rho\sigma = \rho^{-1} \right\rangle.$$

Problem 15. For each of the following two maps $f : Y = \mathbb{P}^1(\mathbb{C}) \to X = \mathbb{P}^1(\mathbb{C})$, find the ramification/branch locus

$$Y \supset R(f) \longrightarrow Br(f) \subset X$$

and the deck transformation group $\mathrm{Aut}(Y/X)$, and decide whether this branched cover is Galois:

(a) $f(z) = z^3 - 3z$,

(b) $f(z) = (z^2 + 1)^2$.

Bonus problem. If X is a Riemann surface and $G \to \mathrm{Aut}(X)$ is a freely discontinuous group action by biholomorphic automorphisms, check that the quotient $Y = X/G$ is again a Riemann surface. Determine $\pi_1(Y, y)$ when X is simply connected.