

Please hand in your solutions at the beginning of the problem class on Friday.

**Problem 19.** Let  $F : Y \rightarrow X$  be a branched cover of degree  $n$ .

- (a) Check that the following properties are equivalent:
- $[\mathcal{M}(Y) : \mathcal{M}(X)] = n$ ,
  - there is a function  $t \in \mathcal{M}(Y)$  that is injective on a general fiber  $F^{-1}(x)$ .
- (b) Verify these properties when  $F : Y \rightarrow \mathbb{C}$  is obtained by glueing two copies of the complex plane along the countably many slits  $[2n, 2n + 1]$  with  $n \in \mathbb{Z}$ .
- (c) Can you extend the double cover in (b) to a cover of the Riemann sphere?

**Problem 20.** Let  $\Lambda \subset \mathbb{C}$  be a lattice.

- (a) Show that on any compact subset  $K \subset \mathbb{C}$  the series

$$\wp_{\Lambda}(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left( \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right)$$

can be made uniformly convergent by removing finitely many summands, and that  $\wp_{\Lambda} \in \mathcal{M}(\mathbb{C})$  is a well-defined meromorphic function with poles only in  $\Lambda$ .

- (b) Look at derivatives to see that  $\wp_{\Lambda}$  is  $\Lambda$ -periodic. Using that  $\wp_{\Lambda}(z) = \wp_{\Lambda}(-z)$  is an *even* function of  $z$ , determine its multiplicity at every point.

**Problem 21.** Let  $\Lambda \subset \mathbb{C}$  be a lattice and  $f \in \mathcal{M}(\mathbb{C})$  a  $\Lambda$ -periodic function.

- (a) If  $f(z) = f(-z)$  is an even function of  $z$ , show that
- for  $f^{-1}(\infty) \subseteq \Lambda$  one has  $f = P(\wp_{\Lambda})$  for some polynomial  $P \in \mathbb{C}[x]$ ,
  - in any case one has  $f = R(\wp_{\Lambda})$  for a rational function  $R \in \mathbb{C}(x)$ .

- (b) In general, show that

$$f = R(\wp_{\Lambda}) + S(\wp_{\Lambda}) \cdot \wp'_{\Lambda} \quad \text{for rational functions } R, S \in \mathbb{C}(x).$$

- (c) Find  $R$  and  $S$  when  $f(z) = (\wp'_{\Lambda}(z))^2$  and explain what this says about  $\mathbb{C}/\Lambda$ .