

Please hand in your solutions at the beginning of the problem class on Friday.

**Problem 16.** Check that the affine algebraic curves

$$C = \{(x, y) \in \mathbb{C}^2 \mid y^2 = 3 + 10x^4 + 3x^8\}, \quad D = \{(z, w) \in \mathbb{C}^2 \mid w^2 = z^6 - 1\}$$

are both smooth. How many points must be added to make them compact Riemann surfaces? Show that

$$f: C \setminus \{(\pm 1, \pm 4)\} \longrightarrow D, \quad (x, y) \mapsto \left( \frac{1+x^2}{1-x^2}, \frac{2xy}{(1-x^2)^3} \right)$$

is well-defined and extends to an unramified cover between these Riemann surfaces.

**Problem 17.** For  $d \in \mathbb{N}$ , find all  $f \in \mathbb{C}[x]$  so that the polynomial  $y^d - f(x) \in \mathbb{C}[x, y]$  is irreducible. In the irreducible case consider the algebraic curve

$$\{(x, y) \in \mathbb{C}^2 \mid y^d = f(x)\}.$$

Show that for the associated compact Riemann surface  $Y$  the map  $x: Y \rightarrow \mathbb{P}^1(\mathbb{C})$  is a branched Galois cover. Find its deck transformation group, and decompose the field  $\mathcal{M}(Y)$  of meromorphic functions into eigenspaces for the action of this group.

**Problem 18.** For  $F \in \mathbb{C}[x, y, z]$  irreducible and homogenous of positive degree, show there is a unique compact Riemann surface with a holomorphic generically injective map  $f: X \rightarrow \mathbb{P}^2(\mathbb{C})$  whose image is

$$f(X) = \{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) \mid F(x, y, z) = 0\} \subset \mathbb{P}^2(\mathbb{C}).$$

Find this Riemann surface and the map  $f$  for the polynomial  $F(x, y, z) = y^2z - x^3$ .

**Bonus problem.** Let  $f \in \mathbb{C}[x, y]$ . Viewing  $f$  as a polynomial over the field  $K = \mathbb{C}(x)$ , let  $\alpha_1, \dots, \alpha_n$  be its roots in an algebraic closure of  $K$ , including multiplicities. Show that

$$\Delta := \prod_{i < j} (\alpha_i - \alpha_j)^2 \in \mathbb{C}[x]$$

and that for any  $x_0 \in \mathbb{C}$ , one has  $\Delta(x_0) = 0$  iff  $f(x_0, y) \in \mathbb{C}[y]$  has a multiple root.