Next Friday we celebrate Grothendieck's 90th birthday at the BMS, so there will be no problem class. Please hand in your solutions before the lecture on Monday.

Problem 22. Two simple applications of the Riemann-Hurwitz formula:

(a) Let $f : X \to X$ be a branched cover of degree $d > 1$ from a compact Riemann surface to itself. What can you say about the genus of the surface and the number of ramification points? Find examples for all cases.

(b) For $n \in \mathbb{N}$, compute the genus of the compact Riemann surface attached to the algebraic curve $\{(x, y) \in \mathbb{C}^2 \mid x^n + y^n = 1\}$. Can you generalize this?

Problem 23. Let X be a Riemann surface, and take a finite group $G \subset \operatorname{Aut}(X)$ of biholomorphic automorphisms.

(a) Define the structure of a Riemann surface on the topological space $Y = X/G$.

Hint: First divide out the action of the stabilizer $G_p = \{g \in G \mid gp = p\}$ in some neighborhood of $p \in X$. Show that if z is a local coordinate centered at p then a local coordinate on the quotient is given by $\prod_{g \in G_p} g^*(z)$.

(b) If X is compact, show that

$$\chi(X) \;=\; |G| \cdot \chi(Y) - \sum_{p \in X} \big(|G_p| - 1 \big).$$

(c) Deduce that if a compact Riemann surface of genus two has an automorphism of prime order p, then $p \in \{2, 3, 5\}$. Look at hyperelliptic Riemann surfaces to find examples where these three primes actually occur.

Problem 24. Let X be a Riemann surface. For open subsets $U \subseteq X$ and $a \neq b \in X$ put

$$\begin{aligned}
\mathscr{E}(U) &= \mathscr{O}_X^*(U)/\exp(\mathscr{O}_X(U)),\\
\mathscr{F}(U) &= \{\, f \in \mathscr{O}_X(U) \mid f(a) = 0 \text{ if } a \in U \,\},\\
\mathscr{G}(U) &= \{\, f \in \mathscr{O}_X(U) \mid f(a) = f(b) \text{ if } a, b \in U \,\},\\
\mathscr{H}(U) &= \{\, \text{bounded continuous functions } f : U \to \mathbb{C} \,\}.
\end{aligned}$$

Which of these presheaves are sheaves? For the others, what is the associated sheaf?